LINEAR TOPOLOGICAL SPACES AND ITS PSEUDO-NORMS.*)

By

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Linear topological spaces were studied by A. Kolmogoroff,[1] J. v. Neumann,[2] H. Hyers[3] and many other authers. Concerning relations among these investigations, J. V. Wehausen[4] proved the equivalency of linear topological spaces of Neumann and Kolmogonoff, and Hyers gave a new defintion of linear topological spaces equivalent to them. After him to any linear topological space we can associate a cernain directed system. When we examine this directed system, we see that the directed system can be replaced by a semi-join-lattice, and the linear topological space is characterized by the family of new topologies which form a semi-join-lattice (§ 2). In § 3 we show that this semi-lattice can be replaced by the semi-meet-attice. The norm of the convex linear topological space satisfies the triangular inequality. But the "Norm" of § 3 does not necessarily satisfy it. In § 4 we consider that the "Norm" satisfying the triangular inequality actually characterizes the convex linear topological space.

1. Definitions. Kolmogoroff's Definition (Definition K). Let L be a linear Hausdorff space. If the vector operations $x+y$ and $t\cdot x$ are continuous with respect to this topology, then L is said to be a linear topological space.

Neumann's Definition (Definition N). Let L be a linear space. If L has family A of subsets U in L satisfying the following conditions, it is said to be a linear topological space, and is denoted by $L(A)$. A and U are said to be the neighbourhood system and neighbourhood, respectively.

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1) Kolmogoroff, Zur Normierbarkeit Eines Allgemeinen Topologischen Linear Raumes (Studia Math., Tom. V).

2) von Neumann, On complete Topological spaces (Trans. Amer. Math. Soc. XXXVII (1935)).

3) Hyers, Pseudo-normal Linear Space and Abelian Groups (Duke Math. Journ. Vol. 5 (1939)).

4) Wehausen, Transformations in Linear Topological space (Duke Math. Journ. Vol. 4 (1938)).

(N. 1) Only common point of all U is θ.

(N. 2) For any U_1 and U_2 there exists U_3 such that

$$U_3 \subset (U_1, U_2).$$

(N. 3) For any U_3 and numerical t (but $|t| \leqq 1$) there exists

$$U_1 \text{ such as } tU_1 \subset U.$$

(N. 4) For any U there exists U_1 such as $U_1 + U \subset U$.

(N. 5) For any point $x \in L$ and U there exists numerical value t such that $x \in tU$.

Hyers' Definition (Definition H). Let L be a linear spece and D a directed system. When there exists a real valued function $|x|_d$ (called pseudonorm) on the domain $L \times D$ satisfying the following conditions, L is said to be a linear topological space, and is denoted by $L(D)$.

(H. 1) $|x|_d \geqq 0$, if $|x|_d = 0$ for all $d \in D$ then $x = \theta$.

(H. 2) $|tx|_d = |t| \cdot |x|_d$.

(H. 3) For $\varepsilon > 0$ and $d \in D$ there exist $\delta > 0$ and $e \in D$ such that

$$|x|_e < \delta \text{ and } |y|_e < \delta \text{ imply } |x+y|_d < \varepsilon.$$

(H. 4) If $d > e$ then $|x|_d = |x|_e$.

Definition 1. Let S be a subset of the linear space L. Then two real valued functions $|x|_S$ and $\|x\|_S$ are defined by

$$|x|_S = \underset{\lambda > 0,\ x \in \lambda S}{\text{gr. l. b.}}\ \lambda,$$

and

$$\|x\|_S = \underset{\gamma(x)}{\text{gr. l. b.}} \sum_{k=1}^{n} |x_{k-1} - x_k|_S,$$

where $\gamma(x)$ is a finite set such as $\gamma(x) = \{\theta = x_0, x_1, \cdots x_n = x\}$.

Theorem 1. If $S \subset T \subset L$, then

(1) $|x|_S \geqq |x|_T$,

(2) $\|x\|_S \geqq \|x\|_T$,

(3) $\|x\|_S \leqq |x|_T$.

