NOTES ON FOURIER ANALYSIS (XVIII): ABSOLUTE SUMMABILITY OF SERIES WITH CONSTANT TERMS.*)

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The object of this paper is to prove some theorems concering absolute summability systematically. In § 1, key theorems are proved, from which theorems of the remaining sections are derived. One of the key theorems reads as follows: when (x_n) is a given sequence and (y_n) is defined by

$$y_n = a_{n,0}\, x_0 + a_{n,1}\, x_1 + \cdots\cdots + a_{n,m}\, x_m + \cdots\cdots,$$

where $(a_{n,k})$ is an infinite matrix, then

$$\sum_{n=0}^{\infty} |a_{n+1,m} - a_{n,m}| < M \ (m=1, 2, \cdots\cdots)$$

is the necessary and sufficient condition that any $\sum_{n=0}^{\infty} |x_n| < \infty$ implies $\sum_{n=0}^{\infty} |\Delta y_n| < \infty$. By this and the similar key theorems we prove theorems of Mercerian type (in § 3), inclusion relation between absolute Riesz summations of different types (in § 4) and Tauberian theorems (in § 5).

§ 1. Key theorems. Let (x_n) be a sequence of real number and its linear transformation be

$$(1) \qquad y_n = \sum_{k=0}^{\infty} a_{n,k}\, x_k.$$

Theorem 1. In order that any $\sum_{n=0}^{\infty} |x_n| < \infty$ implies $\sum_{n=0}^{\infty} |\Delta y_n| < \infty$, it is necessary and sufficient that

$$(2) \qquad \sum_{n=0}^{\infty} |a_{n+1,m} - a_{n,m}| < M.$$

Proof. Necessity. We have

$$\Delta y_n = y_{n+1} - y_n = \sum_{m=0}^{\infty} (a_{n+1,m} - a_{n,m})\, x_m$$

which is a linear functional on (l). If we put $x \equiv (x_n) \varepsilon (l)$, $\Delta y_n \equiv U_n(x)$, then $W(x) \equiv \sum_{n=0}^{\infty} |U_n(x)|$ satisfies the assumption of the Bosanquet-Kestelman theorem [2]. Hence we have

$$\sum_{n=0}^{\infty} |U_n(x)| \leqq M \|x\|.$$

If we put $x_n = 1 (n=m)$, $x_n = 0$ $(n \neq m)$, then we get (2). Thus the necessity

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of the condition is proved.

Sufficiency. On the other hand, if the condition of the theorem is satisfied, then

$$\sum_{n=0}^{\infty}|\Delta y_n| = \sum_{n=0}^{\infty}\Big|\sum_{m=0}^{\infty}(a_{n+1,m}-a_{n,m})\,x_m\Big|$$
$$\leqq \sum_{m=0}^{\infty}|x_m|\sum_{n=0}^{\infty}|a_{n+1,m}-a_{n,m}| < M\sum_{m=0}^{\infty}|x_m|.$$

Thus the convergence of $\sum_{n=0}^{\infty}|x_n|$ implies that of $\sum_{n=0}^{\infty}|\Delta y_n|$.

We will now remark that the necessity of the condition can be derived from Gelfand's theorem [3]. For, if we put $x\equiv(x_n)$, $y\equiv(y_n)\equiv U(x)$, then U is a linear operation from (l) onto (bv), by Gelfand's theorem. By the representation theorem,

$$y_n = \sum_{m=0}^{\infty} a_{n,m}\, x_m$$

where $A_n=(a_{1,m}, a_{2,m},\ldots\ldots, a_{n,m},\ldots\ldots)$ lies in (bv) and the norm of A_m in (bv) is uniformly bounded, which is nothing but the condition of the theorem.

More generally, we will consider the transformation,

$$\Phi(z) = \sum_{m=0}^{\infty}\varphi_m(z)\, x_m. \tag{3}$$

Then we get

Theorem 2. In order that any $\sum_{m=0}^{\infty}|x_m|<\infty$ implies the existence of

$$\lim_{z\to z_0}\int^{z}|d\,\Phi(z)|, \tag{4}$$

it is necessary and sufficient that

$$\lim_{z\to z_0}\int^{z}|d\,\varphi_m(z)|<M. \tag{5}$$

Proof runs similarly as that of Theorem 1.

