NOTES ON FOURIER ANALYSIS (X): ABSOLUTE CESÀRO SUMMABILITY OF FOURIER SERIES.*)

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§ 1. Introduction. Concerning the absolute convergence of the Fourier series of function in the Lip(α, p) class, Hardy and Littlewood proved the following theorem:

Theorem A[1]. It $f \varepsilon$ Lip (α, p), where

$$0<\alpha\leqq 1,\ 1\leqq p\leqq 2,\ \alpha p>1,$$

then the Fourier series of $f(x)$ converges absolutely. Further

$$\Sigma\,|c_n|^{\beta}<\infty,$$

where c_n is the n-th Fourier cofficient of the Fourier series (of the complex form) of $f(x)$ and

$$\beta>p/(p+\alpha p-1).$$

Theorem B[2]. If $f(x)\ \varepsilon$ Lip$(1/p,p)$ and $f(x)\ \varepsilon$ Lip α, where

$$0<\alpha\leqq 1,\ 1\leqq p<2,$$

then the Fourier series of $f(x)$ converges absolutely.

Concerning the absolute Cesàro summability of the Fourier series of functions in the Lip(α,p) class, we will prove the following theorems.

Theorem 1. If $f\ \varepsilon$ Lip (α,p), where

$$0<\alpha\leqq 1, 1\leqq p\leqq 2,\ \alpha p\leqq 1,$$

then the Fourier series of $f(x)$ is summable $|C, \frac{1}{p}-\alpha+\varepsilon|$ almost everywhere, ε being any positive number.

Theorem 2. If $f\ \varepsilon$ Lip (α,p), where

$$0<\alpha\leqq 1,\ \alpha p>1,$$

then the Fourier series of $f(x)$ is summable $|C, \frac{1}{q}+\varepsilon|$.

If $f\ \varepsilon$Lip α, then $f\ \varepsilon$ Lip $(\alpha,2)$, and theorem 1 implies.

Theorem 3. If $f\ \varepsilon$ Lip α, $0<\alpha\leqq 1/2$, then the Fourier series of $f(x)$ is summable $|C, 1/2-\alpha+\varepsilon|$.

This is due to Hyslop[3].

Functions in Lip(1.1) are equivalent to functions of bounded variation.

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Putting $\alpha=p=1$ in Theorem 1, we get

Theorem 4. If $f(x)$ is of bounded variation, then the Fourier series of $f(x)$ is summable $|C, \varepsilon|$, ε being any positive number.

This is due to Bosanquet.

In the proof of these theorems we used the complex method due to Hardy and Littlewood. Therefore the proof is quitely different from that of Bosanquet and Hyslop.

§ 2. Lemmas. Let $f \varepsilon$ Lip (α, p), $0<\alpha\leqq 1$, $p\geqq 1$, and

$$f(x)\sim\sum_{-\infty}^{\infty} c_n e^{inx} \equiv \sum_{-\infty}^{\infty} C_n. \tag{1}$$

Let the power series component of (1) be

$$F(e^{ix})\sim\sum_{0}^{\infty} c_n e^{inx},\ F_2(e^{ix})\sim\sum_{-\infty}^{-1} c_n e^{inx},$$

which are boundary functions of

$$F_1(z)=\sum_{0}^{\infty} c_n z^n,\ F_2(z)=\sum_{-\infty}^{-1} c_n z^n,$$

regular in $|z|<1$.

It is known that

Lemma 1. It $f \varepsilon$ Lip (α,p), then $F_i(e^{ix})$ ε Lip (α,p) and

$$\left(\int_{-\pi}^{\pi} |F_i'(z)|^p dx\right)^{1/p} = O\left((1-r)^{\alpha-1}\right)$$

where $i=1,2$, and $z=re^{ix}$.

If we put $z=\rho e^{it}$ and

$$g(z)=F(ze^{ix})=\sum_{0}^{\infty} c_n z^n e^{inx}=\sum_{0}^{\infty} C_n z^n,$$

then we have

$$\left(\int_{-\pi}^{\pi} |g(e^{i(t+h)})-g(e^{it})|^p dt\right)^{\frac{1}{p}}=\left(\int_{-\pi}^{\pi} |F_1(e^{i(t+h+x)})\right.$$
$$\left.-F(e^{i(x+t)})^p|dt\right)^{\frac{1}{p}}$$

which is $O(h^\alpha)$ by Lemma 1. Thus we get

Lemma 2. If $f \varepsilon$ Lip (α,p), then $g(e^{it})$ ε Lip (α,p).