Proof. There exists a suitable sequence ε_n ($\varepsilon_n \downarrow 0$) such that

$$x \in (|x|_S + \varepsilon_n)\ S \text{ for } n = 1, 2, \cdots.$$

Hence $x \in (|x|_S + \varepsilon_n) S \subset (|x|_S + \varepsilon_n)\ T$ for $n = 1, 2, \cdots$.

This implies (1), (2) and (3) are evident by

$$\|x\|_S = \underset{\gamma(x)}{\text{gr. l. b.}}\ \Sigma |x_k - x_{k-1}|_S \geqq \underset{\gamma(x)}{\text{gr. l. b.}}\ \Sigma |x_k - x_{k-1}|_T = \|x\|_T$$

and

$$\|x\|_S = \underset{\gamma(x)}{\text{gr. l. b.}}\ \Sigma |x_k - x_{k-1}|_S \leqq \Sigma |x|_S = |x|_S.$$

2. Characterization of the linear topological space $L(A)$ depends on the semi-join-lattice.

Let A' be a class of all U' such that

$$U'=U(U,\alpha)\equiv\{tx;\ x\in U,\ |t|\leqq\alpha\},$$

where $U\in A$ and α is a positive number. B is a class of all V such that

$$V=\overset{n}{\underset{i=1}{D}}\ U',\qquad U'\in A',\ n=1,\ 2,\cdots.$$

In this B if $V_1\supset V_2$ we write $V_1<V_2$ and if $V_1>V_2>V_1$ then write $V_1\equiv V_2$. By this classification of B we have a new set (B), whose point is (V) having V as a representation.

Evidently $(V_1)\vee(V_2)=(D(V_1,V_2))$.

Theorem 2. For any linear topological space $L(A)$ there exists a semi-join-lattice (B) and A is topologically equivalent[5] to B.

Proof. The first part of the theorem is evident. Let $V\in B$, $V=\overset{n}{\underset{i=1}{D}}U'_i$ and $U'_i=U'_i(U_i,\ \alpha_i)$, then there exists a U such as $U\subset\overset{w}{\underset{1}{D}}\ U'_i$. If we take $\alpha=\min(\alpha_1\alpha_2\cdots\alpha_n)$, then $U'=U'(U,\alpha)\subset V$ and $U'\in A'$. Since $A'<B$, A' is topologically equivalent to B. Conseqnently A is topologically equivalent to B.

Theorem 3. B satisfies the following conditions.

(1) If $V\in B$ and $\beta\neq0$ then $\beta V\in B$.

(2) If $|\beta|\leqq1$ then $\beta V<V$.

(3) $V=-V$

(4) B satisfies (N. 1),$\cdots$, (N. 5).

Proof is easy.

Theorem 4. If $V=D\ (U_1',\ U_2')$ then $|x|_V=\max\ (|x|_{U'_1},\ |x|_{U'_2})$.

Proof. Let $|x|_{U'_1}\leqq|x|_{U'_2}$. Thenthere exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n>0$ and $x\in(|x|_{U'_2}+\varepsilon_n)\ U'_2$ for $n=1,\ 2.\cdots$. Again by theorem 3 (2) there is $\varepsilon'>0$ such that

$$|x|_{U'_1}+\varepsilon'\leqq|x|_{U'_2}+\varepsilon_n \text{ and } x\in(|x|_{U'_1}+\varepsilon')U'_1.$$

Consequently $\quad x\in(|x|_{U'_1}+\varepsilon')\ U'_2\subset(|x|_{U'_2}+\varepsilon_n)\ U_1'$

and $\quad x\in D[(|x|_{U'_2}+\varepsilon_n)U'_1,\ (|x|_{U'_2}+\varepsilon_n)U'_2]$

and then $\quad =(|x|_{U'_2}+\varepsilon_n)\ D(U'_1,\ U_2')=(|x|_{U'_2}+\varepsilon_n)\ V,\ |x|_V\leqq|x|_{U'_2}$.

On the other hand we have $|x|_V\geqq|x|_{U'_2}$ evidently.

Corollary. If $V=\overset{n}{\underset{1}{D}}\ U_i'$, then $|x|_V=\max\ (|x|_{U'_1},\cdots\cdots,\ |x|_{U'_n})$.

