NOTES ON FOURIER ANALYSIS (XXXVII): ON THE CONVERGENCE FACTOR OF THE FOURIER SERIES AT A POINT*)

By

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1. Let $f(x)$ be an L-integrable function, and denote its Fourier series by

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x). \tag{1}$$

Hardy[1] has proved the following theorem.

(1. 1) *if*

$$\int_0^t |\varphi(u)|\, du = o(t) \tag{2}$$

then the series $\Sigma A_n(x)/\log n$ *converges at the point* x, *where*

$$\varphi(t) = \frac{1}{2}\left\{ f(x+t) + f(x-t) - 2f(x)\right\}.$$

On the other hand Wang[2] has proved the following theorems.

(1. 2) *If*

$$\int_0^t \varphi(u)\, du = o(t) \tag{3}$$

then the series $\Sigma A_n(x)/n^{1/2}$ *converges at the point* x.

(1. 3) *Conversely if for* $0 < \rho < 1$ *the series*

$$\sum_{n=2}^{\infty} A_n(x)/n^{\rho}$$

converges, then

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1) G.H.Hardy, Proc. London Math. Soc., **13** (1912).

2) F.T.Wang, Tōhoku Science Report, **24** (1935).

$$\varphi_2(t) = \int_0^t du \int_0^u \varphi(v)\, dv = O(t^{2-\rho}).$$

The object of this paper is to prove the following theorems concerning (1. 2) and (1. 3).

THEOREM 1. *There exists a function such that*

$$\int_0^t \varphi(u)\, u^{-(1+r)}\, du \tag{4}$$

exists by the Cauchy sense, but $\Sigma A_n(x)/n^{\delta}$ *is not convergent, where* $r > 0$, $0 < \delta < 1/(2+r)$.

THEOREM 2. *If for any* $r \geqq 0$

$$\int_0^t \varphi(u)\, du = O(t^{1+r}), \tag{5}$$

then the series $\Sigma A_n(x)/n^{1/(2+r)}$ *converges.*

THEOREM 3. *There exists a function* $f(x)$ *such that*

$$\sum_{n=1}^{\infty} A_n(x)/n^{\rho} \tag{6}$$

converges, and

$$\varphi_2(t) \neq O(t^{2-\rho'}) \tag{7}$$

where $0 \leqq \rho' < \rho$.

2. In Theorem 2 if $r = 0$ then we have Wang's result (1. 2).

LEMMA. *There exists an even function* $\varphi(x)$ *such that for any* δ, $(2+r)^{-1} > \delta > 0$, $\varlimsup_{n\to\infty} s_n(0, \varphi)/n^{\delta} = \infty$, *and*

$$\int_0^t \varphi(u)/u^{(1+r)}\, du$$

converges by the Cauchy sense, where $r > 0$.

PROOF. Let $\{p_k\}$, $\{q_k\}$ and $\{\mu_k\}$ be three increasing sequences and

$$p_0 = q_0 = 1, \qquad p_k = q_k\, \mu_k.$$

Then the even function $\varphi(t)$ is defined by

$$\varphi(t) = c_k \sin p_k t,$$

if t is a point of the interval $J_k = (\pi/q_k, \pi/q_{k-1})$, where $\{c_k\}$ is some positive sequence determined later.

1°. *The condition for which $\varphi(t)$ is integrable.*

$$\int_0^\pi |\varphi(t)|\, dt \leqq \sum_{k=1}^\infty c_k \int_{J_k} |\sin p_k t|\, dt \leqq \pi \sum_{k=1}^\infty c_k/q_{k-1}.$$

Hence if this last series is convergent, then $\varphi(t)$ is integrable.

