ON THE SPACES WITH NORMAL CONFORMAL CONNEXIONS AND SOME IMBEDDING PROBLEM OF RIEMANNIAN SPACES, II*)

BY

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In the previous paper[1] we have studied the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere. The main results that we have obtained are as follows:

(1) If the group of holonomy of an $(n+1)$-dimensional space C_{n+1} with a normal conformal connexion fixes a hypersphere $\mathfrak{S}_n$, the C_{n+1} is a space with a normal conformal connexion corresponding to the class of Riemannian spaces conformal to each other including an Einstein space with a negative, vanishing or positive scalar curvature according as the $\mathfrak{S}_n$ is real, point or imaginary. The converse is also true.

(2) For $n = 2m+1$ $(m \geqq 1)$ and 2 any Riemannian space V_n, and for $n = 2m$ $(m \geqq 2)$ any Riemannian space V_n satisfying the condition $L^{\nu}_{\lambda} = 0$ can be imbedded in a Riemannian space V_{n+1} conformal with some Einstein space as a hypersurface which is the image of a hypersphere $\mathfrak{S}_n$ invariant under the group of holonomy of the space C_{n+1} with the normal conformal connexion associated with this V_{n+1}.

But the meaning of the immersion of a given Riemannian space V_n in a V_{n+1} as a hypersurface of it as stated above is that at each point P of V_{n+1}, the invariant hypersphere $\mathfrak{S}_n$ in the tangent Möbius space $M_{n+1}(P)$ at P under the group of holonomy of C_{n+1} *contain the point at infinity in* $M_{n+1}(P)$ (with respect to the natural frame of C_{n+1}), and the image of $\mathfrak{S}_n$ in V_{n+1} is the set of points P such that P as a point in $M_{n+1}(P)$ is contained in $\mathfrak{S}_n$.

In the present paper, we shall investigate the same problem to imbed a given Riemannian space V_n in an V_{n+1} as stated above *without the restriction such that* $\mathfrak{S}_n$ *contains the point at infinity in the tangent Möbius space* $M_{n+1}(P)$ *at each point* P *of* V_{n+1}, in other words, without any restriction with respect to the scalar y^0 (in the previous paper, no. 1, 2).

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1) Tominosuke Ôtsuki, On the spaces with normal conformal connexions and some imbedding problem of Riemannian spaces, I, Tôhoku Math. Jour., 2nd. S., Vol. 1, No. 2, 1950, pp. 194-224. We shall refer this paepr by [I] in the present paper.

We shall use the same notations as those in Part I for the geometrical objects with some exceptions.

§1. The space with a normal conformal connexion whose group of holonomy fixes a real hypersphere.

Let there be given an n-dimensional space with a normal conformal connexion C_n. If we take normal frames R^*: $(A_0^*, A_i^*, A_\infty^*)$[2] composed of the hyperspheres such that

$$A_0^* A_i^* = A_i^* A_\infty^* = 0, \qquad A_0^* A_\infty^* = -1, \qquad A_i^* A_j^* = \delta_{ij}$$
$$(i, j = 1, 2, \ldots, n)^{3)},$$

where δ_{ij} is the Kronecker's δ, the connexion of the space is given by the following equations:

$$\begin{aligned} dA_0^* &= \omega^{*0}_{\ 0} A_0^* + \omega^{*i} A^*_i, \\ dA_i^* &= \omega^{*0}_{\ i} A_0^* + \omega^{*k}_{\ i} A_k^* + \omega^{*i} A_\infty^*, \\ dA_\infty^* &= \omega^{*0}_{\ i} A_i^* - \omega^{*0}_{\ 0} A_\infty^*, \\ &\omega^{*j}_{\ i} + \omega^{*i}_{\ j} = 0, \end{aligned}$$

where $\omega^{*0}_{\ 0}$, ω^{*i}, $\omega^{*i}_{\ j}$, $\omega^{*0}_{\ j}$ are Pfaffian forms. Suppose that the group of holo omy of C_n fixes a real hypersphere $\mathfrak{S}_{n-1}$. If we express it by $X = x^0 A_0^* + x^i A_i^* + x^\infty A_\infty^*$ with respect to the normal frame R^* $(A_0^*, A_i^0, A_\infty^*)$ in the tangent Möbius space $M_n(P)$ $(P = A_0^*)$ at each point P of C_n, then $dX = \pi X$, where π is a Pfaffian form. Since we have

$$\begin{aligned} dX = &(dx^0 + x^0 \omega^{*0}_{\ 0} + x^k \omega^{*0}_{\ k}) A_0^* \\ &+ (dx^i + x^0 \omega^{*i} + x^k \omega^{*i}_{\ k} + x^\infty \omega^{*0}_{\ i}) A_i^*. \\ &+ (dx^\infty + x^k \omega^{*k} - x^\infty \omega^{*0}_{\ 0}) A_\infty^*, \end{aligned}$$

the system of Pfaffian equations

$$\begin{aligned} \frac{dx^0 + x^0 \omega^{*0}_{\ 0} + x^k \omega^{*0}_{\ k}}{x^0} &= \frac{dx^i + x^0 \omega^{*i} + x^k \omega^{*i}_{\ k} + x^\infty \omega^{*0}_{\ i}}{x^i} \\ &= \frac{dx^\infty + x^k \omega^{*k} - x^\infty \omega^{*0}_{\ 0}}{x^\infty} \end{aligned}$$

must be integrable. The converse is also true. Since $XP = XA_0^* = -x^\infty$, the hypersurface $\mathfrak{F}_{n-1}$ of the image of $\mathfrak{S}_{n-1}$ in C_n, that is, the locus of points

2) E. Cartan, Les espaces à connexion conforme, Ann. Soc. Pol. Math., 2 (1923), pp. 171-221.