Since $|x|_V$ takes the same value for all $V\in(V)$ we define by $|x|_V$.

Theorem 5. If $(V_1),\ (V_2)\in(B)$, then $|x|_{(V_1)\cup(V_2)}=\max\ (|x|_{(V_1)},\ |x|_{(V_2)})$.

Proof is easy from above corollary and the definition.

Theorem 6. Each linear topological space $L(A)$ is characterized by the real valued function $|x|_l$ on the domain $L \times L_1$ where L_1 is a semi-join-lattice, satisfying the following conditions.

(1) $|x|_l \geqq 0$ and if $|x|_l = 0$ for all $l \in L_1$ then $x = \theta$.

(2) $|tx|_l = |t| \cdot |x|_l$.

(3) For $\varepsilon > 0$ and $l \in L_1$ there exist $\delta > 0$ and $l_2 \in L$ such that $|x|_{l_2} < \delta$ and $|y|_{l_2} < \delta$ imply $|x+y|_{l_1} < \varepsilon$.

(4) $|x|_{l_1 \cup l_2} = \max(|x|_{l_1}, |x|_{l_2})$.

Proof. Evidently the function $|x|_{(V)}$ satisfies (1)–(4), conversely in $L(L_1)$ if we put

$$U = U(l, \varepsilon) \equiv \{x : |x|_l < \varepsilon\}.$$

Then the class of all U satisfies (N. 1)-(N. 5). Again the neighbourhood system A of $L(A)$ is topologically equivalent to the class $\{U(V), \varepsilon\}$. For if $0 < \varepsilon_1 < \varepsilon$, $\varepsilon_1 V < U((V), \varepsilon)$ and $U((V), 1) \subset V$. Hence B is topologically equivalent A as well as $\{U((V), \varepsilon)\}$.

By this Theorem we can understand the linear topological space in the following space. Let L be the linear space and L_1 be the semi-join-lattice. Then to each element l of L_1 there corresponds a norm topology of L safisfying (1)-(3), which we call (l)-togology and if we order these (l)-topologies by their implication, it becomes a semi-join-lattice, homeomorphic to L_1.

3. Characterization of the linear topological space $L(A)$ depends on the semi-meet-lattice.

Let B be the class of all W such that

$$W = \overset{n}{\underset{i=1}{S}} U_i', \quad U_i' \in A' \quad n = 1, 2, \ldots\ldots$$

If (W) is a set of all (W) which is analogous to (V) of (B),

$$(W_1) \wedge (W_2) = (S(W_1, W_2)).$$

Theorem 7. To each linear topological space $L(A)$ there corresponds a semi-meet-lattice (B) and A is topologically equivalent to L.

Theorem 8. B satisfies the conditions (1)-(3) of theorem 3.

Proof of these two theorems are analogous to those of B.

Theorem 9. If $W = S(U_1', U_2')$, then $|x|_W = \min(|x|_{U'_1}, |x|_{U'_2})$.

Proof. Let $|a|_{U'_1} \leqq |a|_{U'_2}$. For some positive sequence $\{\varepsilon_n\}$ converging to 0,

$$x \in (|x|_W + \varepsilon_n) W = (|x|_W + \varepsilon_n) \circ (U'_1, U'_2)$$
$$= S[(|x|_W + \varepsilon_n) U'_1, (|x|_W + \varepsilon_n) U'_2] \quad (n = 1, 2, \ldots\ldots\ldots).$$

Firstly, if $x \in (|x|_W + \varepsilon_n) U'_1$, then $|x|_{U'_1} \leqq |x|_W$, and secondly if $x \in (|x|_W + \varepsilon_n) U'_2$, then $|x|_{U'_1} \leqq |x|_{U'_2} \leqq |x|_W$. Consequently

$$|x|_W \geqq |x|_{U'_1}.$$

On the otherhand, $|x|_W \leqq |x|_{U'1}$ is evident. Hence we have

$$|x|_W = |x|_{U'1} = \min(|x|_{U'1}, |x|_{U'2}).$$

Corollary. If $W = \overset{n}{\underset{1}{S}} U'_i$ then $|x|_W = \min(|x|_{U'1}, |x|_{U'2}, \cdots\cdots, |x|_{U'n})$.