We have also

Theorem 3.(*) In order that any $\sum_{n=0}^{\infty}|\Delta x_n|<\infty$ implies $\sum_{n=0}^{\infty}|\Delta y_n|<\infty$, it is necessary and sufficient that $\sum_{k=1}^{\infty}a_{n,k}$ converges for all n and

$$\sum_{n=0}^{\infty}\Big|\sum_{k=1}^{m}(a_{n+1,k}-a_{n,k})\Big|<M\quad(m=1, 2,\ldots\ldots). \tag{6}$$

Proof. After S. Izumi [5] we have

(*) After prepared this note, I have learned this theorem is proved by F. M. Mears, Absolute regularity and Nörlund mean, Annals of Math., 38 (1937) 549–601.

$$y_n = \sum_{n=0}^{\infty} A_{n,m}\, \Delta x_m + x\, A_n,$$

where

$$A_{n,m} = \sum_{k=1}^{m} a_{n,k}, \qquad A_n = \lim_{m\to\infty} A_{n,m}, \text{ and } x = \lim_{n\to\infty} x_n.$$

Thus we get the theorem by Theorem 1.

Theorem 4. In order that any $\sum_{n=0}^{\infty} |\Delta x_n| < \infty$ implies (4), it is necessary and sufficient that $\sum_{k=0}^{\infty} \varphi_k(z)$ converges for all z and

$$\lim_{z\to z_0} \int^{z} |d\,\Phi_m(z)| < M, \qquad \Phi_m(z) = \sum_{k=0}^{m} \varphi_k(z). \tag{7}$$

§ 2. Absolutely regular transformation. Linear transfomation (1) (or (3)) is absolutely regular provided that it transforms all absolutely convergent series into absolutely summable ones.

Theorem 5. Riesz's method of summation is absolutely regular.

Proof. Riesz mean of $\sum_{n=0}^{\infty} a_n$ is defined by

$$C_\lambda^k(\omega)/\omega^k = \sum_{\lambda_m<\omega} \left(1 - \frac{\lambda_m}{\omega}\right)^k a_m.$$

This is a transformation of the type (3). Now the condition (5) is satisfied, since

$$\lim_{\omega\to\infty} \int_{\lambda_m}^{\omega} \left| d\left(1 - \frac{\lambda_n}{\omega}\right)^k \right| = \lim_{\omega\to\infty} k \int_{\lambda_m}^{\omega} \left(1 - \frac{\lambda_m}{\omega}\right)^{k-1} \frac{\lambda_m}{\omega^2}\, d\omega$$

$$= \lim_{\omega\to\infty} \left[\left(1 - \frac{\lambda_m}{\omega}\right)^k\right]_{\lambda_m}^{\omega} = 1.$$

The following theorems are proved by the similar method.

Theorem 6. Abel's method of summation is absolutely regular. More generally the summation by Dirichlet series is also.

Theorem 7. In (1), if Toeplitz condition is satisfied and $(a_{m,n})$ is a monotone sequence of n for each m, then (1) is absolutely regular.

Corollary. Riesz's (R, p_n)-summation is absolutely regular, if $p_n > 0$ and $P_n \to \infty$.

(R, p_n)-summation is defined by

$$y_n = (p_1 s_1 + p_2 s_2 + \cdots\cdots + p_n s_n)/P_n,$$

where

$$s_k = \sum_{m=1}^{k} x_m, \qquad P_n = \sum_{m=1}^{n} p_m.$$

§ 3. Theorems of Mercerian type.

Theorem 8. If $y_n = (1 + a_n) x_n - a_n x_{n-1}$ $(a_n > 0)$, then $\sum_{n=0}^{\infty} |\Delta y_n| < \infty$ implies

$\sum_{n=0}^{\infty}|\Delta x_n|<\infty$.