3) In §§ 1-2, we assume that indices take the following values.
$i, j, k, h, \ldots = 1, 2, \ldots, n,$
$a, b, c, \ldots, \lambda, \mu, \ldots = 1, 2, \ldots, n\text{-}1.$

P which are on $\mathfrak{S}_{n-1}$ is given by $x^\infty = 0$. As $\mathfrak{S}_{n+1}$ is real, $XX = x^i x^i - 2x^0 x^\infty > 0$. Accordingly, on $\mathfrak{F}_{n-1}$ we have $x^i x^i > 0$. Hence, in a coordinate neighborhood of each point on $\mathfrak{F}_{n-1}$, we may assume that $x^n \neq 0$. Now, if we put

$$y^a = \frac{x^a}{x^n}, \qquad y^0 = \frac{x^0}{x^n}, \qquad y^\infty = \frac{x^\infty}{x^n},$$

we can choose frames such that

$$y^a = 0$$

by virtue of the equations of structure of C_n[4]. Then, y^0 and y^∞ become scalars and satisfy the following relations

$$(1) \qquad \begin{cases} dy^0 + \omega^{*0}_n - y^0(y^0 \omega^{*n} + y^\infty \omega^{*0}_n) = 0, \\ dy^\infty + \omega^{*n} - y^\infty(y^0 \omega^{*n} + y^\infty \omega^{*0}_n) = 0, \\ \omega^{*a}_n + y^0 \omega^{*a} + y^\infty \omega^{*a}_n = 0. \end{cases}$$

§ 2. The image of the invariant hypersphere.

1. $\mathfrak{F}_{n-1}$ and natural frames (Veblen's frames). From (1) we get

$$y^0 y^0 \omega^{*n} - (1 - y^0 y^\infty)\, \omega^{*0}_n = dy^0,$$
$$-(1 - y^0 y^\infty)\, \omega^{*n} + y^\infty y^\infty \omega^{*0}_n = dy^\infty,$$

hence we get

$$\omega^{*n} = -\frac{y^\infty y^\infty dy^0 + (1 - y^0 y^\infty) dy^\infty}{1 - 2y^0 y^\infty} = -dy^\infty + y^\infty d\log(1 - 2y^0 y^\infty)^{1/2},$$
$$\omega^{*0}_n = -\frac{(1 - y^0 y^\infty) dy^0 + y^0 y^0 dy^\infty}{1 - 2y^0 y^\infty} = -dy^0 + y^0 d\log(1 - 2y^0 y^\infty)^{1/2}.$$

If we put

$$1 - 2y^0 y^\infty = \psi^2, \qquad \frac{y^\infty}{\psi} = -y, \qquad \frac{y^0}{\psi} = -z,$$

we get by virtue of the above equations and the last one of (1)

$$(2) \qquad \omega^{*n} = \psi\, dy, \qquad \omega^{*0}_n = \psi\, dz,$$

and

4) [I], § 1, no. 1, (3).

$$\omega_n^{*a} - \psi(z\omega^{*a} + y\,\omega_a^{*0}) = 0. \tag{3}$$

Between the scalars ψ, y, z there exist the following relation

$$\frac{1}{\psi^2} - 2yz = 1, \tag{4}$$

and $\psi \neq 0$ by virtue of the assumption that $\mathfrak{S}_{n-1}$ is real.

Now, let us denote the integrals of the system of Pfaffian equations

$$\omega^{*1} = \omega^{*2} = \ldots = \omega^{*n-1} = 0$$

by $x^1, x^2, \ldots, x^{n-1}$. There will happen no confusion of these notations with those of the components of $\mathfrak{S}_{n-1}$ with respect to R* $(A_0^*, A_i^*, A_\infty^*)$. Then, by (2) we may consider $x^1, x^2, \ldots, x^{n-1}, x^n(=y)$ as a coordinate system.

On the other hand, let us suppose that the space C_n corresponds to a Riemannian space V_n whose line element is

$$ds^2 = g_{ij}(x)\,dx^i\,dx^j.$$

Then, the connexion of C_n with respect to the natural frame R(A^0, A_i, A_∞) is given, as is well known, by the following equations:

$$\begin{aligned} dA &= dx^i A_i, \qquad (A = A_0), \\ dA &= \omega_i^0 A + \omega_i^k A_k + \omega_i^\infty A_\infty, \\ dA_\infty &= \omega_\infty^i A_i \end{aligned}$$

and

$$\begin{cases} \omega_i^k = \Gamma_{ij}^k\,dx^j, & \omega_i^\infty = g_{ij}\,dx^j, \\ \omega_i^0 = \Pi_{ij}^0\,dx^j, & \omega_\infty^k = g_{kj}\,\omega_j^0, \end{cases} \tag{5}$$