Further we have

Lemma 3. If $g(x)$ ε Lip (α,p), then $g_1(t)=\int_{-\pi}^{\pi} g(x+t)dt$ ε Lip (α, p).

For

$$\left(\int_{-\pi}^{\pi} |g_1(t+h)-g_1(t)|^p dt\right)^{\frac{1}{p}}=\left(\int_{-\pi}^{\pi} dt\left|\int_{-\pi}^{\pi} [g(x+t+h)\right.\right.$$

$$-g(x+t)]dx|^p\Big)^{\frac{1}{p}}$$

$$\leqq A\int_{-\pi}^{\pi}dt\Big(\int_{-\pi}^{\pi}|g(x+t+h)-g(x+t)|^p dx\Big)^{\frac{1}{p}}$$

$$=A\int_{-\pi}^{\pi}dth^{\alpha}\leqq Ah^{\alpha}.$$

Lemma 4. If we denote by σ_n^r and τ_n^r the r-th Cesàro mean of Σu_n and $\{un^n\}$ respectively, then

$$\sigma_{n+1}^r-\sigma_n^r=\tau_n^r/n.$$

From above lemmas we see that, in proving Theorem 1 and 2, we can suppose that the Fourier series of $f(x)$, is of power series type and it is sufficient to prove that $\Sigma|\tau_n^r|/n<\infty$ for required r.

§ 3. Proof of Theorem 1. We will distiguish two cases. Firstly we will exclude the case $p=1$. By T_n^r we denote the r-th Cesàro sum of $\{nc_n\}$, which is given by the expansion coefficient of $zg'(z)/(1-z)^r$.
Thus

$$(2)\qquad T_{n+1}^r=\frac{1}{2\pi i}\int_{-\pi}^{\pi}zg'(z)/(1-z)^r z^{n+2}\,dt \qquad (z=\rho e^{it})$$

$$=\frac{1}{2\pi\rho^n}\int_{-\pi}^{\pi}g'(\rho e^{it})e^{-int}/(1-\rho e^{it})^r dt.$$

$$T_{n+1}^r=\frac{1}{2\pi\rho^n}\int_{-\pi}^{\pi}\frac{e^{-int}}{(1-\rho e^{it})^r}\;g'(\rho e^{it})dt.$$

Let us put, for the sake of simplicity,

$$\tau_n\equiv T_{n+1}^r,$$
$$H(t)\equiv H(\rho e^{it})\equiv 1/(1-\rho e^{it})^r,$$
$$G(t)\equiv G(\rho e^{it})\equiv g'(\rho e^{it}),$$

By Lemma 1

$$\int_{-\pi}^{\pi}|g'(\rho e^{it})|^p dt=O\Big((1-p)^{(\alpha-1)p}\Big),$$

and then

$$(3)\qquad \Big(\int_{-\pi}^{\pi}|G(\rho e^{it})|^p dt\Big)^{1/p}=\Big(\int_{-\pi}^{\pi}dt\Big|\int_{-\pi}^{\pi}g'(\rho e^{i(t+x)})dx\Big|^p\Big)^{1/p}$$

$$\leqq\int_{-\pi}^{\pi}dx\Big(\int_{-\pi}^{\pi}|g'(\rho e^{i(t+x)})|^p\,dt\Big)^{\frac{1}{p}}=O\Big((1-\rho)^{\alpha-1}\Big).$$

For the proof of Theorem 1 it is sufficient to prove that

$$\sum_{1}^{\infty}|T_n^r|/n^{r+1}<\infty,\ a.e.$$

or
$$\int_{-\pi}^{\pi}\sum_{1}^{\infty}|\tau_n|/n^{r+1}dx<\infty.$$

Since we can suppose $r<1/p$, we have by the Hausdorff-Young theorem

$$(4)\qquad (\Sigma|\tau_n|^q\rho^{nq}|\sin nh|^q)^{1/q}\leqq A(\int_{-\pi}^{\pi}|H(t+h)G(t+h)-H(t-h)G(t-h)|^p dt)^{1/p}$$
$$\leqq A(\int_{-\pi}^{\pi}|H(t+h)|^p|G(t+h)-G(t-h)|^p dt)^{1/p}+A\int_{-\pi}^{\pi}|H(t+h)-H(t-h)^p||G(t-h)|^p dt)^{1/p},$$

where $1/p+1/q=1$. We will denote by P_1 and P_2 the integrals on the right hand side.