2°. *The condition for which the condition* (5) *is satisfied.* Let $\pi/q_l \leqq \varepsilon < \pi/q_{l-1}$, $\pi/q_k \leqq \delta < \pi/q_{k-1}$ and $k < l$.

$$\left|\int_\varepsilon^\delta \varphi(t)\, t^{-(1+r)}\, dt\right| \leqq \left|\int_\varepsilon^{\pi/q_{l-1}}\right| + \sum_{i=l-1}^{k+1} \left|\int_{\pi/q_i}^{\pi/q_{i-1}}\right| + \left|\int_{\pi/q_k}^{\delta}\right|$$

$$\leqq \pi^{-(r+1)} \left(c_l q_l^{r+1} p_l^{-1} + \sum_{i=k+1}^{l-1} c_i q_i^{r+1} p_i^{-1} + c_k q_k^{r+1} p_k^{-1}\right) = \pi^{-(r+1)} \sum_{i=k}^{l} c_i\, q_i^{r+1} p_i^{-1}.$$

Hence if this last series is $o(1)$ as $k \to \infty$, then the condition (4) is satisfied.

3°. *The condition for which*

$$\lim_{n\to\infty} s_n(0, \varphi)/n^\delta = \infty.$$

If we consider especially the sequence $\{s_{p_k}(0, \varphi)/p_k^\delta\}$,

$$\int_0^\pi \varphi(t)(\sin p_k t/t)\, dt = \left(\int_0^{\pi/q_k} + \int_{\pi/q_k}^{\pi/q_{k-1}} + \int_{\pi/q_{k-1}}^{\pi}\right) \equiv S_1 + S_2 + S_3,$$

say.

$$|S_1| = \left|\sum_{i=k+1}^\infty \frac{c_i}{2} \int_{\pi/q_i}^{\pi/q_{i-1}} \left[\left\{\cos(p_i - p_k)t - \cos(p_i + p_k)t\right\}\Big/ t\right] dt\right|$$

$$\leqq \frac{1}{2} \sum_{i=k+1}^\infty c_i\, q_i\, 2p_i\, (p_i^2 - p_k^2)^{-1} \leqq \sum_{i=k+1}^\infty c_i\, q_i\, p_i^{-1} = \sum_{i=k+1}^\infty c_i/\mu_i.$$

$$S_2 = \frac{c_k}{2} \int_{\pi/q_k}^{\pi/q_{k-1}} \left[(1 - \cos 2p_k t)/t\right] dt$$

$$= \frac{c_k}{2} \log(q_k/q_{k-1}) - \frac{c_k}{2} \int_{\pi/q_k}^{\pi/q_{k-1}} \left[\cos 2p_k t/t\right] dt.$$

Hence

$$S_2 \geqq \frac{c_k}{2} \log (q_k/q_{k-1}) - c_k/\mu_k.$$

By the similar way as $|S_1|$,

$$|S_3| \leqq \sum_{i=1}^{k-1} c_i/\mu_i.$$

Consequently

$$\left| \int_0^\pi \varphi(t) \sin p_k t/t \, dt \right| \geqq S_2 - |S_1| - |S_3|$$

$$\geqq \frac{c_k}{2} \log (q_k/q_{k-1}) - \sum_{n=1}^{\infty} c_n/\mu_n.$$

That is, it is sufficient to prove that

(a) $$\sum_{k=1}^{\infty} c_k/q_{k-1} < \infty,$$

(b) $$c_k p_k^{-\delta} \log q_k/q_{k-1} \to \infty \qquad (0 < \delta < 1/(2+r)),$$

(c) $$\sum_{k=1}^{\infty} c_k q_k^r/\mu_k < \infty.$$

If we put

$$p_k = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k+1), \qquad \mu_k = p_k^{(r+\delta)/(1+r)} \cdot k,$$

$$q_k = p_k^{(1-\delta)/(1+r)} k^{-1}, \qquad \text{and } c_k = p_k^\delta / \sqrt{\log (2k+1)},$$

then the left hand side of (b) is

$$\frac{p_k^\delta}{\sqrt{\log (2k+1)}} \frac{1}{p_k^\delta} \left\{ \frac{1-\delta}{r+1} \log (2k+1) + \log \frac{k-1}{k} \right\} \to \infty, \ (k \to \infty).$$

Thus the condition (b) is satisfied.