where

$$\Gamma_{ij}^k = \frac{1}{2} g^{kh}\left(\frac{\partial g_{ih}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^h}\right), \tag{6}$$

$$\Pi_{ij}^0 = -\frac{1}{n-2}\left(K_{ij} - \frac{K}{2(n-1)} g_{ij}\right), \tag{7}$$

$$K = g^{ij} K_{ij}, \tag{8}$$

$$K_{ij} = K_i{}^h{}_{jh}, \tag{9}$$

$$K_i{}^h{}_{jk} = \frac{\partial \Gamma_{ij}^h}{\partial x^k} - \frac{\partial \Gamma_{ik}^h}{\partial x^j} + \Gamma_{ij}^s \Gamma_{sk}^h - \Gamma_{ik}^s \Gamma_{sj}^h. \tag{10}$$

Now, we can put

$$\omega^{*a} = f^a_\lambda dx^\lambda.$$

From $A_0 = A^*_0 = A$ and

$$\begin{aligned} dA^*_0 &= \omega^{*i} A^*_i = f^a_\lambda dx^\lambda A^*_\lambda + \psi\, dy\, A^*_n \\ &= dA_0 = dx^\lambda A_\lambda + dy\, A_n, \end{aligned}$$

we get

$$A_\lambda = f^a_\lambda A^*_a, \qquad A_n = \psi A^*_n.$$

Then, from the above equations we get

$$\begin{aligned} dA_\lambda &= df^a_\lambda A^*_a + f^a_\lambda dA^*_a \\ &= f^b_\lambda \omega^{*0}_b A^*_0 + (df^a_\lambda + f^b_\lambda \omega^{*a}_b) A^*_a \\ &\quad + f^b_\lambda \omega^{*n}_b A^*_n + f^b_\lambda \omega^{*b} A^*_\infty \\ &= \omega^0_\lambda A_0 + \omega^i_\lambda A_i + \omega^\infty_\lambda A_\infty \\ &= \omega^0_\lambda A^*_0 + \omega^\mu_\lambda f^a_\mu A^*_a + \omega^n_\lambda \psi A^*_n + \omega^\lambda_\infty A_\infty . \end{aligned}$$

Putting $A_\infty = A^*_\infty$, we get

$$\begin{aligned} dA_n &= d\psi A^*_n + \psi(\omega^{*0}_n A^*_0 + \omega^{*a}_n A^*_a + \omega^{*n} A^*_\infty) \\ &= \omega^0_n A_0 + \omega^i_n A_i + \omega^\infty_n A_\infty \\ &= \omega^0_n A^*_0 + \omega^\lambda_n f^a_\lambda A^*_a + \omega^n_n \psi A^*_n + \omega^\infty_n A^*_\infty . \end{aligned}$$

Hence, by these relations and (2) we obtain the following relations:

$$(11) \qquad \begin{cases} \omega^0_\lambda = f^a_\lambda \omega^{*0}_a, \qquad \omega^\infty_\lambda = f^b_\lambda \omega^{*b}, \qquad \psi\, \omega^n_\lambda = f^b_\lambda \omega^{*n}_b, \\ \omega^\mu_\lambda f^a_\mu = df^a_\lambda + f^b_\lambda \omega^{*a}_b, \\ \omega^0_n = \psi\, \omega^{*0}_n, \qquad \omega^\infty_n = \psi^2 dy, \qquad \psi\, \omega^n_n = d\psi, \\ \omega^\lambda_n f^a_\lambda = \psi\, \omega^{*a}_n. \end{cases}$$

Now, since the line element of the Riemannian space V_n is given by

$$\begin{aligned} ds^2 &= dA^*_0 dA^*_0 = \omega^{*i} \omega^{*i} \\ &= f^a_\lambda f^a_\mu dx^\lambda dx^\mu + \psi^2 dy\, dy \\ &= dA_0 dA_0 = A_i A_j dx^i dx^j, \end{aligned}$$

we have

$$(12)\qquad \begin{cases} g_{\lambda\mu} = A_\lambda A_\mu = f^a_\lambda f^a_\mu, \\ g_{\lambda n} = A_\lambda A_n = 0, \\ g_{nn} = A_n A_n = \psi^2. \end{cases}$$

If we substitute these equations in the second equation of (2), we obtain

$$\begin{aligned} \psi\, dz &= \omega^{*0}_{\ n} = \frac{1}{\psi}\,\omega^0_{\ n} \\ &= \frac{1}{\psi}\left\{-\frac{1}{n-2}K_{ni}\,dx^i + \frac{K}{2(n-1)(n-2)}\,g_{ni}\,dx^i\right\}, \end{aligned}$$

that is

$$(13)\qquad \psi^2 dz + \frac{1}{n-2}K_{n\lambda}\,dx^\lambda + \left\{\frac{K_{nn}}{n-2} - \frac{K\psi^2}{2(n-1)(n-2)}\right\}dy = 0.$$

