TOPOLOGICAL METHOD FOR TAUBERIAN THEOREM.*)**)

By
Masanori Fukamiya.

1. Introduction. The present paper deals with some algebraic-topological methods in the theory of Fourier analysis, especially with the so-called "Tauberian Theorems" of N. Wiener [1]. Our point of stand is that of the theory of normed rings, recently developped [2] by I. Gelfand, A. Kolmogoroff, D. Raikov, M.-S. Krein and others, where the Fourier transformations are considered as the ring-isomorphic representants of original functions. The topological commutative groups are taken as the field of our analysis; it is rather inevitable to reconsider the Pontrjagin topology of character groups [3].

The author is grateful to Prof. S. Izumi, Prof. K. Yosida for their kind criticisms and encouragements, to which the present paper owes very much.

2. The Ring L. Let G be a commutative topological group, locally bicompact or separable. There exists the (unique up to the constant factor) invariant Lebesgue measure[4] $m(A)$ on G, which has the following properties:

(a) The outer measure $m(A)$ is defined for all subsets $A \subset G$, and $0 \leqq m(A) \leqq +\infty$;

---

*) Received Oct. 5th, 1942.

**) Read before the Annual Meeting of the Math.-Phys. Soc. of Japan. July, 1942.

1) N. Wiener, Tauberian Theosrems, Annals of Math., 33 (1932); The Fourier Integral and certain of its Applications, Cambridge, 1933; especially. Chap. II.

N. Wiener-R.E.A.C. Paley, Fourier Transforms in the Complex Domain, 1934.

2) I. Gelfand, Normierte Ringe, Recueil Math. de Moscou, 9(51) (1941);

I. Gelfand-G. Silov, Uber verschiedene Methoden der Einfuhrung der Topologie in die Menge der Maximalen Ideale eines normierten Ringes, ibidem.

I. Gelfand, Uber absolut konvergente trigonometrische Reihen und Integrale, ibidem;

I. Gelfand-A. Kolmogoroff, On rings of continuous functions on topological space, C.R.URSS, 23(1939).

3) L. Pontrjagin, Topological Groups, Princeton, 1939; especially; Chapt. V.

(b) If $A \subset \Sigma A_n$, then $m(A) \leqq \Sigma m(A_n)$ (the sum being finite or enumerable);

(c) For the empty set Λ, $m(\Lambda)=0$.

Following Caratheodory's definition of the measurability and the measure, the latter being denoted by $m(A)$;

(d) All open sets O are measurable and $m(O)>0$; all bicompact (closed) sets C are of finite measure;

(e) $m(A)=\inf m(B)$, where B are open sets covering A;

(f) The measure is invariant: $m(A+g)=m(A)$ for all A and for all $g \in G$;

(g) The measure is symmetric: $m(-A)=m(A)$ for all A;

(h) The measure is continuous with respect to translations, that is, $m(A+g)$ is a continuous function on G;

(i) There exit the sets $E_n, n=1,2,\cdots$, each of finite measure and $G=\Sigma E_n$. It is possible to define for complex valued functions on G (i) a notion of measurability, (ii) a notion of summability, (iii) an integral $\int_G x(g)dg$; all the formal properties of these notions hold good as in the usual Lebesgue theory of integration on the real axis.

The set L_1 of all summable complex-valued functions on G forms a commutative normed ring, defined by Mazur and Gelfand[2], where we understand the following:

(1) The addition is the usual function-addition: $x+y(g)=x(g)+y(g)$;

(2) The product of two elements x, y is the "convolution product (Faltung)":

$$x \circ y(g)=\int_G x(g-h)y(h)\, dh=y \circ x(g),$$

where the integral exists for almost all g in virtue of Fubini's theorem;

(3) The norm is defined and denoted by

$$\| x(g) \| =\int_G |x(g)|\, dg.$$

Generally, two cases are possible: (i) Every point $g \in G$ has a positive measure, in which case the group is discrete, and L_1 has an unit element e:

---

4) A.Haar, Der Massbegriff in der Theorie der kontinuierlichen Gruppen, Annals of Math., 34(1933);

J.von Neumann, Zum Haarschen Mass in Topologischen Gruppen, Comp. Math., 1(1934);

A. Weil, L'intégration dans les groupes topologique et ses applications. Actualité 863, Paris. 1940.

$\epsilon(g)=1$ or 0 according as $g=0$ or not: (ii) Every point has the measure 0, in which case L_1 has not the unit, but by formally adjoining the unit element e, we may consider the extended ring L in stead of L_1.