Now, we have

$$\sum_{k=1}^{\infty} c_k q_k^r / \mu_k = \sum_{k=1}^{\infty} \frac{p_k^\delta}{\sqrt{\log (2k+1)}} \cdot \frac{p_k^{r(1-\delta)/(r+1)} k^{-r}}{p_k^{(r+\delta)/(r+1)} k}$$

$$= \sum_{k=1}^{\infty} (\log (2k+1))^{-1/2} k^{-(r+1)} < \infty.$$

This is the condition (c). Lastly the left hand side of (a)

$$\sum_{k=1}^{\infty} p_k^{\delta} (\log (2k+1))^{-1/2} (k-1)\, p_{k-1}^{-(1-\delta)/(r+1)}$$

$$\leqq \sum_{k=1}^{\infty} (\log (2k+1))^{-1/2} k^{(2+r-\delta)/(r+1)} p_k^{-\{1-(r+2)\delta\}/(r+1)}.$$

Since

$$\frac{1-(r+2)\delta}{r+1} > \frac{1}{r+1} - \frac{r+2}{r+1} \cdot \frac{1}{r+2} = 0,$$

and the inequality

$$\frac{2+r-\delta}{r+1} < x \frac{1-(r+2)\delta}{r+1}$$

has a solution, that is

$$x > \frac{2+r-\delta}{1-(r+2)\delta} > \left(2+r-\frac{1}{r+2}\right) > 1+r > 0,$$

the condition (a) is satisfied, and thus the Lemma is proved.

3. We prove Theorem 1.

If $\varphi(t)$ satisfies (4) we can easily prove $\sigma_n = o(1)$,

$$\sum_{k=1}^{n} A_k(0)/k^{\delta} = \sum_{k=1}^{n-2} (k+1)\sigma_k(0)\,\Delta^2(1/k^{\delta}) + n\,\sigma_{n-1}(0)\,\Delta(1/(n-1)^{\delta})$$

$$-\sigma_0(0) + s_n(0)/n^{\delta} - s_0(0)$$

$$\geqq s_n(0)/n^{\delta} - O(1).$$

Since there exists a function satisfying (4) such as

$$\overline{\lim_{n\to\infty}}\, s_n(0)/n^{\delta} = \infty$$

for $0 < \delta < 1/(r+2)$, our theorem is proved.

Proof of Theorem 2 is similar as that of Wang's.

4. We will pass to the proof of Theorem 3. Without loss of generality we can suppose that $f(t)$ is even and $x = 0$. As the theorem is proved by the same way as the following theorem, we prove only

THEOREM 3′. *There exists an even and integrable function $f(x)$ such that the series*

$$\sum_{k=1}^{\infty} A_k(0)/k^\rho.$$

is convergent for $0 < \rho < 1$, *and*

$$f_2(t) = \int_0^t du \int_0^u f(v)\,dv \neq O(t^2).$$

Let $f(x)$ be an even, periodic and integrable function with $a_0 = 0$. Generally

$$\sum_{k=1}^{n} A_k(0)/k^\rho = \frac{2}{\pi}\int_0^\pi f(t)\,(\sum_{k=1}^{n} \cos kt/k^\rho)\,dt$$

$$= \frac{2}{\pi}\int_0^\pi f(t)\Big(\sum_{k=1}^{n-1} D_k(t)\,\Delta(1/k^\rho)\Big)\,dt + \frac{2}{\pi}n^{-\rho}\int_0^\pi f(t)\,D_n(t)\,dt \equiv P + Q,$$

say. If $s_n(0) = O(n^\varepsilon)$ for $\rho > \varepsilon > 0$, then $Q = o(1)$, and

$$\sum_{k=1}^{\infty} |s_k(0)\,\Delta(1/k^\rho)| \leq \sum_{k=1}^{\infty} k^{-\rho-1+\varepsilon} < \infty.$$