From (3) we obtain

$$\omega^\lambda_{\ n} f^a_\lambda - \psi^2 (z f^a_\lambda\,dx^\lambda + y p^\lambda_a\,\omega^0_{\ \lambda}) = 0,$$

where the matrix (p^λ_a) is the inverse of (f^a_λ). Since we have

$$p^\lambda_a p^\mu_a = g^{\lambda\mu},$$

we obtain by virtue of the above relation and (5)

$$\begin{aligned} &\omega^\lambda_{\ n} - \psi^2 (z\,dx^\lambda + y g^{\lambda\mu}\,\omega^0_{\ \mu}) \\ &= \omega^\lambda_{\ n} - \psi^2 z\,dx^\lambda + y\psi^2\left\{\frac{1}{n-2}K^\lambda_i - \frac{K}{2(n-1)(n-2)}\,\delta^\lambda_i\right\}dx^i = 0, \end{aligned}$$

that is

$$(14)\qquad \frac{1}{\psi^2}\,\Gamma^\lambda_{n\mu} = y\left\{-\frac{1}{n-2}K^\lambda_\mu + \frac{K}{2(n-1)(n-2)}\,\delta^\lambda_\mu\right\} + z\,\delta^\lambda_\mu,$$

$$(15)\qquad \frac{1}{\psi^2}\,\Gamma^\lambda_{nn} = -\frac{y}{n-2}K^\lambda_n.$$

Hence we obtain the following theorem:

THEOREM 1. *If the group of holonomy of the space with a normal conformal connexion corresponding to a Riemannian space V_n fixes a real hypersphere $\mathfrak{S}_{n-1}$, there exist a scalar y with the following properties. The image of $\mathfrak{S}_{n-1}$ in V_n*

is hhe hypersurface determined hy the equation $y=0$. *If we put the line element of* V_n

$$ds^2 = g_{\lambda\mu}(x, y)\, dx^\lambda\, dx^\mu + (\psi(x, y)\, dy)^2$$

hy means of the family of hypersurfaces $\mathfrak{F}_{n-1}(y)$ *such that* y *is a constant on every* $\mathfrak{F}_{n-1}(y)$ *and their orthogonal trajectories and define* z *by*

$$1 = \psi^2(1 + 2yz),$$

y, z, ψ *satisfy the equations* (13), (14), (15). *The converse is also true.*

2. The family of hypersurfaces $\mathfrak{F}_{n-1}(y)$. In a Riemannian space with line element such that

$$ds^2 = g_{\lambda\mu}(x^a, y)\, dx^\lambda\, dx^\mu + (\psi(x^a, y)\, dy)^2$$

in a coordinate neighborhood $x^1, \ldots, x^{n-1}, y$, let $V_{n-1}(y)$ be the Riemannian space induced from the ambient space on the hypersurface $\mathfrak{F}_{n-1}(y)$: $y =$ a constant. Let us denote the Christoffel's symbols of $V_{n-1}(y)$ determined by its fundamental tensor $g_{\lambda\mu}$ by $\{{}^{\lambda}_{\mu\nu}\}$ and the covariant differentiation of $V_{n-1}(y)$ by a comma.

Now, the unit normal vector n_i on the hypersurface $\mathfrak{F}_{n-1}(y)$ has their components such that $(0, \ldots, 0, \psi)$. Hence the second fundamental tensor of $\mathfrak{F}_{n-1}(y)$ is given by

$$h_{ab} = -\, n_{i;j} \frac{\partial x^i}{\partial x^a} \frac{\partial x^i}{\partial x^b} = -\, n_{a;b} = n_i\, \Gamma^i_{ab} = \psi \Gamma^n_{ab},$$

where the symbol " ; " denotes the covariant differentiation of V_n. On the other hand, by (6), (12), we have

$$\Gamma^n_{ab} = -\frac{1}{2} g^{nn} \frac{\partial g_{ab}}{\partial y} = -\frac{1}{2\psi^2} \frac{\partial g_{ab}}{\partial y}.$$

Hence, we have

$$\frac{\partial g_{ab}}{\partial y} = -2\psi\, h_{ab} \qquad \text{or} \qquad \frac{\partial g_{ab}}{\partial y} = 2\psi\, h^{ab} \tag{16}$$

where $h^{ab} = g^{a\lambda} g^{b\mu} h_{\lambda\mu}$.

On the other hand, by (6), (12), (16). we can easily prove the following relations:

$$
(17)\qquad
\begin{aligned}
&\Gamma^a_{bc} = \left\{ {a \atop bc} \right\},\\
&\Gamma^n_{ab} = \frac{1}{\psi} h_{ab},\ \Gamma^a_{bn} = -\psi h^a_b = -\psi g^{a\lambda} h_{b\lambda},\\
&\Gamma^n_{an} = \frac{1}{\psi}\psi_{,a},\quad \Gamma^a_{nn} = -\psi g^{a\lambda}\psi_{,\lambda},\ \Gamma^n_{nn} = \psi \frac{\partial\psi}{\partial y}.
\end{aligned}
$$