Proof. Putting $\alpha_n=1/a_n$ and expressing x_n by (y_n),

$$x_n=\frac{\alpha_1}{\prod_{\nu=1}^{n}(1+\alpha_\nu)}y_1+\frac{(1+\alpha_1)\alpha_2}{\prod_{\nu=1}^{n}(1+\alpha_\nu)}y_2+\cdots\cdots+\frac{(1+\alpha_1)\cdots(1+\alpha_{n-1})\alpha_n}{\prod_{\nu=1}^{n}(1+\alpha_\nu)}y_n$$

$$=a_{n,1}y_1+a_{n,2}y_2+\cdots\cdots+a_{n,n}y_n,$$

say. By Theorem 3, it is sufficient to prove that the transformation satisfies the condition (6).

$$\sum_{k=1}^{m}a_{n,k}=\left\{\alpha_1+(1+\alpha_1)\alpha_2+\cdots\cdots+(1+\alpha_1)\cdots\cdots(1+\alpha_{m-1})\alpha_m\right\}\Big/\prod_{\nu=1}^{n}(1+\alpha_\nu)$$

$$=\begin{cases}\left[\prod_{\nu=1}^{m}(1+\alpha_\nu)-1\right]\Big/\prod_{\nu=1}^{n}(1+\alpha_\nu) & (m\leqq n),\\ 1-1\Big/\prod_{\nu=1}^{n}(1+\alpha_\nu) & (m>n).\end{cases}$$

$$\sum_{n=1}^{\infty}\left|\sum_{k=1}^{m}(a_{n+1,k}-a_{n,k})\right|=\frac{\alpha_1}{1+\alpha_1}-\left\{\prod_{\nu=1}^{m}(1+\alpha_\nu)-1\right\}\Big/\prod_{\nu=1}^{\infty}(1+\alpha_\nu),$$

which is evidently bounded uniformly. This theorem has been proved by the author [6] by direct calculation.

Corollary. If $q>-1$, $p_n>0$, $P_n=\sum_{m=1}^{n}p_m\to\infty$ and

$$y_n=x_n+q\,(p_1x_1+\cdots\cdots+p_nx_n)/P_n,$$

then $\sum_{n=0}^{\infty}|\Delta y_n|<\infty$ implies $\sum_{n=0}^{\infty}|\Delta x_n|<\infty$.

For put

$$X_n=(p_1x_1+\cdots\cdots+p_nx_n)\,/\,(p_1+\cdots\cdots+p_n),$$

then we have

$$y_n=\left(q+1+\frac{P_n}{p_n}\right)X_n-\frac{P_{n-1}}{p_n}X_{n-1}.$$

This theorem includes Bosanquet's [1] and Hayashi's results [4].

§ 4. Inclusion relation of absolute Riesz's summations of different types.

Let us suppose that $p_n>0$, $P_n=\sum_{m=1}^{\infty}p_m\to\infty$ and put

$$y_n=(p_1s_1+p_2s_2+\cdots\cdots+p_ns_n)/P_n,$$

then we have

$$s_n=-\frac{P_{n-1}}{p_n}y_{n-1}+\frac{P_n}{p_n}y_n. \tag{8}$$

Theorem 9. In order that $\sum_{n=0}^{\infty}|\Delta y_n|<\infty$ implies $\sum_{n=0}^{\infty}|\Delta s_n|<\infty$, it is necessary and sufficient that $P_n/P_{n-1}>\alpha>1$.

Proof. It is sufficient to prove that the condition (6) in Theorem 3 is satisfied. If we put (8) in the form

$$s_n = \sum_{m=1}^{\infty} a_{n,\,m}\, y_m,$$

then

$$a_{n,\,n-1} = -P_{n-1}/p_n,\quad a_{n,\,n} = P_n/p_n,$$

and the other $a_{n,m}$ becomes zero. Thus the left-hand side of the condition (6) becomes

$$\sum_{n=1}^{\infty}\left|\sum_{k=1}^{m}(a_{n+1,\,k}-a_{n,\,k})\right| = \left| -\frac{P_m}{p_{m+1}}+1\right| + \frac{P_m}{p_{m+1}} \leqq 2\,\frac{P_m}{p_{m+1}}+1,$$

which is bounded if and only if

$$P_n/p_{n+1}<M, \text{ i.e., } \quad P_m/(P_{m+1}-P_m)<M.$$

This is equivalent to the condition of the Theorem.

Theorem 10. If $p_{n+1}/P_n<q_{n+1}/Q_n$, then the $|R, q_n|$-summation implies $|R, p_n|$-summation.

