ARITHMETIC MEANS OF SUBSEQUENCES*)

By

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Introduction. Let $\{s_n\}$ be a sequence of real numbers which is summable $(C, 1)$ to s: $(s_1 + s_2 + \cdots + s_n)/n \to s$ as $n \to \infty$. Let $\{r_n(x)\}$ be the Rademacher system. If the limit of

$$(1) \qquad \varphi_n(x) = \Big(\sum_{k=1}^{n} s_k \frac{1 + r_k(x)}{2}\Big) \Big/ \Big(\sum_{k=1}^{n} \frac{1 + r_k(x)}{2}\Big)$$

for $n \to \infty$, exists for almost all x, we shall say that *almost all the subsequences* of $\{s_n\}$ are summable $(C, 1)$; if the limit of (1) does not exist for almost all x, we say that *almost all the subsequences* of $\{s_n\}$ are not summable $(C, 1)$ (cf. [2]). These two cases are the all which may occur, since the existence set of the limit of (1) is homogeneous. If the limit of (1) exists only for x belonging to a set of the first category, it is called that *nearly all the subsequences* of $\{s_n\}$ are not summable $(C, 1)$.

R. C. Buck and H. Pollard [2] proved the following theorem.

Theorem. *If $\{s_n\}$ is summable $(C, 1)$ to s, then in order that almost all the subsequences of $\{s_n\}$ are summable $(C, 1)$, it is sufficient that*

$$(2) \qquad \sum_{k=1}^{\infty} s_k^2/k^2 < \infty,$$

and it is necessary that

$$(3) \qquad \sum_{k=1}^{n} s_k^2 = o(n^2) \qquad as\ n \to \infty.$$

In § 1 of this paper we shall give another sufficient condition, and in § 2 we shall construct an example which shows not only that this condition is the best possible one in a sense but also give a negative answer for the Buck-Pollard problem [2] whether the condition (3) is a sufficient one. In the last § we shall concern ourselves the summability $(C, 1)$ of nearly all the subsequences.

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§ 1. By easy consideration, we may see that the existence almost everywhere of the limit of (1) is equivalent to:

$$(4) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} s_k r_k(x) = 0$$

almost everywhere, provided that $\{s_n\}$ is summable $(C, 1)$ (See [2]).

Theorem 1. *If $\{s_n\}$ is summable $(C, 1)$ to s, and if*

$$(5) \qquad \sum_{k=1}^{n} s_k^2 = o(n^2/\log\log n) \qquad \textit{as } n \to \infty,$$

then almost all the subsequences of $\{s_n\}$ are summable $(C, 1)$ to s.

Proof. Let us put

$$B_n = \sum_{k=1}^{n} s_k^2, \quad S_n(x) = \sum_{k=1}^{n} s_k r_k(x) \quad \text{and} \quad S_n^*(x) = \max_{1\le k\le n} |S_k(x)| \qquad (k = 1, 2, \cdots).$$

For $\delta > 0$, we denote by E_k $(k = 1, 2, \cdots)$ the set of all x such that $|S_n(x)| > n\delta$ for at least one value of n, $2^{k-1} < n \leqq 2^k$. If we put

$$G_k = \left[x\,;\ S_{2^k}^*(x) > 2^{k-1}\delta\right] \qquad (k = 1, 2, \cdots)$$

we have evidently $F_k \subset G_k$ $(k = 1, 2, \cdots)$. Hence if the inequality

$$(6) \qquad \sum_{k=1}^{\infty} |G_k| < \infty$$

holds for every $\delta > 0$ we can deduce that $|S_n(x)|/n \to 0$ as $n \to \infty$ almost everywhere, and by the remark at the beginning of this § we may complete the proof. To prove (6), we use the Marcinkiewicz-Zygmund inequelity ([4]; [5] Remark 1 § 3)

$$(7) \qquad \int_0^1 \exp(a\, S_n^*(x))\, dx \leqq 32 \exp\left(\frac{1}{2} a^2 B_n\right), \qquad a = a_n > 0.$$

From this we have

$$|G_k| \exp(a 2^{k-1}\delta) \leqq \int_0^1 \exp(a S_{2^k}^*(x))\, dx \leqq 32 \exp\left(\frac{1}{2} a^2 B_{2^k}\right),$$

and if we take $a = 2^{k-1}\delta/B_{2^k}$, we have

$$(8) \qquad |G_k| \leqq 32 \exp\Big(-\frac{\delta^2}{2}\frac{2^{2(k-1)}}{B_{2k}}\Big) = 32 \exp\Big(-\frac{\delta^2}{8}\frac{(2^k)^2}{B_{2k}}\Big).$$

On the other hand, from (5) it follows that

$$B_{2k}/(2^k)^2 \leqq \delta^2/(16 \log\log 2^k)$$

for large $k (> k_0$ say). Consequently we have from (8)

$$|G_k| \leqq 32 \exp(-2 \log\log 2^k) = 32/(k \log 2)^2 \qquad \text{for } k > k_0,$$

which is a term of a convergent series, and (6) is proved, q. e. d.