If we denote the components oi Riemann tensor, the components of Ricci tensor, and the scalar curvature of $V_{n-1}(y)$ by

$$
\begin{aligned}
R_{bcd}^{\ a} &= \frac{\partial \Gamma^a_{bc}}{\partial x^d} - \frac{\partial \Gamma^a_{bd}}{\partial x^c} + \Gamma^\lambda_{bc}\Gamma^a_{\lambda a} - \Gamma^\lambda_{bd}\Gamma^a_{\lambda c},\\
R_{bc} &= R_{b\ c\lambda}^{\ \lambda},\\
R &= g^{\lambda\mu} R_{\lambda\mu}
\end{aligned}
$$

respectively, we have, by means of the formulas of Gauss-Codazzi, the following relations:[5]

$$
\begin{aligned}
K_{acbd} &= R_{acbd} - h_{ab} h_{cd} + h_{ad} h_{cb},\\
\psi K^n_{abc} &= h_{ab,c} - h_{ac,b}.
\end{aligned}
$$

Furthermore, by (10), (17), we get easily

$$
K_{a\ bn}^{\ n} = \frac{1}{\psi}\frac{\partial h_{ab}}{\partial y} + h^\lambda_a h_{b\lambda} - \frac{1}{\psi}\psi_{,ab}.
$$

Hence, obtain the following relations:

$$
(18)\qquad
\begin{cases}
K_{ab} = g^{\lambda\mu} K_{a\lambda b\mu} + K_{a\ bn}^{\ n}\\
\qquad = \dfrac{1}{\psi}\dfrac{\partial h_{ab}}{\partial y} - h\, h_{ab} + 2 h^\lambda_a h_{b\lambda} + R_{ab} - \dfrac{1}{\psi}\psi_{,ab},\\
K_{an} = g^{\lambda\mu} K_{a\lambda n\mu} = \psi\,(h_{,a} - h^\lambda_{a,\lambda}),\\
K_{nn} = K_{n\ n\lambda}^{\ \lambda} = \psi \dfrac{\partial h}{\partial y} - \psi^2 h^\lambda_\mu h^\mu_\lambda - \psi g^{\lambda\mu}\psi_{,\lambda\mu},\\
K\ = g^{ij} K_{ij} = \dfrac{2}{\psi}\dfrac{\partial h}{\partial y} - h\,h - h^\mu_\lambda h^\lambda_\mu + R - \dfrac{2}{\psi} g^{\lambda\mu}\psi_{,\lambda\mu},
\end{cases}
$$

where $h = g^{\lambda\mu} h_{\lambda\mu}$.

3. Relations between $\mathfrak{F}_{n-1}(y)$ and the invariant hypersphere. Now, if we substitute (18) in (13), we get

5) Schouten-Struik, Einf:hrung in die neuren Methoden der Differentialgeometrie, II, p. 122.

$$\zeta_{,a} = -\frac{1}{(n-2)\psi}(h_{,a} - h^{\lambda}_{a,\lambda}) \tag{19}$$

and

$$\begin{aligned} \frac{\partial \zeta}{\partial y} &= -\frac{1}{(n-2)\psi}\left\{K_{nn} - \frac{K\psi^2}{2(n-1)}\right\} \\ &= -\frac{1}{(n-1)\psi}\left(\frac{\partial h}{\partial y} - g^{\lambda\mu}\psi_{,\lambda\mu}\right) - \frac{h^2 - R}{2(n-1)(n-2)} \\ &\quad + \frac{2n-3}{2(n-1)(n-2)} h^{\mu}_{\lambda} h^{\lambda}_{\mu}. \end{aligned} \tag{20}$$

Substituting (17), (18) in (14), we get

$$\begin{aligned} -\frac{1}{\psi} h_{ab} = y\Big[&-\frac{1}{n-2}\left(\frac{1}{\psi}\frac{\partial h_{ab}}{\partial y} - h\, h_{ab} + 2h^{\lambda}_{a} h_{b\lambda} + R_{ab} - \frac{1}{\psi}\psi_{,ab}\right) \\ &+ \frac{1}{2(n-1)(n-2)} g_{ab}\left(\frac{2}{\psi}\frac{\partial h}{\partial y} - h^2 - h^{\mu}_{\lambda} h^{\lambda}_{\mu} + R - \frac{2}{\psi} g^{\lambda\mu}\psi_{,\lambda\mu}\right)\Big] \\ &+ \zeta g_{ab}, \end{aligned}$$

that is

$$\begin{aligned} \frac{\partial h_{ab}}{\partial y} &= \frac{n-2}{y}(h_{ab} + \psi\zeta g_{ab}) + \psi(h\, h_{ab} - 2h^{\lambda}_{a} h_{b\lambda} - R_{ab}) + \psi_{,ab} \\ &\quad + \frac{g_{ab}}{2(n-1)}\left\{2\frac{\partial h}{\partial y} - \psi(h^2 + h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R) - 2g^{\lambda\mu}\psi_{,\lambda\mu}\right\}. \end{aligned} \tag{21}$$

Substituting (17), (18) in (15), we get

$$\psi_{,a} = \frac{y\psi^2}{n-2}(h_{,a} - h^{\lambda}_{a,\lambda}).$$

However, by means of (4), (19), we get

$$\psi_{,a} = -y\,\psi^3 \zeta_{,a} = \frac{y\psi^2}{n-2}(h_{,a} - h^{\lambda}_{a,\lambda}).$$

Hence, the system of equations (19), (20), (21) is equivalent to the system of eqations (13), (14), (15).

