ON A CERTAIN MOTION IN THE EUCLIDEAN SPACE*)

BY

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In the n-dimensional Euclidean space we consider a moving simplex of dimension r. Let the vertices of the simplex at an instant t $(t_1 \leqq t \leqq t_2)$ be $x_0, x_1, \cdots, x_r$, whose coordinates are assumed to be functions of t. Any point of the simplex can be expressed as $x = \sum_{i=0}^{r} \lambda_i x_i$, where $\sum_{i=0}^{r} \lambda_i = 1$, $\lambda_i \geqq 0$ and at inner points all λ_i's > 0. Let the arc lengths described by these points be L_0, $L_1, \cdots, L_r$ and L. Then

$$L = \int_{t_1}^{t_2} (\dot{x}, \dot{x})^{1/2}\, dt = \int_{t_1}^{t_2} \left(\sum_{i=0}^{r} \lambda_i \dot{x}_i, \sum_{i=0}^{r} \lambda_i \dot{x}_i \right)^{1/2} dt, \tag{1}$$

where dots mean the differentiation with respect to t. Now there exists the relation

$$L \leqq \sum_{i=0}^{r} \lambda_i L_i. \tag{2}$$

The proof can easily be accomplished on account of the convexity of the function $(\dot{x}, \dot{x})^{1/2}$ in the velocity vector $\dot{x}$, which is the integrand of (1). The equality holds good for inner points when and only when the velocities of all the vertices $x_0, x_1, \cdots, x_r$ at any instant are positive multiples of the same vector function b, that is

$$\dot{x}_i = \beta_i\, b \qquad (i = 0, 1, \cdots, r), \tag{3}$$

where β_i are scalar functions. In the present paper we shall compute this case and find that the paths of moving points in such a motion are generally orthogonal trajectories of the osculating planes of a certain curve. We remark that, if the above equality holds for an inner point of the simplex, the same is true for any other point of the simplex and also for the vertices and points of every inner simplex.

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2. First, if all β_i's are equal, our motion is *a translation at every instant* (type (A)).

Next we see by virtue of (3) that, if some β_i is not equal to some β_j,

$$(b, x_i - x_j) = \frac{1}{\beta_i - \beta_j}(\dot{x}_i - \dot{x}_j, x_i - x_j) = 0, \tag{4}$$

because the distance between x_i and x_j is independent of t. Now consider a variable point of the carrier r-plane of the simplex

$$X = \sum_{i=0} \alpha_i x_i,$$

where α_i's are functions of t and

$$\sum_{i=1}^{r} \alpha_i = 1. \tag{5}$$

Then we have by differentiation $\dot{X} = \Sigma \dot{\alpha}_i x_i + (\Sigma \alpha_i \beta_i) b$. (For the present we assume that the summation ranges from 0 to r). If we take α_i such that $(\alpha, \beta) = \Sigma \alpha_i \beta_i = 0$, we have

$$\dot{X} = \Sigma \dot{\alpha}_i x_i$$

and then $\ddot{X} = \Sigma \ddot{\alpha}_i x_i + \Sigma \dot{\alpha}_i \dot{x}_i = \Sigma \ddot{\alpha}_i x_i + (\dot{\alpha}, \beta) b$. In this way if we can take α_i such that

$$(\alpha, \beta) = 0, (\dot{\alpha}, \beta) = 0, \ldots, (\overset{(r-1)}{\alpha}, \beta) = 0, \tag{6}$$

we have

$$X = \Sigma \alpha_i x_i, \quad \dot{X} = \Sigma \dot{\alpha}_i x_i, \ldots, \overset{(r)}{X} = \Sigma \overset{(r)}{\alpha}_i x_i. \tag{7}$$

By a slight calculation we see that (6) is equivalent to the following equations

$$(\alpha, \beta) = 0, (\alpha, \dot{\beta}) = 0, \ldots, (\alpha, \overset{(r-1)}{\beta}) = 0. \tag{8}$$

Now we go into the determination of α_i's which satisfy (5) and (8). We put

$$\Delta \equiv \begin{vmatrix} 1 & \cdots\cdots & 1 \\ \beta_0 & \cdots\cdots & \beta_r \\ \cdots & \cdots\cdots & \cdots \\ \overset{(r-1)}{\beta_0} & \cdots\cdots & \overset{(r-1)}{\beta_r} \end{vmatrix} \tag{9}$$

CASE I. $\Delta \neq 0$.