In L, the generic element is of the form $z=\alpha e+x(g)$ with the norm $\|z\| =|\alpha|+\|x(g)\|$, where α is any complex number and $x(g)$ is any element of L_1.

The following considerations are carried out for the ring L, but they remain valid wtih corresponding modifications also in the case (i), when the group G is discrete and L_1 has an unit element [5].

3. The Riemann-Lebesgue lemma. We consider the maximal ideals of L. Let M_0 be any maximal ideal of L. The number, into which the generic element z passes under the homomorphism $L\to L/M$, we shall denote by (z,M_0). The set of all maximal ideals of L will be denoted by $\mathbf{M}$.

The maximal ideals of L are closely connected with the continuous characters of G; in fact, I. Gelfand and D. Raikov [6] proved the following

Therem 1. Every maximal ideal $M\neq L_1$, induces certain continuous character of G, i.e.

(1) $$\chi(h)=\frac{(x(g+h),M)}{(x(g),M)},\ x(g) \text{ being any element of } L_1;$$

and conversely, every continuous (or even measurable) character $\chi(g)$ induces certain maximal ideal of L, i.e.

(2) $$(z,M)=(\alpha e+x(g),\ M)=\alpha+\int_G x(g)\overline{\chi}(g)\ dg.$$

(1) and (2) are involutery; the correspondence between the maximal ideals, $\neq L_1$, and the continuous characters is one-to-one.

In what follows, we shall denote by X the charcter group, in the sence of Pontrjagin, hence X is also a commutative group, locally bicompact or separable. For every bicompact set $F\subset G$, and for every $\varepsilon>0$ every subset $U=[\chi/\chi(F)-1<\varepsilon]$ constitute the neighborhood system $\mathfrak{U}$ of the unit element

---

5) D.Raikov, Positive definite functions on commutative groups with an invariant measure, C.R.URSS., 28(1940), pp. 7-11.

D.Raikov, Positive definite functions on discrete commutative groups, C.R.URSS., 27 (1940), pp. 324–328.

6) I.Gelfand-D. Raikov, On the theory of characters of commutative topological groups, C.R.URSS 28(1940);

I. Gelfand, Zur Theorie der Charactere der Abe'schen topologischen Gruppen. Recueil math. de Moscou, 9(51)(1941).

in X.

By Theorem 1, the set $\mathbf{M}-(L_1)$ is identical with the set X, hence a group. However, we may define the weak topology on $\mathbf{M}-(L_1)$[7], in teams of L, and we prove that the so defined topological space $\mathbf{M}-(L_1)$ is homeomorphic to X. At first, we prove

Lemma 1. (Generalistion of Riemann-Lebesgue lemma) If $x(g)\varepsilon L_1$, then

$$\lim_{\chi\to(\chi_\infty)}\int_G x(g)\chi(g)dg=0,$$

where the sign $\lim_{\chi\to(\chi_\infty)}$ are taken in the sense that, for every $\varepsilon>0$, there exists a bicompact set $F\subset K$, such that

$$\left|\int_G \chi(g)\bar{\chi}(g)\ dg\right|<\varepsilon,\quad \text{for all } \chi\bar{\varepsilon}F.$$

Proof. we have

$$\int_G x(g+h)\bar{\chi}(g)dg=\chi(h)\int_G x(g)\bar{\chi}(g)dg,$$

and

$$|\chi(h)-1|\left|\int_G x(g)\bar{\chi}(g)\ dg\right|\leqq\int_G |x(g+h)-x(g)|\ dg,$$

and, for appropriately chosen neighborhood V (with bicompact $\bar{V}$) of the unit element of G,

$$\int_G |x(g+h)-x(g)|\,dg<\varepsilon^2,\ \text{for all } h\varepsilon V,$$

hence if we put $H=\left[\chi\Big/\left|\int_G x(g)\chi(q)\ dg\right|\geqq\varepsilon\right]$, then

$$|\chi(h)-1|<\varepsilon\quad \text{for all } \chi\varepsilon H,\quad \text{for all } h\varepsilon V,$$

that is,

$$H\subset\left[\chi/|\chi(V)-1|<\varepsilon\right],$$

and, since the right-hand side has bicompact closure, this proves the lemma.