Now, by virtue ot (16), (21) we have

$$\begin{aligned} \frac{\partial h}{\partial y} &= \frac{\partial}{\partial y}(g^{\lambda\mu} h_{\lambda\mu}) \\ &= 2\psi\, h_{\lambda\mu} h^{\lambda\mu} + \frac{n-2}{y}\left\{h + (n-1)\psi\zeta\right\} \end{aligned}$$

$$+\psi(h^2 - h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R) + g^{\lambda\mu}\psi_{,\lambda\mu}$$
$$+\left\{\frac{\partial h}{\partial y} - \frac{1}{2}\psi(h^2 + h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R) - g^{\lambda\mu}\psi_{,\lambda\mu}\right\},$$

that is

$$\frac{n-2}{y}\left\{h + (n-1)\psi z\right\} + \frac{1}{2}\psi(h^2 - h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R) = 0. \tag{22}$$

Lastly, substituting (20) in (21), we get

$$\frac{\partial h_{ab}}{\partial y} = \frac{(n-2)}{y}(h_{ab} + \psi z g_{ab})$$
$$+\psi(h\, h_{ab} - 2h^{\lambda}_{a} h_{b\lambda} - R_{ab}) + \psi_{,ab}$$
$$+\frac{\psi g_{ab}}{2(n-1)(n-2)}\left\{R - h^2 + (2n-3)h^{\mu}_{\lambda} h^{\lambda}_{\mu}\right\} - \psi g_{ab}\frac{\partial z}{\partial y}$$
$$-\frac{\psi g_{ab}}{2(n-1)}(h^2 + h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R),$$

that is

$$\frac{\partial h_{ab}}{\partial y} \quad \frac{(n-2)}{y}(h_{ab} + \psi z g_{ab}) + \psi(h\, h_{ab} - 2h^{\lambda}_{a} h_{b\lambda} - R_{ab}) \tag{23}$$
$$-\frac{\psi g_{ab}}{2(n-2)}(h^2 - h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R) + \psi_{,ab} - \psi g_{ab}\frac{\partial z}{\partial y}.$$

Thus, we obtain the system of equations (19), (22), (23) which is equ-i valent to the system of equations (13), (14), (15) and is represented by means of the quantities of $\mathfrak{F}_{n-1}(y)$. Accordingly, Theorem 1 is reduced to the following

THEOREM 1′, *In a Riemannian space V_n, take a coordinate system such that the line element of V_n is given by*

$$ds^2 = g_{\lambda\mu}(x^a, y)\,dx^{\lambda}\,dx^{\mu} + (\psi(x^a, y)\,dy)^2,$$

then a necessary and sufficient condition that the hypersurface $y=0$ is the image of a hypersphere invariant under the group of holonomy of the space with a normal conformal connexion corresponding to V_n is that the fundamental tensors $g_{ab}(x, y)$, $h_{ab}(x, y)$ of the hypersurfaces $y = a$ constant and the scalar determined by (4) *satisfy the equations* (19), (22), (23).

§ 3. The invariant hypersphere and an imbedding problem.

1. A fundamental system of equations and quantities ξ_a, ζ. In the following paragraphs we shall assume that the indices take the following values:

$$a, b, c; \lambda, \mu, \nu, \ldots = 1, 2, \ldots n.$$

Let us now investingate the problem *to imbed a given Riemannian space V_n in a suitable Riemannian space V_{n+1} such that the group of holonomy of the space with a normal conformal connexion corresponding to V_{n+1} fixes a real hypersphere $\mathfrak{S}_n$ and V_n is the image of $\mathfrak{S}_n$ in V_{n+1}.*

According to Theorem 1', *a necessary and sufficient condition that a given Riemannian space V_n with line element*

$$ds^2 = g_{\lambda\mu}(x^a)\,dx^\lambda\,dx^\mu$$

is the image of the invariant hypersphere $\mathfrak{S}_n$ in the above-mentioned sense is that we can solve the following system of equations:

$$(\mathrm{I}_1) \qquad \frac{\partial}{\partial y} g_{ab} = -2\psi\, h_{ab},$$

$$(\mathrm{I}_2) \qquad \frac{\partial}{\partial y} h^a_b = \frac{n-1}{y}(h^a_b + \psi\,\chi\delta^a_b) + \psi\,(h\,h^a_b - \mathrm{R}) - \frac{\psi}{2(n-1)}\,\delta^a_b\,(h^2 - h^\mu_\lambda h^\lambda_\mu - \mathrm{R}) + g^{a\lambda}\,\psi_{,b\lambda} - \psi\,\delta^a\,\frac{\partial\chi}{\partial y},$$

provided that the conditions

$$(\mathrm{II}_1) \qquad \xi_a \equiv (h_{,a} - h^\lambda_{a,\lambda}) + (n-1)\,\psi\,\chi_{,a} = 0,$$

$$(\mathrm{II}_2) \qquad \zeta \equiv \frac{1}{y}(h + n\,\psi\chi) + \frac{\psi}{(n-1)}(h^2 - h^\mu_\lambda h^\lambda_\mu - \mathrm{R}) = 0$$

and the initial conditions

$$\Big[g_{ab}(x, y)\Big]_{y=0} = g_{ab}(x)$$

are satisfied. Then, the line element of V_{n+1} is

$$ds^2 = g_{\lambda\mu}(x, y)\,dx^\lambda\,dx^\mu + (\psi(x, y)\,dy)^2,$$

and the hypersurface $y = 0$ is the image of $\mathfrak{S}_n$.