By hypothesis $\beta_i \neq \beta_j$ for every pair of indices i, j such that $i \neq j$, hence (4) holds. As $\Delta \neq 0$, x_i 's are determined by (8). From $X = \Sigma\alpha_i x_i$ we get by virtue of (7) and (5)

$$= \sum_{i=0}^{r} \overset{(j)}{\alpha}_i x_i = \sum_{k=1}^{r} \overset{(j)}{\alpha}_k (x_k - x_0). \tag{10}$$

Further we put

$$D \equiv \begin{vmatrix} \dot{\alpha}_1 \cdots\cdots\cdots \dot{\alpha}_r \\ \cdots\cdots\cdots\cdots \\ \overset{(r)}{\alpha}_1 \cdots\cdots\cdots \overset{(r)}{\alpha}_r \end{vmatrix},$$

and divide the case I into the following two subcases:

CASE Ia. $D \neq 0$.

In this case $\overset{(i)}{X}$'s are linearly independent by virtue of (10), and the simplex in question is on a plane spanned by vectors $\overset{(i)}{X}$ with X as their origin. So if we put

$$x = \sum_{i=1}^{r} \gamma_i \overset{(i)}{X} + X, \tag{11}$$

x lies on this r-plane. Taking γ_i suitably, let us show conversely that the motion in question, namely the one for which the equality holds in (2), is really possible. From (11) we get

$$\dot{x} = (1 + \dot{\gamma}_1)\dot{X} + (\gamma_1 + \dot{\gamma}_2)\ddot{X} + \ldots\ldots + (\gamma_{r-1} + \dot{\gamma}_r)\overset{(r)}{X} + \gamma_r \overset{(r+1)}{X}.$$

Now let x be a fixed point of our simplex. On account of (3), (4) and (10) $\dot{x}$ should be perpendicular to $\dot{X}, \ldots, \overset{(r)}{X}$, hence we must have from the last equation the following relation:

$$(1 + \dot{\gamma}_1)\mu_{i1} + (\gamma_1 + \dot{\gamma}_2)\mu_{i2} + \ldots + (\gamma_{r-1} + \dot{\gamma})\mu_{ir} = -\gamma_r \mu_{ir+1}$$
$$(i = 1, 2, \ldots\ldots, r), \tag{12}$$

where we have put $(\overset{(i)}{X}, \overset{(j)}{X}) = \mu_{ij}$. As the determinant $|\mu_{ij}| \neq 0$ $(i,j = 1, 2, \ldots, r)$ by virtue of the linear independence of $\dot{X}, \ldots, \overset{(r)}{X}$, we get from (12)

$$1 + \dot{\gamma}_1 = \nu_1 \gamma_r, \ \gamma_1 + \dot{\gamma}_2 = \nu_2 \gamma_r, \ \ldots\ldots, \ \gamma_{r-1} + \dot{\gamma}_r = \nu_r \gamma_r, \tag{13}$$

where $\nu_1, \ldots, \nu_r$ are functions of μ_{ij} $(i = 1, \ldots, r; j = 1, \ldots, r+1)$. When we write the general solution of the differential equation (13) in the vector form $\gamma = (\gamma, \ldots, \gamma_r)$, we obtain

$$\gamma = \sum_{i=1}^{r} \lambda_i \gamma^i + \gamma^0, \tag{14}$$

where λ_i's are constants and γ^0 is a special solution. As λ_i's vary, the point (11) moves on the osculating r-plane. By calculation we have

$$\dot{x} = \frac{(-1)^r}{|\mu_{ij}|} \begin{vmatrix} \dot{X} & \ddot{X} & \cdots\cdots & \overset{(r+1)}{X} \\ \mu_{11} & \mu_{12} & \cdots\cdots & \mu_{1r+1} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \mu_{r1} & \mu_{r2} & \cdots\cdots & \mu_{rr+} \end{vmatrix} \gamma_r.$$