Since $|x|_W$ takes the same value for all $W \epsilon (W)$ we define $|x|_{(W)}$ by $|x|_W$. This definitions is analogous to the case of (B).

Theorem 10. If $(W_1), (W_2) \epsilon (W)$ then $|x|_{(W_1)\frown(W_2)} = \min(|x|_{(W_1)}, |x|_{(W_2)})$.

Lemma 1. $|tx|_W = |t| \cdot |x|_W$.

Proof. If $W = \overset{n}{\underset{1}{S}} U'_1$ then we have

$$|tx|_W = \min(|tx|_{U'1}, \cdots\cdots, |tx|_{U'n})$$
$$= |t| \min(|x|_{U'1}, \cdots\cdots, |x|_{U'n})$$
$$= |t| \cdot |x|_W.$$

Theorem 11. Each linear topological space $L(A)$ is characterized by the real valued function $|x|_l$ on the domain $L \times L_2$, where L_2 is a semi-meet-lattice and is also a directed system satisfying the following conditions.

(1) $|x|_l \geqq 0$ and if $|x|_l = 0$ for all $l \epsilon L_2$ then $x = 0$.

(2) $|tx|_l = |t| \cdot |x|_l$.

(3) For $\varepsilon > 0$ and $l_1 \epsilon L_2$ there exist $\delta > 0$ and $l_2 \epsilon L_2$ such that $|x|_{l_2} < \delta$ and $|y|_{l_2} < \delta$ imply $|x+y|_{l_1} < \varepsilon$.

(4) $|x|_{l_1 \wedge l_2} = \min(|x|_{l_1}, |x|_{l_2})$.

Proof. By the construction we can easily see that (W) determined by $L(A)$ and $|x|_{(W)}$ astisfies the conditions (1)—(4). Conversely, in $L \times L_2$ the class of all $U = U(l, \varepsilon) = (x; |x|_l \leqq \varepsilon)$ satisfies (N. 1)=(N. 5), and moreover A is topologically equivalent to $\{U((W), \varepsilon)\}$.

Corollary. In Theorem 11, we can replace the word "directed system" by the condition:

(5) For any x and $l_1, l_2 \epsilon L_2$ there exists $l \epsilon L_2$ such that
$$\max(|x|_{l_1}, |x|_{l_2}) < |x|_l.$$

4. Convex linear topological space.

In definition N, if any neighbourhood U satisfies the following condition

$$\text{(N. 6)} \qquad U + U \subset 2U,$$

then L is said to be convex.

In Definition K if for any neighbourhood U_θ there exists a convex neighbourhood V_θ such that $V_\theta \subset U_\theta$, then L is said to be locally convex.[5]

Two neighbourhood-systems B and Z are called topologically equivalent if for any $U \epsilon A$ there exists $V \epsilon B$ such that $V \subset U$ and converse.

In Definition H, of the Pseuedo-norms $|x|_d$ satisfies the following condition

$$\text{(H. 5)} \quad |x+y|_d \leqq |x|_d + |y|_d \text{ for all } d \in D.$$

We say that the pseudo-norm satisfies the triangular inequality. It is well-known that these three notions are mutually equivalent.

If $L(A)$ is a convex linear topological space, then $|x|_{(V)}$ satisfies the triangular inequality[6], but $|x|_{(W)}$ does not.

We will now replace $|x|_W$ by an equivalent $\|x\|_W$ satisfying the triangular inequality.

We will put

$$\|x\|_W = \text{gr. l. b.} \sum |x_k - x_{k-1}|_W,$$

where gr. l. b. is taken for all chain $\{\theta, x_1, x_2, \cdots\cdots, x_n = x_n\}$.

Then we have

Lemma 2. $\|tx\|_W = |t| \cdot \|x\|_W$.