§2. Theorem 2. *There exists a sequence* $\{s_n\}$ *summable* $(C, 1)$, *which satisfies the condition*

$$(9) \qquad \sum_{k=1}^{n} s_k^2 = O(n^2/\log\log n) \qquad as\ n \to \infty,$$

and such that almost all the subsequences of this sequence are not summable $(C, 1)$.

This theorem gives us a negetive answer for the Buck-Pollard problem. and comparing Theorem 1 and 2, we may say that the condition (5) is the best possible one of this form.

For the proof we will construct an example.

Let us put $s_1 = 0$ and $s_n = (-1)^n \sqrt{n/\log\log n}$ $(n = 1, 2, \cdots)$, then, as easily be seen, $\{s_n\}$ is summable $(C, 1)$ to 0. We have

$$B_n = \sum_{k=1}^{n} s_k^2 = \sum_{k=1}^{n} k/\log\log k \sim n^2/\log\log n \qquad \text{as } n \to \infty^{1)},$$

and (9) is satisfied. Since $B_n \to \infty$ and $s_n = o(\sqrt{B_n/\log\log B_n})$ as $n \to \infty$, tq conditions of the law of the iterated logarithm are fulfilled [3]. Henc $\limsup_{n\to\infty} S_n(x)/\sqrt{2B_n \log\log B_n} = 1$, that is, $\limsup_{n\to\infty} S_n(x)/n = \text{constant} \neq 0$ almost everywhere. Thus the example was established.

§3. Theorem 3. *If* $\{s_n\}$ *is summable* $(C, 1)$ *but not convergent, then nearly all the subsequences of* $\{s_n\}$ *are not summable* $(C, 1)$.

Proof. If all the subsequences of $\{s_n\}$ are summable $(C, 1)$, then $\{s_n\}$ must be convergent (See. e. g. [1]), hence from the assumption of the theorem there exists a subsequence $\{s_{n_i}\}$ which is not summable $(C, 1)$. Let $\{s_{n_i}\} = \Big\{s_n \dfrac{1+r_n(x_0)}{2}\Big\}$,

1) $P_n \sim Q_n$ means that P_n and Q_n are of the same order as $n \to \infty$. $P_n \simeq Q_n$ means that $P_n/Q_n \to 1$ as $n \to \infty$.

$0 < x_0 < 1$, where the terms with indices n such that $\frac{1}{2}\left\{1 + r_n(x_0)\right\} = 0$, are regarded to be omitted; evidently x_0 belongs to the set R of all dyadic irrationals.

Since $\{\varphi_n(x_0)\}$ is divergent, there exists a positive integer p_0 and a sequence of positive integers $m_1 < n_1 < m_2 < n_2 < \cdots \to \infty$, such that

$$|\varphi_{m_i}(x_0) - \varphi_{n_i}(x_0)| > \frac{1}{p_0} \qquad (i = 1, 2, \cdots). \tag{10}$$

If we put $E_{pq} = \mathrm{R} \cap [x;\ |\varphi_m(x) - \varphi_n(x)| \leqq 1/p\ (m, n > q)]$ $(p, q = 1, 2, \cdots)$, then the set of $x \in \mathrm{R}$ for which the limit of (1) exists, may be represented as

$$E = \bigcap_{p=1}^{\infty} \bigcup_{q=1}^{\infty} E_{pq}.$$

If we suppose that the set E is of the second category in R, so is the set $\bigcup_{q=1}^{\infty} E_{2p_0, q}$ and then for some q_0, the set E_{2p_0, q_0} is still of the second category in R. The function $\varphi_n(x)$ being continuous in R, the set $E_{2p_0 q_0}$ is closed in R, and hence it contains an interval $I \subset \mathrm{R}$. Since there is a point $x_1 \in I$ such that the difference $|x_0 - x_1|$ is dyadically rational, we have

$$\frac{1}{n}\sum_{k=1}^{n} r_k(x_0) \simeq \frac{1}{n}\sum_{k=1}^{n} r_k(x_1), \qquad \frac{1}{n}\sum_{k=1}^{n} s_k r_k(x_0) \simeq \frac{1}{n}\sum_{k=1}^{n} s_k r_k(x_1)$$

as $n \to \infty$. Hence from (10) we have

$$|\varphi_{m_i}(x_1) - \varphi_{n_i}(x_1)| > \frac{1}{2p_0}$$

for large i, which contradicts the fact $x_1 \in I \subset E_{2p_0 q_0}$.

Consequently the set E is of the first category in R. The complement of R, $(0,1) - \mathrm{R}$ being enumerable, the set E is of the second category in $(0,1)$, q. e. d.

References

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