In the course of the above proof, we have

Lemma 2. (Raikov)[8] Every bicompact set F of characters forms the

---

7) I, Gelfand-G. Silov, loc. cit.;

A. Tychonoff, Uber einen Funktionenraum, Math. Ann, 111(1935);

S. Kakutani, Weak Topology, Bicompact set and the Principle of Duality, Proc. Imp. Acad. Japan, 16(1940).

8) D.Raikov, Positive definite functions on commutative group with an invariant measure, C.R.URSS, 28(1940).

set of equi-uniformly continuous characters, that is, for every $\varepsilon>0$, there exists a neighborhood V(with bicompact closure) of the unit element of G, such that

$$\operatorname*{Sup}_{g\in G} |\chi(g+h)-\chi(g)| < \varepsilon, \quad \text{for all } h\varepsilon V, \text{ for all } \chi\varepsilon F.$$

In what follows, we shall use the space $\underline{\underline{X}}$ instead of X.
The space $\underline{\underline{X}}$ is the bicompact Hausdorff space obtained by adjoining an ideal point, (χ_∞), to X, and by defining its system of neighborhoods as the totality of sets $\left[X-\bar{U}\right]$, where U denotes arbitrary neighborhood (with bicompact closure) of $\chi=0$. Those continuous functions which have the properties indicated in Lemma 1 can be continuously extended over $\underline{\underline{X}}$ by defining the value at (χ_∞) to be equal to 0, so we have

Lemma 3. Every function $\int_G x(g)\,\chi(g)\,dg$, with $x(g)\varepsilon L_1$, is continuous over $\underline{\underline{X}}$.

Proof. Evident.

Now, we can prove the following Theorem 2, another proof of which may be obtained by use of Theorem 3.

Theorem 2. **M** and $\underline{\underline{X}}$ are homomorphic.

Proof. As indicated in Theorem 1,(1) and (2), $\mathbf{M}-(L_1)$ and X correspond in an one-to-one manner. We make to correspond the point (χ_∞) to the point (L_1). Now this correspondence is one-to-one between **M** and $\underline{\underline{X}}$. Since both **M** and $\underline{\underline{X}}$ are bicompact H-space, it is sufficient to prove that the mapping $\underline{\underline{X}}\rightarrow$ **M** is continuous, and this is already proved implicitly in Lemma 3, and we have our theorem.

4. Lemmas on the Fourier integral. Many Russian authors [9] have recently treated the Fourier analysis on G. The Bochner's theorem, the

9) D. Raikov, loc. cit.;

A. Powzner, Ueber positive Funktionalen auf einer Abelschen Gruppe, C.R. URSS, 28(1940);

M, Krein, On a special ring of functions, C.R.URSS, 29(1940);

M. Krein. A ring of functions on a topological group, ibidem.;

M. Krein, On almost periodic functions on a topological group, ibidem, 30(1941);

M. Krein, On positive functionals on almost periodic functions, ibidem;

M. Krein, Sur une généralisation du théorème de Plancherel au cas des intégrales de Fourier sur les groupes topologiques commutatifs. ibidem.

Parceval's theorem and the Plancherel's theorem were proved, which we use.

Lemma 4. For every neighborhood $W_1, W_2, W_1 \supset \overline{W}_2$, of the unit element of X, there exists a function $u(\chi)$ satsfying the conditions:

(i) $u(\chi)=0$, for $\chi \bar{\varepsilon} W_1$; $=1$, for $\chi \varepsilon W_2$;

(ii) $0 \leqq u(\chi) \leqq 1$ for all $\chi \varepsilon \underline{\underline{X}}$;

(iii) $u(\chi)=\int_G U(g)\overline{\chi}(g)\, dg$ where $U(g) \varepsilon \dot{L}_1$.