Proof. Let $\gamma(x) = \{\theta, x_1, \cdots\cdots, x_n = x\}$, $\gamma'(tx) = \{\theta, x'_1, \cdots\cdots, x'_m = tx\}$.

$$\|tx\|_W = \underset{\gamma'(tx)}{\text{gr. l. b}} \sum |x'_k - x'_{k-1}|_W \leqq \text{gr. l. b.} \sum |t| \cdot |x_k - x_{k-1}|_W$$

$$= |t| \text{ gr. l. b.} \sum |x_k - x_{k-1}|_W = |t| \cdot \|x\|_W.$$

If we replace x and t by tx and $\frac{1}{t}$, we get

$$\|tx\|_W \geqq |t| \cdot \|x\|_W.$$

Hence

$$\|tx\|_W = |t| \cdot \|x\|_W.$$

Lemma 3. $\|x+y\|_W \leqq \|x\|_W + \|y\|_W$.

Proof. Let $\gamma(x) = \{\theta, x_1, x_2, \cdots\cdots, x_k, \cdots\cdots x_m = x\}$,

$$\gamma(y) = \{\theta, y_1, \cdots\cdots, y_l, \cdots\cdots y_n\}$$

and $y'_l = x + y_l$. We have

$$\|x\|_W + \|y\|_W = \underset{\gamma(x)}{\text{gr. l. b.}} \sum |x_k - x_{k-1}|_W + \underset{\gamma(y)}{\text{gr. l. b.}} \sum |y_l - y_{l-1}|_W$$

$$= \underset{\gamma(x)}{\text{gr. l. b.}} \sum |x_k - x_{k-1}|_W + \underset{\gamma(y)}{\text{gr. l. b.}} \sum |y'_l - y'_{l-1}|_W$$

$$\geqq \underset{\gamma(x)}{\text{gr. l. b.}} \sum |x_k - x_{k-1}|_W + \underset{\gamma(x,\, x+y)}{\text{gr. l. b.}} \sum |y''_p - y''_{p-1}|_W$$

$$\geqq \underset{\gamma(x+y)}{\text{gr. l. b.}} \sum |z_q - z_{q-1}|_W = \|x+y\|_W.$$

(7) If we put $V = \overset{n}{\underset{1}{D}} U'_i$, then

$$|x+y|_{(V)} = |x+y|_V = \max(|x+y|_{U'_1}, \cdots |x+y|_{U'_n})$$

$$\leqq \max(|x|_{U'_1} + |y|_{U'_1}, \cdots\cdots, |x|_{U'_n} + |y|_{U'_n}),$$ On the other hand

$$|x|_{U'_i} + |y|_{U'_i} \leqq \max(|x|_{U'_1}, \cdots\cdots, |x|_{U'_n}) + \max(|y|_{U'_1}, \cdots\cdots, |y|_{U'_n})$$

$$= |x|_V + |y|_V \qquad (i = 1, 2, \cdots\cdots, n).$$

Hence $\quad |x+y|_V \leqq |x|_{(V)} + |y|_{(V)}$.

6) Tychonoff, Ein Fixpunktsatz (Math. Ann., Vol. 111 (1935)).

Lemma 4. $\|x\|_{U'} = |x|_{U'}$.

Proof. $\|x\|_{U'} \leqq |x|_{U'}$ is evident.

Conversely $|x|_{U'} \leqq \mathrm{gr, l, b,} \underset{\gamma(v)}{\Sigma} |x_k - x_{k-1}|_{U'} = \|x\|_{U'}$.

Hence $\|x\|_{U'} = |x|_{U'}$.

Theorem 12. $\|x\|_W = \|x\|_{W^{conv}}$.