Here $|\mu_{ij}|$ is a determinant and the right side is a linear combination of vectors $\dot{X}, \ldots, \overset{(r+1)}{X}$. From the last equation we get $(\dot{x}, \dot{x})^{1/2} = \theta\,|\gamma_r|$, where θ is a function which is independent of λ_i's. So far as γ^r *is of the same sign,* $(\dot{x}, \dot{x})^{1/2}$ is linear in λ_i. So if we denote the points corresponding to γ^0, $\gamma^0 + \gamma^1, \ldots, \gamma^0 + \gamma^r$ and γ by $x_0, x_1, \ldots, x_r$ and x respectively, we have by (13)

$$x = \sum_{i=1}^{r} \lambda_i x_i + \left(1 - \sum_{i=1}^{r} \lambda_i\right) x_0 = \sum_{i=0}^{r} \lambda_i x_i \qquad \left(\lambda_0 = 1 - \sum_{i=1}^{r} \lambda_i\right).$$

Hence if γ^r corresponding to $x_0, x_1, \ldots, x_r$ and x are of the same sign, we have $L = \sum_{i=0}^{r} \lambda_i L_i$. Thus the case Ia is settled. Our motion is the *one whose paths are the orthogonal trajectories of the osculating r-planes of a curve* (type (B)).

3. CASE Ib. $D = 0$.

Before proceeding further, we state here the lemma which will be used repeatedly later.

LEMMA. *Let* $z_0, z_1, \ldots, z_n$ *be functions of* t, *and* $a_0, a_1, \ldots, a_n$ *be constants not all zero. If*

$$\begin{vmatrix} a_0 & a_1 & \cdots\cdots & a_n \\ z_0 & z_1 & \cdots\cdots & z_n \\ \dot{z}_0 & \dot{z}_1 & \cdots\cdots & \dot{z}_n \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \overset{(n-1)}{z_0} & \overset{(n-1)}{z_1} & \cdots\cdots & \overset{(n-1)}{z_n} \end{vmatrix} = 0,$$

then there exist constants c_i's not all zero such that $\sum_{i=0}^{n} c_i z_i = 0$. Especially if $a_0 = a_1 = \dots = a_n$, we have in addition the relation $\sum_{i=0}^{n} c_i = 0$.

PROOF. As at least one of a_i's is not zero, we assume $a_0 \neq 0$. Now we obtain from the given equality

$$\begin{vmatrix} a_0 & 0 & \dots & 0 \\ z_0 & z_1 - \frac{a_1}{a_0} z_0 & \dots & z_n - \frac{a_n}{a_0} z_0 \\ \dot{z}_0 & \dot{z}_1 - \frac{a_1}{a_0}\dot{z}_0 & \dots & \dot{z}_n - \frac{a_n}{a_0}\dot{z}_0 \\ \dots & \dots & \dots & \dots \\ z_0^{(n-1)} & z_1^{(n-1)} - \frac{a_1}{a_0} z_0^{(n-1)} & \dots & z_n^{(n-1)} - \frac{a_n}{a_0} z_0^{(n-1)} \end{vmatrix} = a_0 \begin{vmatrix} z_1 - \frac{a_1}{a_0} z_0 & \dots & z_n - \frac{a_n}{a_0} z_0 \\ \dot{z}_1 - \frac{a_1}{a_0}\dot{z}_0 & \dots & \dot{z}_n - \frac{a_n}{a_0}\dot{z}_0 \\ \dots & \dots & \dots \\ z_1^{(n-1)} - \frac{a_1}{a_0} z_0^{(n-1)} & \dots & z_n^{(n-1)} - \frac{a_n}{a_0} z_0^{(n-1)} \end{vmatrix} = 0.$$

Hence by Wronski's theorem there exist constants c_i's not all zero such that

$$\sum_{i=1}^{n} c_i \left(z_i - \frac{a_i}{a_0} z_0 \right) = 0.$$

Putting $c_0 = -\sum_{i=1}^{n} c_i a_i / a_0$ we obtain $\sum_{i=0}^{n} c_i z_i = 0$. If $a_0, a_1, \dots, a_n$ are all equal, we have $\sum_{i=0}^{n} c_i = 0$.