Proof. In order to construct $u(\chi)$, it is sufficient to take a neighborhood V_1 of the unit element such that $W_2 - V_1 - V_1 \subset W_1$ and to put

$$u(\chi)=u_{W_1,W_2}(\chi)=\frac{1}{m(V_1)}\int_X \varphi_{V_2}(\chi-\chi_1)\varphi_{V_1}(\chi_1)d\chi_1,$$

where $V_2 = W_2 - V_1$, and $\varphi_{V_1}(\chi)$, $\varphi_{V_2}(\chi)$ denote the characteristic function of V_1, V_2 respectively.

Now, (i) is evident. In order to see (ii), we notice that $\varphi_{V_1}(\chi), \varphi_{V_2}(\chi)$ is the functions of class L_2 on X, whence the Fourier integral $U(g)$ of $u(\chi)$ is the function of L_1 on G by Parseval's relation, and we have, by Plancherel's theorem,

$$u(\chi)=\int_G U(g)\chi(g)\, dg$$

under the appropriate normalization of the measure on X.

Lemma 5. For every $\chi_0 \varepsilon X$, and for every pair of neighborhood W_1', W_2', $W_1' \supset W_2'$, of χ_0, there exists the function $u(\chi)=u_{W_1',W_2'}(\chi,\chi_0)$ satisfying the conditions (i), (ii) in Lemma 4, and $\underline{\underline{u}}=u(\chi)\varepsilon L$.

Proof. Case (a), when $\chi_0=(\chi_\infty)$, we put $W_1 = X - \overline{W}'_2 = X - \overline{W}'_1$, and apply Lemma 4 for $\chi=0$, then $\underline{\underline{u}}=1-u_{W_1,W_2}(\chi)$ is the required one.

Case(b), when $\chi_0=(0)$, Lemma 4.

Case (c), when $\chi_0 \neq (0)$, (χ_∞). If we write $W_1'=W_1+\chi_0$, $W_2'=W_2+\chi_0$, and put $\underline{\underline{u}}_{\chi_0}=u_{\chi_0}(\chi)=u(\chi+\chi_0)$, where u is the function in Lemma 4, then $\underline{\underline{u}}_{\chi_0}$ is the required one.

Lemma 6. Let $O_1 \subset O_2 \subset O_3 \subset \ldots\ldots$ be the sequence of bicompact open sets on X such that

(1) $$\lim_{n\to\infty}\frac{1}{mO_n}m(O_n+\chi)\ominus O_n), \quad \text{for all } \chi\varepsilon X^{10)}.$$

10) Y.Kawada, Weil ノ Mass ノアル Abel 群ニツイテ, 全國紙上數學談話會, 238 (1942).

and let us put

$$(2) \qquad E_n(g)=\frac{1}{mO_n}\left|\int_{O_n}\overline{\chi}(g)\,d\chi\right|^2.$$

If $K(g)$ is any function of L_1, then, for any given number $\varepsilon>0$, we can find an integer n_0 such that

$$(3) \qquad \int_G\left|K(g)-\int_G K(g-h)F_n(h)\,dh\right|dg<\varepsilon, \quad \text{for all } n>n_0.$$

Proof. It is earsily seen, by Plancherel's theorem, that

$$(4) \qquad \text{(i)} \quad F_n(O)=m(O_n), \quad \text{(ii)} \int_G F_n(g)\,dg=1,$$

and (iii)

$$(5) \qquad \int_G |1-\chi_1(g)|^2\cdot\frac{1}{mO_n}\left|\int_{O_n}\overline{\chi}(g)\,d\chi\right|^2 dg=\frac{1}{mO_n}m\Big((O_n+\chi)\ominus O_n\Big).$$

Hence, by the assumption and the continuity of $m\,(O_n\ominus(O_n+\chi))$ (with respect to χ), we have: for any $\varepsilon>0$, and for any given neighborhood V of $g=0$, there exists an integer n_0 and a neighborhood $V_1\subset V$ such that

$$(6) \qquad \int_{CV_1} F_n(g)\,dg<\varepsilon, \text{ for all } n>n_0.$$

By (4) and (6), following the usual device in the Lebesgue theory of integration, we have (3).