Proof. Let us put

$$t_n \equiv (q_1 s_1 + q_2 s_2 + \cdots\cdots + q_n s_n)/Q_n,$$

which is equivalent to

$$s_n = -\frac{Q_{n-1}}{q_n}\, t_{n-1} + \frac{Q_n}{q_n}\, t_n.$$

We have

$$(p_1 s_1 + p_2 s_2 + \cdots\cdot + p_n s_n)/P_n = \frac{1}{P_n}\sum_{m=1}^{n-1}\left(\frac{p_m}{q_m}-\frac{p_{m+1}}{q_{m+1}}\right) Q_m t_m$$
$$+\frac{Q_n}{P_n}\,\frac{p_n}{q_n}\, t_n = \sum_{m=1}^{n} a_{n,\,m}\, t_m,$$

say. Then for $m<n$ we have

$$\sum_{k=1}^{m} a_{n,\,k} = \begin{cases} \dfrac{1}{P_n}\displaystyle\sum_{k=1}^{m}\left(\frac{p_k}{q_k}-\frac{p_{k+1}}{q_{k+1}}\right) Q_k, & n>m \\ \dfrac{1}{P_n}\left(P_m-\dfrac{p_{m+1}}{q_{m+1}}\, Q_m\right), & n\leqq m \end{cases}$$

where the left-hand side is equal to 1 for $m\geqq n$. Hence we have

$$\sum_{n=1}^{\infty}\left|\sum_{k=1}^{m}(a_{n+1,\,k}-a_{n,\,k})\right|$$
$$\leqq \left|\frac{P_m}{P_{m+1}}-\frac{p_{m+1}}{q_{m+1}}\,\frac{Q_m}{P_{m+1}}-1\right| + \left(P_m-\frac{p_{m+1}}{q_{m+1}}\,Q_m\right)\sum_{k=m+1}^{\infty}\left(\frac{1}{P_k}-\frac{1}{P_{k+1}}\right)$$
$$\leqq 2\,\frac{P_m}{P_{m+1}}\left(1-\frac{p_{m+1}}{q_{m+1}}\,\frac{Q_m}{P_m}\right)\leqq 2.$$

By Theorem 3 we get the implication relation required.

If we denote the consequence of Theorem 10 by

$$|R,\, p_n| \supset |R,\, q_n|,$$

symbolically. Then we have, by Theorem 10,

$$|R, 1/n \log n \log_2 n| \supset |R, 1/n \log n| \supset |R, 1/n|$$
$$\supset |R, 1| = |C, 1| \supset |R, k^n| = \text{absolute convergence.}$$

§ 5. A Tauberian theorem. Hyslop [7] has proved a Tauberian tneorem for absolute Abel summability, which may be generalized in the following form:

Theorem 11. If

(1°) $\varphi(x,t)$ is continuous and $0 \leqq \varphi(x,t) \leqq 1$,

(2°) $\varphi(x,t)$ is monotonic with respect to x,

(3°) $\int_0^u \varphi(x,t) d\lambda(t) - \lambda(x)$ is bounded variation in any finite interval of x uniformly to u, and monotonic with respect to x from a fixed x,

(4°) $\int_0^u \varphi(u,t) d\lambda(t) - \lambda(u) = O(1)$,

(5°) $\lim_{x\to\infty} \int_0^u \{\varphi(x,t) - 1\} d\lambda(t)$ exists and is bounded then, any function $s(t)$ which is absolutely summable Φ:

$$\Phi(x) = \int_0^\infty \varphi(x,t) ds(t) \varepsilon BV(0,\infty),$$

$\Phi(x)$ being an absolutely regular transformation, is absolutely convergent $(s(t) \varepsilon BV(0,\infty))$, provided that

$$\frac{1}{\Lambda(x)} \int_0^x \Lambda(t)\ ds(t) \varepsilon BV(0,\infty),$$

where

$$\Lambda(x) = e^{\lambda(x)}.$$

Proof. If we put

$$\delta(x) = \frac{1}{\Lambda(x)} \int_0^x \Lambda(t)\ ds(t), \tag{1}$$

then

$$\int_0^\infty |d\delta(x)| \leqq M. \tag{2}$$

Solving (1) with respect to $s(x)$, we have

$$\begin{aligned} s(x) &= \delta(x) + \int_0^x \delta(t) \frac{d\Lambda(t)}{\Lambda(t)} \\ &= \delta(x) + \int_0^x \delta(t)\, d\lambda(t). \end{aligned} \tag{3}$$

Substituting this into $\Phi(x)$, we get

$$\Phi(x)=\int_0^\infty \varphi(x,t)\, d\delta(t) + \int_0^\infty \varphi(x,t)\, \delta(t)\, d\lambda(t)=I_1+I_2,$$

say. Since Φ is absolutely regular, I_1 is absolutely summable and then

$$\int_0^\infty \varphi(x,t)\, \delta(t)\, d\lambda(t) \varepsilon BV(0,\infty).$$