Now we proceed to investigate the case Ib. Owing to $D = 0$, there exist constants μ_i not all zero such that $\sum_{i=1}^{r} \mu_i \dot{\alpha}_i = 0$. So we have $\sum \mu_i \alpha_i + \mu = 0$ (μ constant). From (5), (8) and the last equation we eliminate α_i's, then we get after a slight calculation

$$\begin{vmatrix} \mu_1 & \mu_2 & \dots & \mu_r & \mu \\ \dot{\beta}_1 - \dot{\beta}_0 & \dot{\beta}_2 - \dot{\beta}_0 & \dots & \dot{\beta}_r - \dot{\beta}_0 & \dot{\beta}_0 \\ \dots & \dots & \dots & \dots & \dots \\ \beta_1^{(r-1)} - \beta_0^{(r-1)} & \beta_2^{(r-1)} - \beta_0^{(r-1)} & \dots & \beta_r^{(r-1)} - \beta_0^{(r-1)} & \beta_0^{(r-1)} \end{vmatrix} = 0.$$

By virtue of the lemma $\beta_1 - \beta_0, \beta_2 - \beta_0, \dots, \beta_r - \beta_0$ and β_0 are linearly dependent with constant coefficients, and consequently $\sum_{i=0}^{r} \nu_i \beta_i = 0$ (ν_i constants, not all zero).

Here it is evident that $\sum_{i=0}^{r} \nu_i \neq 0$ by virtue of the assumption $\Delta \neq 0$. If we take ν_i such that $\sum_{i=0}^{r} \nu_i = 1$, then $x = \sum_{i=0}^{r} \nu_i x_i$ is a fixed point in the space as well as in the plane of the simplex, because by (3) we have $x = \sum_{i=0}^{r} \nu_i x_i = \sum_{i=0}^{r} \nu_i \beta_i b = 0$. Let us take this point as the origin and denote the r points, which are fixed to the simplex in question and are forming together with the origin the vertices of a certain simplex of dimension r, by $y_1, \ldots, y_r$. As y_i's are linear combinations of $x_0, x_1, \ldots, x_r$ with constant coefficients, we get by (3)

$$\dot{y}_i = \delta_i b, \tag{15}$$

where δ_i are real functions of t. Now we put

$$X = \sum_{i=1}^{r} \alpha_i y_i.$$

If we can find $\alpha_i (i = 1, \ldots, r)$ such that

$$(\alpha, \delta) = 0, \ (\alpha, \dot{\delta}) = 0, \ \ldots, \ (\alpha, \overset{(r-2)}{\delta}) = 0 \tag{16}$$

where $(\alpha, \delta) = \sum_{i=1}^{r} \alpha_i \delta_i$ etc., we will have

$$X = \sum_{i=1}^{r} \alpha_i y_i, \ \ldots, \ \overset{(r-1)}{X} = \sum_{i=1}^{r} \overset{(r-1)}{\alpha_i} y_i, \tag{17}$$

because (16) is equivalent to the following relations:

$$(\alpha, \delta) = 0, \ (\dot{\alpha}, \delta) = 0, \ \ldots, \ (\overset{(r-2)}{\alpha}, \delta) = 0.$$

Now the rank of the matrix

$$\begin{pmatrix} \delta_1 & \delta_2 & \cdots\cdots\cdots\cdots & \delta_r \\ \dot{\delta}_1 & \dot{\delta}_2 & \cdots\cdots\cdots\cdots & \dot{\delta}_r \\ \cdots & \cdots & \cdots\cdots\cdots\cdots & \cdots \\ \overset{(r-2)}{\delta_1} & \overset{(r-2)}{\delta_2} & \cdots\cdots\cdots\cdots & \overset{(r-2)}{\delta_r} \end{pmatrix} \tag{18}$$

is $r-1$. If not, there would be constants ρ_i not all zero such that $\sum_{i=1}^{r-1} \rho_i \delta_i = 0$, so the point $\sum_{i=1}^{r-1} \rho_i y_i$ would be a fixed point and the direction from the origin to this point would be constant. On the other hand, by the lemma and (3), $\Delta = 0$ is equivalent to the existence of constants μ_i's not all zero such that $\sum_{i=1}^{r} \mu_i (\beta_i - \beta_0) = 0$, which shows us by virtue of (3) the existence of a fixed direction.

By the hypothesis $\Delta \neq 0$ the rank of (18) is $r-1$. Hence we can find $\alpha_1, \ldots, \alpha_r$ such that (16) holds.