The Fourier integral of $F_n(g)$ are

$$\frac{1}{mO_n}\int_X \varphi_{O_n}(\chi+\chi_1)\varphi_{O_n}(\chi_1)\,d\chi_1,$$

where $\varphi_O(\chi)$ denotes the characteristic function of the set O. This function vanishes except for $\chi\varepsilon[O_n+O_n]$.

5. Division in abstract rings. Let R be an abstract ring possessing the unit element e. A subset I of R is called an ideal if $x, y\,\varepsilon\, I$ and $z, z'\,\varepsilon\, R$ imply $x+y$, $zxz'\,\varepsilon\, I$. The zero-element set (0) and R are ideals, the other ideals are called proper. A ring, with the unit, having no proper ideal is called simple.

If an ideal M is not contained in any other proper ideal, we call M a maximal ideal. The residue class ring R/M by the maxmal ideal M is simple, we shall denote it by R_M, and the element of R_M, into which the generic element x passes under the homomorphism $R\rightarrow R/M$, will be denoted by $x(M)$. The ring R is represented ring-homomorphically by the function of M; for the isomorphism of this representation it is necessary and sufficient that the intersection of all maximal ideals is the zero ideal only, in which case we call R semi-simple.

A ring, not simple, has at least a maximal ideal. Let $\mathbf{M}$ be the totality of all maximal ideals of R, whose generic element will be denoted by M, N,...... and whose generic subset by $\mathbf{A}, \mathbf{B}, \mathbf{N}, \mathbf{O}$......

A maximal ideal M is defined to be a points of contact of $\mathbf{A}$ [11], when $M \supset \bigwedge_{N \in \mathbf{A}} N$; the set of all points of contact to $\mathbf{A}$, is called the closure of $\mathbf{A}$, and denoted by $\overline{\mathbf{A}}$, then we have

Lemma 7. Let R be an abstract ring possessing the unit element, and let $\mathbf{M}$ be the totality of its maximal ideals. Then the closure operation defined above satisfies the conditions:

$$(1)\ \mathbf{A} \subset \overline{\mathbf{A}},\quad (2)\ \overline{\overline{\mathbf{A}}} = \overline{\mathbf{A}},\quad (3)\ \overline{\mathbf{A} \vee \mathbf{B}} = \overline{\mathbf{A}} \vee \overline{\mathbf{B}},\quad (4)\ \overline{(M)} = (M)$$

and with this topology $\mathbf{M}$ is a bicompact T_1-space.

Proof. We shall prove the relation $\overline{\mathbf{A} \vee \mathbf{B}} \subset \overline{\mathbf{A}} \vee \overline{\mathbf{B}}$ and the bicompactness of $\mathbf{M}$. Let $M_0 \in \overline{\mathbf{A} \vee \mathbf{B}}$ and $M_0 \notin \overline{\mathbf{A}} \vee \overline{\mathbf{B}}$. Then we have

(i) $M_0 \supset \bigwedge_{\mathbf{A}} N \wedge \bigwedge_{\mathbf{B}} N$, and

(ii) For some $z, z' \in R$, $z \notin M_0$, $z \in \bigwedge_{\mathbf{A}} N$; $z' \notin M_0$, $z' \in \bigwedge_{\mathbf{B}} N$.

As $zRz'R \subset \bigwedge_{\mathbf{A}} N \wedge \bigwedge_{\mathbf{B}} N \subset M_0$, and $M_0 + Rz'R = R$, we have $z \in zR = zM_0 + zRz'R \subset M_0$, which contradicts $z \notin M_0$, and therefore $\overline{\mathbf{A} \vee \mathbf{B}} \subset \overline{\mathbf{A}} \vee \overline{\mathbf{B}}$.

Thus $\mathbf{M}$ is a T_1-space.

To prove the bicompactness, let $\{\mathbf{F}_\alpha\}$ be any family of closed subsets of $\mathbf{M}$. Form the ideal $I_\alpha = \bigwedge_{\mathbf{F}_\alpha} N$, and let I be the ideal which consists of all possible finite sums of elements from each I_α; moreover, let us suppose that the intersection of any finite number of these closed sets $\mathbf{F}_\alpha$ is not empty. Now, I is contained in a maximal ideal M_0. For, if $I = R$, a finite set of elements $[x_i, i = 1, 2, \ldots, n]$, $x_i \in I_{\alpha_i}$, would exist such that $\sum x_i = e$, and consequently we would have $\sum I_{\alpha_i} \ni e$, which contradicts the assumption that $\bigwedge_i \mathbf{F}_{\alpha_i}$ is empty.