Then we have

$$\int_{-\pi}^{\pi}P_1^p\text{bx}\leqq A\int_{-\pi}^{\pi}|H(t+h)|^p\int_{-\pi}^{\pi}|g'(\rho e^{i(t+h)})-g'(\rho e^{i(t-h)})|^p dxdt$$
$$\leqq A\int_{-\pi}^{\pi}|H(t+h)|^p\left\{\int_{-\pi}^{\pi}|g'(\rho e^{i(t+h)})|^p dx+\int_{-\pi}^{\pi}|g'(\rho e^{i(t-h)})|^p dx\right\}dt$$
$$\leqq A(1-\rho)^{(\alpha-1)p}\int_{-\pi}^{\pi}|H(t+h)|^p dt$$
$$\leqq A(1-\rho)^{(\alpha-1)p}\int_{-\pi}^{\pi}\frac{dt}{\{(1-\rho)^2+\rho(t+h)\}^{pr/2}}$$
$$\leqq A(1-\rho)^{(\alpha-1)p},$$

and

$$\int_{-\pi}^{\pi}P_2^p dx\leqq A\int_{-\pi}^{\pi}|H(t+h)-H(t-h)|^p\int_{-\pi}^{\pi}|g'(\rho e^{i(t-h)})|^p dxdt$$
$$\leqq A(1-\rho)^{(\alpha-1)p}(\int_{-\pi}^{0}+\int_{0}^{\pi})|H(t+h)-H(t-h)|^p dt,$$

where the second integral on the right hand side is

$$\int_{0}^{\pi}|H(t+h)-H(t-h)|^p dt=\int_{-h}^{\pi-h}|H(t+2h)-H(t)|^p dt$$
$$=\int_{-h}^{h}+\int_{h}^{\pi-h}\equiv Q_1+Q_2,$$

say. Firstly

$$Q_1\leqq A\int_{-h}^{h}|H(t+h)|^p dt+A\int_{-h}^{h}|H(t)|^p dt$$
$$\leqq A\int_{-h}^{h}\frac{dt}{(t+2)^p h^r}+A\int_{-h}^{h}\frac{dt}{t^{rp}}\leqq A(h^{1-rp}).$$

$$Q_2 \leqq A\int_h^\pi |H(t+2h)-H(t)|^p dt$$

$$\leqq A\int_h^\pi \left| \frac{1}{(1-\rho e^{i(t+2h)})^r} - \frac{1}{(1-\rho e^{it})^r} \right|^p dt$$

$$\leqq Ah^p\int_h^\pi \frac{dt}{(1-\rho e^{it})^{(r+1)p}} \leqq Ah^p\int_h^\pi \frac{dt}{t^{(r+1)p}} \leqq Ah^{1-rp}.$$

Summing up above estimations

$$\int_{-\pi}^{\pi} P_2^p dx \leqq A(1-\rho)^{(\alpha-1)p}h^{1-rp}.$$

Putting $1-\rho=h$, we have

$$\int_{-\pi}^{\pi} P_1^p dx \leqq Ah^{(\alpha-1)p}, \quad \int^{\pi} P_2^p dx \leqq Ah^{(\alpha-1)p+(1-rp)}.$$

Substituting these into (4) we get

$$\int_{-\pi}^{\pi} \left\{\sum_1^\infty |\tau_n|^q \rho^{nq} |\sin nh|^q\right\}^{p/q} \leqq Ah^{(\alpha-1)p},$$

consequently

$$\int_{-\pi}^{\pi} \left\{\sum_{N/2}^{N} |\tau_n|^q \rho^{nq} (\sin nh)^q\right\}^{p/q} \leqq Ah^{(\alpha-1)p}.$$