Next we have

$$D \equiv \begin{vmatrix} \alpha_1 & \cdots\cdots\cdots & \alpha_r \\ \dot{\alpha}_1 & \cdots\cdots\cdots & \dot{\alpha}_r \\ \cdots & \cdots\cdots\cdots & \cdots \\ \overset{(r-1)}{\alpha_1} & \cdots\cdots\cdots & \overset{(r-1)}{\alpha_r} \end{vmatrix} \neq 0.$$

If not, there would be constants ρ_i not all zero such that $\sum_{i=1}^{r} \rho_i \alpha_i = 0$. From the last equation and (16) we get

$$\begin{vmatrix} \rho_1 & \cdots\cdots\cdots & \rho_r \\ \delta_1 & \cdots\cdots\cdots & \delta_r \\ \dot{\delta}_1 & \cdots\cdots\cdots & \dot{\delta}_r \\ \cdots & \cdots\cdots\cdots & \cdots \\ \overset{(r-2)}{\delta_1} & \cdots\cdots\cdots & \overset{(r-2)}{\delta_r} \end{vmatrix} = 0.$$

So by the lemma this would lead to $\Delta = 0$ as above stated which contradicts $\Delta \neq 0$, and hence $D \neq 0$.

By (17) and $D \neq 0$, $X, \ldots, \overset{(r-1)}{X}$ are linearly independent vectors. Now we can show as in Ia that the motion in question is really possible. We put $x = \sum_{i=0}^{r-1} \gamma_i \overset{(i)}{X}$ and proceed as before, and finally get the result that our motion is *the one whose paths are orthogonal trajectories of the r-planes which contain a fixed point and osculating $(r-1)$-planes of a curve*. (*type* (*C*)).

4. CASE II. $\Delta = 0$.

By virtue of the lemma we have in this case $\sum_{i=0}^{r} \mu_i \beta_i = 0$, $\sum_{i=0}^{r} \mu_i = 0$ (μ_i constants not all zero), and hence we get $\sum_{i=1}^{r} \mu_i (\beta_i - \beta_0) = 0$. Accordingly we find by virtue of (3) that $\sum_{i=0}^{r} \mu_i x_i = \sum_{i=1}^{r} \mu_i (x_i - x_0)$ is a constant direction. If the maximum number of the linearly independent ones among such vectors of constant direction is $r-k$, we can take them as $a_{k+1} = x_{k+1} - x_0, \ldots, a_r = x_r - x_0$. Now we consider the point

$$X = \sum_{i=0}^{r} \alpha_i x_i = x_0 + \sum_{i=1}^{k} \alpha_i (x_i - x_0) + \sum_{i=k+1}^{r} \alpha_i a_i.$$

Putting $x_0 = \beta_0 b, (x_i - x_0)^{\cdot} = \beta_i b \, (i = 1, \ldots, k)$ and taking α such that

$$(\alpha, \beta) = 0, \ (\dot{\alpha}, \beta) = 0, \ \ldots, \ (\overset{(k-1)}{\alpha}, \beta) = 0, \tag{19}$$

where $\alpha = (1, \alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_0, \beta_1, \ldots, \beta_k)$, we get

$$\dot{X} = \sum_{i=0}^{k} \dot{\alpha}_i (x_i - x_0) + \sum_{i=k+1}^{r} \dot{\alpha}_i a_i, \ \ldots, \ \overset{(k)}{X} = \sum_{i=1}^{k} \overset{(k)}{\alpha}_i (x_i - x_0) + \sum_{i=k+1}^{r} \overset{(k)}{\alpha}_i a_i \tag{20}$$

(19) is equivalent to the following relations:

$$(\alpha, \beta) = 0, \ (\alpha, \dot{\beta}) = 0, \ \ldots, \ (\alpha, \overset{(k-1)}{\beta}) = 0. \tag{21}$$

Such α exists. For otherwise we would have by virtue of (21)

$$\begin{vmatrix} \beta_1 & \beta_2 & \cdots & \beta_k \\ \dot{\beta}_1 & \dot{\beta}_2 & \cdots & \dot{\beta}_k \\ \cdots & \cdots & \cdots & \cdots \\ \overset{(k-1)}{\beta_1} & \overset{(k-1)}{\beta_2} & \cdots & \overset{(k-1)}{\beta_k} \end{vmatrix} = 0.$$

This would result in $\sum_{i=1}^{k} \mu_i \beta_i = 0$, where μ_i are constants not all zero. Accordingly one more direction $\sum_{i=1}^{k} \mu_i (x_i - x_0)$ would be constant, which contradicts the assumption on k.