It is easy to see that $M_0 \in \mathbf{F}_\alpha$ for all α, which shows, by the definition, the bicompactness of $\mathbf{M}$.

---

11) H. Stone, Application of the theory of Boolean rings to general topology, Trans. Math. Soc., 41(1937);

I. Gelfand-A. Kolmogoroff, loc. cit.;

H. Wallman, Lattices and topological spaces, Annals of Math., 39(1938).

Now we prove the following simple lemma, due to I. Segal[12].

Lemma 8. Let R be an abstract ring possessing the unit element, and semi-simple. Let a mapping of R into R $x \rightarrow x^*$ exist, such that, for any $M \varepsilon \mathbf{M}$, and for any $x_1, x_2, \ldots\ldots, x_n \varepsilon R$, $\Sigma x_i x_i^* \varepsilon M$ implies $x_i \varepsilon M$ for all $i=1,2,\ldots\ldots n$.

Let I be an ideal, and let $\mathbf{N}$ be the totality of $M \varepsilon \mathbf{M}$, such that $M \supset I$, and let $\mathbf{O}$ be any open sêt containiug $\mathbf{N}$. Then we have

$$I \supset \bigwedge_{N \varepsilon \mathbf{O}} N.$$

Proof. For any $x \varepsilon R$, the sets $\left[M/x(M)=0 \right]$ form the basis for closed sets, i,e, any closed set may be represented by taking finite sums and infinite (enumerable or not) products of these closed sets, and open sets $\left[M|x(M) \neq 0 \right]$ cnntaining M_0 form the system of neighborhoods of M_0.

Since $\mathbf{N}$ is closed, thus bicompact, we can select a finite number of M_1, $M_2, \ldots, M_n \varepsilon \mathbf{N}$ and a finite number of elements $x_{M_1}, x_{M_2}, \ldots\ldots, x_{M_n} \varepsilon R$ such that if $y = \sum_{i=}^{n} x_{M_i} x_{M_i}^*$, then $y(N)=0$ for all $N \varepsilon \mathbf{M}-\mathbf{O}$, and $y(N) \neq 0$ for all $N \varepsilon \mathbf{O}_1$, where $\mathbf{O}_1$ is any open set $\mathbf{O}_1 \subset \mathbf{O}$ and $\mathbf{N} \subset \mathbf{O}_1$. Now, $\mathbf{M}-\mathbf{O}_1$ is closed and does not contain $\mathbf{N}$; hence, for each $M \varepsilon \mathbf{M}-\mathbf{O}_1$ an element $x_M \varepsilon I$ exists and satisfies

$$x_M(N) \neq 0, \quad \text{and } x_M(N)=0 \quad \text{for all } N \varepsilon \mathbf{N}.$$

As the set $\left[N/x_M x_M^*(N) \neq 0 \right]$ is open, and $\mathbf{M}-\mathbf{O}_1$, is bicompact, we can find a finite number of $N_1, N_2 \ldots N_m \varepsilon \mathbf{M}-\mathbf{O}_1$, such that if $y' = \sum_{i=1}^{m} x_{N_i} x_{N_i}^*$, then $y'(N) \neq 0$ for all $N \varepsilon \mathbf{M}-\mathbf{O}_1$.

Let $z \varepsilon \bigwedge_{N \varepsilon \mathbf{O}} N$. As $y+y'$ is not contained in any maximal ideal M, it holds that $e=u(y+y')v$ for some $u, v \varepsilon R$, and that $z(M)=(zuyu+zuy'v)(M)=zuy'v(M)$ for all $M \varepsilon \mathbf{M}$. By the semi-simplicity of R, we have $z=zuy'v \varepsilon I$, which proves the lemma.

6. The topology of character groups of locally bicompact commutative groups. As an immediate consequence of the preceeding considerations, we obtain the formal algebraic deduction of the Pontrjagin topologe of the character group, which is equivalent to the measure-using deduction due to *A*. Weil in our commutative case.