Proof. Since $W' \subset W^{conv}$, $\|x\|_W \geqq \|x\|_{W^{conv}}$ is easy. Let $W = \sum_1^n U'_i$, then there exists a sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0$ and $x \varepsilon (\|x\|_{W^{conv}} + \varepsilon_m)$. $W^{conv} = \alpha_m W^{conv}$ $(m = 1, 2 \dots\dots,)$, where $\alpha_m = \|x\|_{W^{conv}} + \varepsilon_m$.
Hence we have a finite sequence of positive numbers $\{t_i\}$ and $x_i \varepsilon U_i'$ such that

$$\Sigma t_i = 1 \qquad \text{and } x = \alpha_m (t_1 x_1 + \dots\dots + t_n x_n).$$

Thus we have

$$\|x\|_W = \alpha_m \|t_1 x_1 + \dots\dots, + t_n x_n\|_W \leqq \alpha_m \Sigma \|t_i x\|_W$$
$$= \alpha_m \Sigma t_i |x|_W \leqq x_n \Sigma t_i \|x_i\|_{U'i} = \alpha_m \Sigma t_i = \alpha_m.$$

Consequently

$$\|x\|_W \leqq \|x\|_{W^{conv}},$$

and then

$$\|x\|_W = \|x\|_{W^{conv}}.$$

Let $[W]$ be a class of all W^{conv}. If we define $W_1^{conv} > W_2^{conv}$ by $W_1^{conv} \subseteq W_2^{conv}$, then (W) and $[W]$ are isomorphic.
Now we say that the function $\|x\|_{W^{conv}}$ defines W^{conv}-topology of L. If

$$\|x\|_{W_1^{conv}} \leqq \|x\|_{W_2^{conv}}.$$

Then we say that W_2^{conv}-topology is not weaker than W^{conv}-topology with this order relation the class of all W^{conv}-topology is a semi-ordered system.

Theorem 13. The class of all W^{conv}-topology and (W) are meet-isomorphic.

Proof. For any W_1^{conv} and W_2^{conv} we have

$$\|x\|_{W_1^{conv} \cap W_2^{conv}} = \|x\|S\,(W_1, W_2)^{conv} = \|x\|S\,(W_1, W_2)$$
$$\leqq \|x\|_{W_i} = \|x\|_{W_i^{conv}} \quad (i = 1, 2, \dots\dots).$$

If $\|x\|_{W^{conv}} \leqq \|x\|_{W_i^{conv}}$ $(i = 1, 2,)$, then $\|x\|_W \leqq \|x\|_{W_i} \leqq |x|_W$ $(i = 1, 2)$.
Hence

$$\|x\|_W \leqq \min (|x|_{W_1}, |x|_{W_2}) = |x| S\,(W_1, W_2),$$
$$\|x\|_W \leqq \sum_1^n \|x_k - x_{k-1}\|_W \leqq \Sigma\, |x_k - x_{k-1}|\, S\,(W_1, W_2),$$
$$\|x\|_W \leqq \|x\|\, S\,(W_1, W_2) \leqq \|x\|\, S\,(W_1, W_2)^{conv} = \|x\|_{W_1^{conv} \cap W_2^{conv}}.$$

Hence we see that the correspondence between (W) and W^{conv}-topoogy is meet-isomorphic.

Theorem 14. Any locally convex linear topological space $L(U)$ is characterized by the real valued function $\|x\|_l$ on the domain $L \times L_3$, where L_3 is a semi-meet-lattice and is also a directed system satisfying the following conditions.

(1) $\|x\|_l > 0$ and if $\|x\|_l = 0$ for all $l \in L$, then $x=\theta$.

(2) $\|tx\|_l = |t| \bullet \|x\|_l$.

(3) $\|x+y\|_l \leqq \|x\|_l + \|y\|_l$.

(4) Meet of l_1 and l_2-topologies is $l_1 \wedge l_2$-topology, where the phrase l-topology is defined by the function $\|x\|_l$.

Proof. If we consider L ([W]) in $L(A)$, $\|x\|_{W^{conv}}$ satisfies (1)—(4). Conversely let $U=U(l,\varepsilon)=(x; \|x\|_l<\varepsilon)$. It is easy that the class of all U satisfies (N. 1)-(N. 6), and A is topologically equivalent to $\{U(W^{conv}, \varepsilon)\}$. For any $U(W^{conv}, \varepsilon)$, $\varepsilon U' \subset U(W^{conv}, \varepsilon)$ where $\varepsilon_1<\varepsilon$ and $U' \subset W$. Conversely for any U' and $0<\varepsilon<1$, $U(U', \varepsilon) \subset U'$.

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