Let $h\equiv\pi/2N$, then $\rho^{nq}=(1-h)^{nq}=(1-\pi/2N)^{nq}>c>0 (N/2\leqq n\leqq N)$. Thus we have

$$\int_{-\pi} (\sum_{N/2}^{N} |\tau_n|^q)^{p/q} \leqq AN^{-(\alpha-1)p}.$$

Putting $N=2^\nu$ and summing up by ν

$$\int_{-\pi}^{\pi} \sum_1^\infty |\tau_n|/n^{1+r} dx = \int_{-\pi}^{\pi} \sum_{\nu=1}^\infty \sum_{n=2^{\nu-1}+1}^{2^\nu} |\tau_n|/n^{1+r} dx$$

$$\leqq \sum_{\nu=1}^\infty \int_{-\pi}^{\pi} (\Sigma\tau_n^q)^{\frac{1}{q}} dx (\sum_{2^{\nu-1}+1}^{2^\nu} \frac{1}{n^{(r+1)p}})^{\frac{1}{p}}$$

$$\leqq A\sum_{\nu=1}^\infty 2^{\nu(\frac{1}{p}-r-1)}\left(\int_{-\pi}^{\pi} (\sum_{2^{\nu-1}+1}^{2^\nu} |\tau_n|^q)^{p/q} dx\right)^{\frac{1}{p}} \leqq A \sum_{\nu=1}^\infty 2^{-\nu(\alpha-\frac{1}{p}+r)}$$

which is convergent for $r>1/p-\alpha$. Thus we get the theorem.

We will now consider the second case $p=1$. In this ease (4) is replaced by

$$\int_{-\pi}^{\pi}|\tau_n\rho^n \sin nh|\,dx \leqq A\int_{-\pi}^{\pi}|G(t+h)-G(t-h)|dx\int_{-\pi}^{\pi}|H(t+h)|dt$$

$$+A\int_{-\pi}^{\pi}|H(t+h)-H(t-h)|\,dx\int_{-\pi}^{\pi}|G(t-h)|dt$$

$$\leqq A(1-\rho)^{\alpha-1}+A(1-\rho)^{\alpha-r}\leqq A(1-\rho)^{\alpha-}$$

as in the former case. Putting $h=\pi/2N$,

$$\int_{-\pi}^{\pi}\sum_1^{\infty}|\tau_n|/n^{r+1}\leqq A\sum_{\nu=1}^{\infty}2^{\nu(1-\alpha-r)}<\infty.$$

§ 4. Proof of Theorem 2. Using the notation of the proof of Theorem 1, we get

$$T_{n+1}^{r}=\frac{1}{2\pi\rho^n}\int_{-\pi}^{\pi}\frac{g'(\rho e^{it})}{(1-\rho e^{it})^r}e^{-int}dt,$$

$$|T_{n+1}^{r}|\leqq A\left(\int_{-\pi}^{\pi}|g'(\rho e^{it})|^p dt\right)^{\frac{1}{p}}\left(\int_{-\pi}^{\pi}\frac{dt}{|1-\rho e^{it}|^{rq}}\right)^{\frac{1}{q}}$$

$$\leqq A(1-\rho)^{\alpha-1}\left(\int_{-\pi}^{\pi}\frac{dt}{((1-\rho)^2+4\rho\sin^2 t/2)^{rq}}\right)^{\frac{1}{q}},$$

where the integral on the right hand side is

$$\int_{-\pi}^{\pi}\frac{dt}{((1-\rho)^2+4\rho\sin^2 t/2)^{rq}}\leqq\int_{-\pi}^{\pi}\frac{dt}{((1-\rho)^2+4\rho t^2/\pi^2)^{rq}}$$

$$\leqq A\int_{-\pi}^{\pi}\frac{dt}{((1-\rho)^2+\rho t^2)^{rq/2}}\leqq A(1-\rho)^{1-rq}\int_{-\pi}^{\pi}\frac{dt}{(1+t^2)^{rq}}$$

$$\leqq A(1-p)^{1-rq},$$

provided that $rq<1$. Hence we have, putting $\rho=1-1/n$,

$$|T_{n+1}^{r}|\leqq A\,(1-\rho)^{1-r-\frac{1}{p}}\leqq A/n^{\alpha-r-1/p},$$

$$\sum_{n=1}^{\infty}\frac{|T_{n+1}^{r}|}{n^{r+1}}\leqq A\sum_1^{\infty}\frac{1}{n^{\alpha+1/q}}<\infty.$$

Thus the theorem is proved.

Reference.

1) Hardy-Littlewood, Math. Zeits., 28 (1928).
2) Hardy-Littlewood, Jurn. London Math. Soc., 3(1928)
3) Hyslop, Proc. London Math. Soc., 41(1936).
4) Bosanquet, ibidem, 43(1937).