Thus the series $\Sigma s_k(0)\,\Delta(1/k^\rho)$ is absolutely convergent, and

$$P = \sum_{k=1}^{n-1} s_k(0)\,\Delta(1/k^\rho) = O(1) \qquad (n \to \infty).$$

Consequently it is sufficient to prove Theorem 3′ that there exists an even function $f(x)$ such that,

$$(8) \qquad f_2(t) \neq O(t^2)$$

and for any $\varepsilon > 0$

$$(9) \qquad s_n(0, f) = o(n^\varepsilon).$$

Let $\{p_k\}$ and $\{q_k\}$ be two increasing sequences such as $p_k > q_k$, $\{\mu_k\}$ be decreasing tending to 1, and if $t \in J_k = (\pi/q_k,\ \pi\mu_k/q_k)$

$$f(t) = 2c_k \sin p_k t + 4c_k p_k t \cos p_k t - c_k p_k^2 t^2 \sin p_k t, \qquad t \in J_k,$$
$$= 0, \qquad \text{elsewhere.}$$

1° *The condition that* $f(x) \in L$. *If* $c_k \geq 0$, *then*

$$\int_0^\pi |f(t)|\,dt \leq \sum_{k=1}^{\infty} c_k \int_{\pi/q_k}^{\pi\mu_k/q_k} \Big\{2\,|\sin p_k t| + 4\,|p_k t \cos p_k t| + |p_k^2 t^2 \sin p_k t|\Big\}dt$$

$$\leq \sum_{k=1}^{\infty} c_k \Big\{2\pi q_k^{-1}(\mu_k - 1) + 2p_k(\pi q_k^{-1})^2(\mu_k^2 - 1) + p_k^2(\pi q_k^{-1})^3\,\mu_k^2(\mu_k - 1)\Big\}$$

$$\leqq \text{const.} \sum_{k=1}^{\infty} c_k (\mu_k - 1) p_k^2 q_k^{-3}.$$

If the last series converges then $f(x) \in L$.

2° *The condition for which* (8) *is satisfied.*

We consider the integral of $f(x)$ in J_k.

$$\int_{J_k} f(u)\, du = \Big[2 c_k t \sin p_k t + c_k p_k t^2 \cos p_k t \Big]_{\pi/q_k}^{\pi\mu_k/q_k}$$

$$= c_k p_k \{(\pi \mu_k/q_k)^2 - (\pi/q_k)^2\} = c_k p_k (\mu_k^2 - 1) \pi^2/q_k^2,$$

where we suppose that q_k is a common divisor of p_k and $\mu_k p_k$, and

(10) $$p_k/q_k = \text{even}, \quad p_k \mu_k/q_k = \text{even}.$$

Consequently if $t \in J_k$, then

$$f_1(t) \equiv \int_0^t f(u)\, du = \sum_{i=k+1}^{\infty} \int_{J_i} f(u)\, du + \int_{\pi q_k}^{t} f(u)\, du$$

$$= \sum_{i=k+1}^{\infty} c_i p_i (\mu_i^2 - 1) \pi^2/q_i^2 - c_k p_k \pi^2/q_k^2 + (2 c_k t \sin p_k t + c_k p_k t^2 \cos p_k t)$$

$$\equiv A_k - B_k + (2c_k t \sin p_k t + c_k p_k t^2 \cos p_k t), \text{ say.}$$

$$\int_{J_i} f_1(u)\, du = (A_i - B_i)(\mu_i - 1) \pi/q_i.$$

Hence if $t \in J_k$, then

$$f_2(t) = \int_0^t f_1(u)\, du = \sum_{i=k+1}^{\infty} \int_{J_i} f_1(u)\, du + \int_{\pi/q_k}^{t} f_1(u)\, du$$

$$= \sum_{i=k+1}^{\infty} (A_i - B_i)(\mu_i - 1) \pi q_i^{-1} + (t - \pi q_k^{-1})(A_k - B_k) + c_k t^2 \sin p_k t.$$

$$t^{-2} f_2(t) = \pi t^{-2} \sum_{i=k+1}^{\infty} (\mu_i - 1)(A_i - B_i) q_i^{-1} + t^{-2}(t - \pi q_k^{-1})(A_k - B_k) + c_k \sin p_k t.$$

$$|A_k - B_k| \leqq \sum_{i=k+1}^{\infty} c_i p_i (\mu_i^2 - 1)/q_i^2 + c_k p_k/q_k^2.$$