12) I.E.Segal, The group-ring of a locally compact group I, Proc. Nat. Acad. Sci., 27 (1941).

Instroduce the topology in $\mathbf{M}$ by the Stone's method of the preceeding section, then $\mathbf{M}$ becomes a bicompact T-space. We prove

Theorem 3. $\mathbf{M}$ and $\underline{\underline{X}}$ are homeomorphic; for any $\underline{\underline{z}} \varepsilon L$, $\underline{\underline{z}}(M)$ is continuous over $\mathbf{M}$.

Proof. $\mathbf{M}$ and $\underline{\underline{X}}$ are both bicompact T_1-space, and $\underline{\underline{X}}$ is a H-space. The function of L is continuous over $\underline{\underline{X}}$ by Lemma 3. The mapping defined in Theorem 2 is one-to-one. We shall prove that this mapping is continuous, thence the homeomorphism of this correspondence automatically follows by a well-known theorem in the set theory [13].

For this, it is sufficient to see whether there exists an open set U on $\mathbf{M}$ for arbitraryly chosen neighborhood O in $\underline{\underline{X}}$, such that $U \subset O$ and this could be seen easily from our Lemma 4 and 5, and by the definition of open sets on $\mathbf{M}$. Therefore, $\mathbf{M}$ and $\underline{\underline{X}}$ are homeomorphic, and hence every function $\underline{\underline{z}}(M)$ of L is continnuous on $\mathbf{M}$.

7. The general tauberian theorem. From Theorem 3 and Lemma 8 follows

Lemma 9. Let $K_1(g)$ and $K_2(g)$ belong to L_1 on G, and let F_1 be the set $\left[\chi/k_1(\chi)=\int_G K_1(g)\overline{\chi}(g)dg=0\right]$, and let us suppose that $k_2(\chi)=\int_G K_2(g)\overline{\chi}(g)\,dg$ vanishes an open set containing F_1 on X. Then $k_2(\chi)/k_1(\chi)$ is the Fourier transform of a function of L_1.

Note. when $k_1=\int_G K_1(g)\chi(g)dg\neq 0$ except for $\chi=\chi_\infty$, we can take the set (χ_∞) and an arbitrary open neighborhood O of (χ_∞) for the above F_1 and the open neighborhood of F_1. If $K_2(g)$ is any function of L_1 such that $k_2(\chi)=0$ for all $\chi \varepsilon O$, then $k_2(\chi)/k_1(\chi)$ is the Fourier transform of some function of L_1.

Now, we prove

Theorem 4. Let $K_1(g)$ belong to L_1 on G, and let us suppose that

$$k_1(\chi)=\int_G K_1(g)\overline{\chi}(g)\;dg\neq 0, \qquad \text{except for } \chi=\chi_\infty \text{ on } X.$$

Then the principal (closed) ideal generated by K_1, $L(K_1)$, is identical with L_1.

Proof. By Lemma 8 or 9 and the above remark it is only to prove that any function $K(g)$ of L_1 can be arbitrarily approximated in the norm by the functions of L_1 whose Fourier transforms vanish over open neighborhood of $\chi=\chi_\infty$, for, $L\ (K_1)\subset L$, and all functions of the type stated form the ideal of

13) Alexandroff-Hopf, Topologie, p. 95, Satz III.

L_1, and is contained in $L(K_1)$. By Lemma 6, we have our theorem.

Theorem 5. If the closed ideal I is contained only in the maximal ideal, L_1, then $I=L_1$.

Theorem 6. Let Σ be the subset of L_1, and let us suppose that for any $\chi \varepsilon X$, a function $K=K_\chi \varepsilon \Sigma$ exists for which $\int_G K_\chi(g)\overline{\chi}(g)\ dg \neq 0$. Then the closed ideal generated by Σ is identical with L_1.

Theorem 4 and 6 are the general tauberian theorems stated by N. Wiener. Various types of theorems shall be stated in some analogous way, but it is omitted here. Theorem 5 states that L_1 is a primary ideal of L, and on the primarity of ideals of L in general we shall discuss on another occasion.

Mathematical Institute,
Osaka Imperial University.

July 1942.