Let us put

$$(\mathrm{I}_2') \qquad \frac{\partial}{\partial y} h_{ab} = \frac{n-1}{y}(h_{ab} + \psi \chi g_{ab}) + \psi(h h_{ab} - 2h^\lambda_a h_{b\lambda} - R_{ab}) - \frac{\psi}{2(n-1)} g_{ab}(h^2 - h^\mu_\lambda h^\lambda_\mu - R) + \psi_{,ab} - \psi g_{ab} \frac{\partial \chi}{\partial y}$$

and

$$(\mathrm{I}_3) \qquad \frac{\partial}{\partial y} h = \frac{n-1}{y}(h + n\psi\chi) + \frac{\psi}{2(n-1)}\left\{(n-2)(h^2 - R) + nh^\mu_\lambda h^\lambda_\mu\right\} + g^{\lambda\mu}\psi_{,\lambda\mu} - n\psi \frac{\partial \chi}{\partial y}$$

which is derived from (I_2).

Now, suppose that $g_{ab}(x, y)$, $h_{ab}(x, y)$ are solutions of the differential equations (I), and consider the quantities ξ_a, ζ determined by these $g_{ab}(x, y)$, $h_{ab}(x, y)$. By means of (6), (I_1), we get easily

$$(24) \qquad \frac{\partial}{\partial y} \Gamma^a_{bc} = g^{a\lambda}(\psi h_{bc})_{,\lambda} - (\psi h^a_b)_{,c} - (\psi h^a_c)_{,b}$$

and

$$\frac{\partial}{\partial y} \Gamma^\lambda_{\lambda a} = -(\psi h)_{,a}.$$

Accordingly, putting

$$(25) \qquad V_a \equiv h_{,a} - h^\lambda_{a,\lambda},$$

we get by (I), (24), (25),

$$\begin{aligned}
\frac{\partial}{\partial y} V_a &= \left(\frac{\partial h}{\partial y}\right)_{,a} - \left(\frac{\partial}{\partial y} h^\lambda_a\right)_{,\lambda} - \frac{\partial}{\partial y}\Gamma^\lambda_{\lambda\mu} h^\mu_a + \frac{\partial}{\partial y}\Gamma^\mu_{a\lambda} h^\lambda_\mu \\
&= \left(\frac{\partial h}{\partial y}\right)_{,a} - \left(\frac{\partial}{\partial y} h^\lambda_a\right)_{,\lambda} + (\psi h)_{,\mu} h^\mu_a - (\psi h^\mu_\lambda)_{,a} h^\lambda_\mu \\
&= \frac{n-1}{y}\left\{h_{,a} + n(\psi_{,a}\chi + \psi\chi_{,a})\right\} \\
&\quad + \frac{1}{2(n-1)}\psi_{,a}\left\{(n-2)(h^2 - R) + nh^\mu_\lambda h^\lambda_\mu\right\} \\
&\quad + \frac{\psi}{2(n-1)}\left\{(n-2)(2h\, h_{,a} - R_{,a}) + 2n\, h^\mu_\lambda h^\lambda_{\mu,a}\right\} \\
&\quad + g^{\lambda\mu}\psi_{,\lambda\mu a} - n\,\psi_{,a}\frac{\partial \chi}{\partial y} - n\psi\frac{\partial}{\partial y}\chi_{,a} \\
&\quad - \frac{n-1}{y}\left\{h^\lambda_{a,\lambda} + (\psi_{,a}\chi + \psi\chi_{,a})\right\} - \psi_{,\lambda}(h h^\lambda_a - R^\lambda_a)
\end{aligned}$$

$$
\begin{aligned}
&- \psi (h_{,\lambda} h^{a}_{\lambda} + h h^{\lambda}_{a,\lambda} - R^{\lambda}_{a,\lambda}) \\
&+ \frac{1}{2(n-1)} \psi_{,a} (h^2 - h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R) \\
&+ \frac{\psi}{2(n-1)} (2h h_{,a} - 2h^{\mu}_{\lambda} h^{\lambda}_{\mu,a} - R_{,a}) \\
&- g^{\lambda\mu} \psi_{,\lambda a\mu} + \psi_{,a} \frac{\partial z}{\partial y} + \psi \frac{\partial}{\partial y} z_{,a} \\
&+ \psi h_{,\mu} h^{\mu}_{a} + \psi_{,\mu} h h^{\mu}_{a} - \psi_{,a} h^{\mu}_{\lambda} h^{\lambda}_{\mu} - \psi h^{\lambda}_{\mu} h^{\mu}_{\lambda,a} \\
= &\frac{n-1}{y} \left\{ V_a + (n-1)(\psi_{,a} z + \psi z_{,a}) \right\} \\
&+ \frac{1}{2} \psi_{,a} (h^2 - h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R) - (n-1) \left(\psi_{,a} \frac{\partial z}{\partial y} + \psi \frac{\partial}{\partial y} z_{,a} \right) \\
&+ \frac{\psi}{2} (2h h_{,a} - R_{,a}) - g^{\lambda\mu} R_{\lambda}{}^{\nu}{}_{\mu a} \psi_{,\nu} + \psi_{,\lambda} R^{\lambda}_{a} \\
&- \psi (h h^{\lambda}_{a,\lambda} - R^{\lambda}_{a,\lambda}).
\end{aligned}
$$