(*) $$t^{-2}(t - \pi q_k^{-1}) |A_k - B_k|$$

$$\leqq (\pi q_k^{-1})^{-2} (\pi q_k^{-1}) (\mu_k - 1) \Big\{ \sum_{i=k+1}^{\infty} c_i p_i q_i^{-2} (\mu_i - 1) + c_k p_k q_k^{-2} \Big\}$$

$$\leqq c_k p_k (\mu_k - 1)/q_k + \pi^{-1} q_k (\mu_k - 1) \sum_{i=k+1}^{\infty} c_i p_i (\mu_i - 1)/q_i^2.$$

$$(**)\quad \left| t^{-2} \sum_{i=k+1}^{\infty} (\mu_i - 1)(A_i - B_i)/q_i \right|$$

$$\leqq (\pi q_k^{-1})^{-2} \sum_{i=k+1}^{\infty} (\mu_i - 1)\Big(\sum_{j=i+1}^{\infty} c_j\, p_j\, q_j^{-2}(\mu_j^2 - 1) + c_i\, p_i\, q_i^{-2} \Big) q_i^{-1}$$

$$= \pi^{-2} q_k^2 \sum_{i=k+1}^{\infty} c_i\, p_i\, q_i^{-3}(\mu_i - 1) + \pi^{-2} q_k^2 \sum_{i=k+1}^{\infty} (\mu_i - 1)\, q_i^{-1} \sum_{j=i+1}^{\infty} c_j\, p_j\, q_j^{-2}(\mu_j^2 - 1)$$

Here if we put

$$p_i = q_i\,(2i)^2,\ q_i = 2^{i^2},\ \mu_i = 1 + i^{-1},\ \text{and}\ c_i = O(1),$$

$$(*) = - t^{-2}(t - \pi\, q_k^{-1})\, c_k\, p_k\, q_k^{-2} + t^{-2}(t - \pi\, q_k^{-1}) \sum_{i=k+1}^{\infty} c_i\, p_i\, q_i^{-2}(\mu_i^2 - 1)$$

$$= - t^{-2}(t - \pi\, q_k^{-1})\, c_k\, p_k\, q_k^{-2} + q_k\,(\mu_k - 1)\,\pi \sum_{i=k+1}^{\infty} c_i\, p_i\, q_i^{-2}(\mu_i^2 - 1)$$

$$= - t^{-2}(t - \pi\, q_k^{-1})\, c_k\, p_k\, q_k^{-2} + O\Big(2^{k^2}\, k^{-1} \sum_{i=k+1}^{\infty} i^2\, 2^{-i^2}\, i^{-1}\Big)$$

$$= - t^{-2}(t - \pi\, q_k^{-1})\, c_k\, p_k\, q_k^{-2} + O\big(2^{k^2-(k+1)^2}\, k^{-1}\big)$$

$$= - t^{-2}(t - \pi\, q_k^{-1})\, c_k\, p_k\, q_k^{-2} + o(1).$$

$$(**) \leqq O\Big(2^{2k^2} \sum_{i=k+1}^{\infty} i\, 2^{-2i^2}\Big) + O\Big(2^{2k^2} \sum_{i=k+1}^{\infty} 2^{-i^2} i^{-1} \sum_{j=i+1}^{\infty} j\, 2^{-j^2} \Big)$$

$$= O\big(2^{2k^2}\, 2^{-2(k+1)^2}\big) + O\Big(2^{2k^2} \sum_{i=k+1}^{\infty} 2^{-i^2}\, 2^{-(i+1)^2}\, i^{-1} \Big) = o(1).$$