Now let us put

$$D_1 \equiv \begin{vmatrix} \dot{\alpha}_1 & \cdots & \dot{\alpha}_k \\ \cdots & \cdots & \cdots \\ \overset{(k)}{\alpha_1} & \cdots & \overset{(k)}{\alpha_k} \end{vmatrix}$$

and divide the case II into the following two subcases.

5. Case IIa. $D_1 \neq 0$.

Then, by virtue of (20), $\dot{X}, \ldots, \overset{(k)}{X}$ are linearly independent. To show the existence of the motion in question we put

$$x = X + \sum_{i=1}^{k} \gamma_i \overset{(i)}{X} + \sum_{i=k+1}^{r} \gamma_i a_i$$

and take $\gamma = (\gamma_1, \ldots, \gamma_r)$ so that $\dot{x}$ is perpendicular to $\dot{X}, \ddot{X}, \ldots, \overset{(k)}{X}$ and $a_{k+1}, \ldots, a_r$. Proceeding in the same way as before we obtain the result that our motion is *the one whose paths are orthogonal trajectories of the r-planes which contain the osculating k-planes of a curve and are parallel to a fixed plane of dimension* $r - k$ (type (D)).

CASE IIb. $D_1 = 0$.

Then there exist constants $\mu_1, \ldots, \mu_k$ not all zero and a constant μ_0 such that

$$\mu_0 + \mu_1 \alpha_1 + \ldots\ldots + \mu_k \alpha_k = 0.$$

From the last equation and (21) we eliminate $\alpha_1, \ldots, \alpha_k$. Then we get, by virtue of the lemma, the following relation

$$\sum_{i=0}^{k} \nu_i \beta_i = 0 \ (\nu_i \text{ constants not all zero}).$$

Hence $\nu_0 x_0 + \sum_{i=1}^{k} \nu_i (x_i - x_0)$ is a constant vector. But as it is not a constant direction on account of the assumption on k, we must have $\nu_0 \neq 0$. So $x_0 + \sum_{i=1}^{k} \nu_i \nu_0 (x_i - x_0)$ is a fixed point in the space. We take it as the origin. Now we put

$$x = \sum_{i=1}^{k} \alpha_i x_i + \sum_{i=k+1}^{r} \alpha_i a_i$$

and proceed in the same way as in Ib. We have $x_i = \delta_i b (i = 1, \ldots, k)$. If we can find α such that

$$(\alpha, \delta) = 0, (\dot{\alpha}, \delta) = 0, \ldots, (\overset{(k-2)}{\alpha}, \delta) = 0, \tag{22}$$

where $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\delta = (\delta_1, \ldots, \delta_k)$, we will have

$$\dot{X} = \sum_{i=1}^{k} \dot{\alpha}_i x_i + \sum_{i=k+1}^{r} \dot{\alpha}_i a_i, \ldots\ldots, \overset{(k-1)}{X} = \sum_{i=1}^{k} \overset{(k-1)}{\alpha_i} x_i + \sum_{i=k+1}^{r} \overset{(k-1)}{\alpha_i} a_i.$$

(22) is equivalent to the following relations:

$$(\alpha, \delta) = 0, (\alpha, \dot{\delta}) = 0, \ldots, (\alpha, \overset{(k-2)}{\delta}) = 0. \tag{23}$$

Here the rank of the matrix

$$\begin{pmatrix} \delta_1 & \delta_2 & \cdots\cdots & \delta_k \\ \dot{\delta}_1 & \dot{\delta}_2 & \cdots\cdots & \dot{\delta}_k \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \overset{(k-2)}{\delta_1} & \overset{(k-2)}{\delta_2} & \cdots\cdots & \overset{(k-2)}{\delta_k} \end{pmatrix}$$

is $k-1$. For otherwise there would be constants $\mu_1, \ldots, \mu_{k-1}$ not all zero such that $\sum_{i=1}^{k-1} \mu_i \delta_i = 0$ and then $\sum_{i=1}^{k-1} \mu_i x_i$ would be a fixed point in the space. As a fixed point has been taken as the origin, this would result in the existence of one more fixed direction except $a_{k+1}, \ldots, a_r$. This contradicts the assumption on k. Thus there exists α such that (23) holds. Next we can find that

$$D_2 \equiv \begin{vmatrix} \alpha_1 & \cdots\cdots & \alpha_k \\ \dot{\alpha}_1 & \cdots\cdots & \dot{\alpha}_k \\ \cdots & \cdots & \cdots \\ \overset{(k-1)}{\alpha_1} & \cdots\cdots & \overset{(k-1)}{\alpha_k} \end{vmatrix} \neq 0.$$