Now

$$\int_0^\infty \varphi(x,t)\, \delta(t)\, d\lambda(t)-s(x)$$

$$=\int_0^\infty \varphi(x,t)\, \delta(t)\, d\lambda(t)-\int_0^x \delta(t)\, d\lambda(t)-\delta(x)$$

$$=\int_0^\infty \psi(x,t)\, \delta(t)\, d\lambda(t)-\delta(x).$$

where

$$\psi(x,t)=\begin{cases}\varphi(x,t)-1, & \text{if } 0\leqq t\leqq x,\\ \varphi(x,t), & \text{if } x<t.\end{cases}$$

Since $\delta(x) \varepsilon BV (0,\infty)$, if we can prove that

$$\int_0^\infty \psi(x,t)\, \delta(t) d\lambda(t) \varepsilon BV (0,\infty), \tag{4}$$

then $s(x) \varepsilon BV(0,\infty)$ which is required. By $\delta(x) \varepsilon BV(0,\infty)$ and Theorem 4, if $\lim_{n\to\infty}\int_0^t \psi(x,t)d\lambda(t)$ exists and

$$\int_0^\infty \left| d_x\int_0^t \psi(x,t)\, d\lambda(t)\right| \leqq M,$$

then we get (4). Now

$$\int_0^\infty \left| d_x\int_0^t \psi(x,t)\, d\lambda(t)\right|$$

$$=\int_0^u \left| d_x\int_0^u \psi(x,t)\, d\lambda(t)\right| + \int_u^\infty \left| d_x\int_0^t \psi(x,t)\, d\lambda(t)\right|$$

$$=I_1+I_2,$$

say.

$$I_1=\int_0^u \left| d_x\left[\int_0^x \{\varphi(x,t)-1\}d\lambda(t)+\int_x^u \varphi(x,t)\, d\lambda(t)\right]\right|$$

$$=\int_0^u \left| d_x\left[\int_0^u \varphi(x,t)d\lambda(t)-\lambda(x)\right]\right|$$

$$\leqq K+\left\{\int_0^u \varphi(u,t)\, d\lambda(t)-\lambda(u)\right\}=O(1),$$

by (4°). Since $\varphi(x,t)$ is monotonic,

$$I_2=\int_u^\infty \left| d_x\int_0^u \left\{\varphi(x,t)-1\right\} d\lambda(t)\right|$$

$$=\lim_{x\to\infty}\int_0^x \left\{\varphi(x,t)-1\right\}d\lambda(t)-\int_0^u \left\{\varphi(u,t)-1\right\}d\lambda(t)$$

$$=O(1),$$

by (4°) and (5°). Thus we get the theorem.

Corollary. If $\Phi(x)=\int_1^\infty e^{-t/x}\, u(t)\, dt \;\varepsilon\; BV(1,\infty)$

and

$$\frac{1}{x}\int_1^x u(t)\, t\, dt \;\varepsilon\; BV(1,\infty),$$

then

$$\int_1^\infty |u(t)|\, dt<\infty.$$

This is a Tauberian theorem for $|A|$-summability, proved by Hyslop [7].

Proof. As by $\Lambda(t)=t$, and $\lambda(t)=\log t$,

$$\int_1^u e^{-t/x}\frac{1}{t}dt-\log x$$

is monotonic from some x, and

$$\int_1^u e^{-t/u}\frac{1}{t}\, dt-\log u=\int_1^u (e^{-t/u}-1)\frac{1}{t}\, dt=O(1)$$

and

$$\lim_{x\to\infty}\int_1^u (e^{-t/x}-1)\frac{1}{t}\, dt=0.$$

Corollary 2. If $\Phi(x)=\int_1^\infty u(t)e^{-(\log t)/x}dt$

$$=\int_1^\infty u(t)t^{-1/x}\, dt \;\varepsilon\; BV(1,\infty),$$

and

$$\frac{1}{\log x}\int_1^x u(t)\log\frac{1}{t}\, dt \;\varepsilon\; BV(1,\infty)$$

then

$$\int_1^\infty |u(t)|\, dt<\infty.$$

This is a Tauberian theorem for the absolute summability of the ordinary Dirichlet series.

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