Consequently

$$\lim_{t\to 0} t^{-2} f_2(t) = \lim_{t\to 0} \Big\{ o(1) - t^{-2}(t - \pi\, q_k)\, c_k\, p_k\, q_k^{-2} + c_k \sin p_k\, t \Big\}$$

$$= O(1) - \lim_{t\to 0} O\big(t^{-2}(t - \pi\, q_k^{-1}) \cdot k^2\, 2^{-k^2}\big) = -\infty.$$

3° *The condition by which* (9) *is satisfied.*

We must prove that

$$(11)\qquad n^{-\varepsilon} \int_0^{\pi} [f(t) \sin nt/t]\, dt = o(1).$$

If $n = p_k$, then

$$\int_0^{\pi} [f(t) \sin p_k t/t]\, dt = \Big(\sum_{i=k+1}^{\infty} \int_{J_i} + \sum_{i=1}^{k-1} \int_{J_i} \Big) + \int_{J_k} \equiv S_1 + S_2 + S_3,$$

say.

$$|S_1| \leqq \sum_{i=k+1}^{\infty} c_i \int_{J_i} \{p_i p_k t + p_i p_k t + p_i^3 p_k t^3\} dt$$

$$\leqq \sum_{i=k+1}^{\infty} c_i p_k \{2p_i (\mu_i - 1) \mu_i (\pi q_i^{-1})^2 + p_i^3 \mu_i^3 (\mu_i - 1) (\pi q_i^{-1})^4\}$$

$$\leqq p_k \sum_{i=k+1}^{\infty} c_i (\mu_i - 1) p_i^3 q_i^{-4} \leqq O(k^2 q_k \sum_{i=k+1}^{\infty} i^5 q_i^{-1})$$

$$= O(k^2 q_k \sum_{i=k+1}^{\infty} i^5 2^{-i^2}) = O(k^6 2^{k^2-(k+1)^2}) = o(1).$$

$$|S_2| \leqq \sum_{i=1}^{k} c_i p_k^{-1} \{q_i \pi^{-1} + 2p_i + c_i p_i^2 \pi \mu_i q_i^{-1}\}$$

$$= O\left(p_k^{-1} \sum_{i=1}^{k} c_i p_i^3 \mu_i q_i^{-1} \right) = O\left(p_k^{-1} \sum_{i=1}^{k} i^4 q_i \right) = k + o(1).$$

$$S_3 = \int_{J_k} c_k (1 - \cos p_k t)/t \, dt + \int_{J_k} 2c_k p_k \sin 2p_k t dt - \int_{J_k} c_k p_k^2 t \sin^2 p_k t \, dt$$

$$= c_k \log \mu_k + O(c_k q_k \mu_k^{-1}) + O(c_k) + O(c_k p_k^2) \int_{J_k} (t + t \cos 2p_k t) dt$$

$$= c_k \log \mu_k + O(1) + O(k^3) + O(k^2) = O(k^3).$$

Hence

$$\int_0^{\pi} [f(t) \sin p_k t/t] \, dt = O(k^3),$$

and

$$S_{p_k}(0)/p_k^{\varepsilon} = O(k^3/k^{2\varepsilon} \cdot 2^{-\varepsilon k^2}) = o(1).$$

If $n \neq p_k$ then for some k, $p_k < n < p_{k+1}$.

$$\int_0^{\pi} [f(t) \sin nt/t] \, dt = \sum_{i=k+2}^{\infty} \int_{J_i} + \int_{J_{k+1}} + \int_{J_k} + \sum_{i=1}^{k-1} \int_{J_i},$$

and by the similar calculation we have

$$s_n = o(n^{\varepsilon}).$$

Thus Theorem 3′ is proved.

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