For otherwise there would be, by the lemma, constants ρ_i not all zero such that $\sum_{i=1}^{k} \rho_i \alpha_i = 0$. From the last equation and (23) we eliminate α_i and by the lemma we would get the relation $\sum_{i=1}^{k} \mu_i \delta_i = 0$, where μ_i are constants not all zero. Then one more direction except $a_{k+1}, \ldots, a_r$ would be constant. This contradicts the assumption on k.

Now by virtue of $D_2 \neq 0, \dot{X}, \ldots, \overset{(k-1)}{X}$ are linearly independent. We put $x = \sum_{i=0}^{k-1} \gamma_i \overset{(i)}{X} + \sum_{i=k+1}^{r} \gamma_i a_i$, and determine $\gamma_0, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots \gamma_r$ so as to satisfy the relation that $\dot{x}$ is perpendicular to $X, \dot{X}, \ldots, \overset{(k-1)}{X}$ and $a_{k+1}, \ldots, a_r$. Thus we find that the motion in question is really possible. Consequently we get the result that our motion is *the one whose paths are the orthogonal trajectories of the r-planes which contain the osculating* $(k-1)$*-planes of a curve and a fixed plane of dimension r-k* (*type* (E)). *Thus our motion, for which the equality holds in* (2), *must be one of the five types* (A), (B), (C), (D) *and* (E). The restriction on the position of the simplex on each plane is evident from the above discussion.

6. Concerning the motion on the sphere of dimension n the result analogous to (2) does not hold. In this case, if n is odd, there exists even a motion by which all points on the sphere describe the arcs with the same length and it is easy to get all the motion of that nature. Let x be a point that moves on the n-dimensional unit sphere and the position of x at an instant $t = 0$ be a. Then x can be represented as $x = Pa$, where P is a proper orthogonal matrix whose elements are functions of t. We denote by ds the arc element of the curve which x describes. When we put $P'\dot{P} = S, S$ is skew and

$$(ds/dt)^2 = (\dot{x}, \dot{x}) = (\dot{P}a, \dot{P}a) = (PSa, PSa) = (S'Sa, a).$$

Now we suppose ds is the same for all a such that $(a, .a) = 1$. Then taking s for the parameter t we get the result $S'S = E$, where E is the unit matrix. As S is skew, there exists an orthogonal matrix such that $S = O'AO$, where

$$A = \begin{pmatrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{pmatrix} \dot{+} \begin{pmatrix} 0 & -\lambda_2 \\ \lambda_2 & 0 \end{pmatrix} \dot{+} \cdots \dot{+} \begin{pmatrix} 0 & -\lambda_m \\ \lambda_m & 0 \end{pmatrix} \qquad (n + 1 = 2m).$$

Then owing to $S'S = E$ we have $S'S = O'A^2O = E$, hence $A^2 = E$, which results in $\lambda_i = \pm 1$. Thus all the matrices S which are orthogonal as well as skew are obtained, namely $S = O'AO$, where A is the skew matrix of the form stated above and O is an arbitrary orthogonal matrix.

Now we solve the matric differential equation

$$\dot{P} = (P^{-1})'S \tag{24}$$

with the initial condition $P = E$ at $t = 0$. Then $\dot{P}'P + P'\dot{P}' = (P'\dot{P}) = 0$ and so $P'P = E$ for all t. Hence we have from (24)

$$\dot{P} = PS. \tag{25}$$

It is more convenient to solve (25) with the initial condition $P = E$ at $t = 0$ than to solve (24). Owing to the uniqueness of the solution of the differential equation the solutions of (24) and (25) with the same initial condition $P = E$ at $t = 0$ are the same. If O is a constant matrix, the motion thus obtained is trivial, while if O contains the parameter t, our motion is of some interest. It is well known that the rotation group of dimension 4 is locally a direct product of the two groups, each of which is locally isomorphic to the rotation group of dimension 3. All the motions of these two groups possess the property stated above, namely all the points on the sphere describe the arcs of the same length.

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