On the other hand, from the Bianchi's identity, we get

$$
R^{\lambda}_{a,\lambda} = \frac{1}{2} R_{,a}
$$

as is well known. Hence, the above equations become

$$
\begin{aligned}
\frac{\partial}{\partial y} V_a = &\frac{n-1}{y} \left\{ V_a + (n-1)(\psi z)_{,a} \right\} \\
&+ \frac{1}{2} \psi_{,a} (h^2 - h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R) + \psi h V_a - (n-1) \left(\psi \frac{\partial z}{\partial y} \right)_{,a}.
\end{aligned}
$$

Then, we obtain easily from (4) the following relations:

$$
\frac{\partial \psi}{\partial y} = - \psi^3 \left(z + y \frac{\partial z}{\partial y} \right), \tag{26}
$$

$$
\psi_{,a} = - y \psi^3 z_{,a} \tag{27}
$$

$$
\psi_{,ab} = - y \psi^3 (z_{,ab} - 3y \psi^2 z_{,a} z_{,b}), \tag{28}
$$

Using these equations, we get

$$
\begin{aligned}
\frac{\partial}{\partial y} \xi_a = &\frac{\partial}{\partial y} V_a + (n-1) \left\{ \psi \frac{\partial}{\partial y} z_{,a} - \psi^3 \left(z + y \frac{\partial z}{\partial y} \right) z_{,a} \right\} \\
= &\left(\frac{n-1}{y} + \psi h \right) V_a + \frac{(n-1)^2}{y} (\psi z_{,a} - y \psi^3 z z_{,a}) \\
&- \frac{1}{2} y \psi^3 z_{,a} (h^2 - h^{\mu}_{\lambda} h^{\lambda}_{\mu} - R)
\end{aligned}
$$

$$
\begin{aligned}
& -(n-1)\left\{\psi\left(\frac{\partial \chi}{\partial y}\right)_{,a}-y\psi^3\frac{\partial \chi}{\partial y}\chi_{,a}\right\} \\
& +(n-1)\left\{\psi\frac{\partial}{\partial y}\chi_{,a}-\psi^3\left(\chi+y\frac{\partial \chi}{\partial y}\right)\chi_{,a}\right\} \\
=& \left(\frac{n-1}{y}+\psi h\right)V_a+\frac{(n-1)^2}{y}\psi\chi_{,a}-n(n-1)\psi^3\chi\chi_{,a} \\
& -\frac{1}{2}y\psi^3\chi_{,a}(h^2-h^\mu_\lambda h^\lambda_\mu-\mathrm{R}).
\end{aligned}
$$

If we substitute ξ_a, ζ in the right hand side of the last equation, we obtain

$$
\begin{aligned}
\frac{\partial}{\partial y}\xi_a =& \left(\frac{n-1}{y}+\psi h\right)\xi_a-(n-1)\psi\left(\frac{n-1}{y}+\psi h\right)\chi_{,a} \\
& +\frac{(n-1)^2}{y}\psi\chi_{,a}-n(n-1)\psi^3\chi\chi_{,a} \\
& -\frac{1}{2}y\psi^3\chi_{,a}(h^2-h^\mu_\lambda h^\lambda_\mu-\mathrm{R}) \\
=& \left(\frac{n-1}{y}+\psi h\right)\xi_a \\
& -(n-1)\psi^3\chi_{,a}\left\{h+n\psi\chi+\frac{y\psi}{2(n-1)}(h^2-h^\mu_\lambda h^\lambda_\mu-\mathrm{R})\right\},
\end{aligned}
$$

that is

$$
\frac{\partial}{\partial y}\xi_a=\left(\frac{n-1}{y}+\psi h\right)\xi_a-(n-1)y\psi^2\chi_{,a}\zeta. \tag{29}
$$

In the next place, let us consider ζ. By (24), (I), we get

$$
\begin{aligned}
\frac{\partial}{\partial y}\mathrm{R} =& \frac{\partial}{\partial y}(g^{\lambda\mu}\mathrm{R}_{\lambda\mu})=2\psi h^{\lambda\mu}\mathrm{R}^{\lambda\mu} \\
& +g^{\lambda\mu}\frac{\partial}{\partial y}\left(\frac{\partial\Gamma^\nu_{\lambda\mu}}{\partial x^\nu}-\frac{\partial\Gamma^\nu_{\lambda\nu}}{\partial x^\mu}+\Gamma^\rho_{\lambda\mu}\Gamma^\nu_\rho-\Gamma^\rho_{\lambda\nu}\Gamma^\nu_{\rho\mu}\right) \\
=& 2\psi h^{\lambda\mu}\mathrm{R}_{\lambda\mu}+g^{\lambda\mu}\left(\frac{\partial}{\partial y}\Gamma^\nu_{\lambda\mu}\right)_{,\nu}-g^{\lambda\mu}\left(\frac{\partial}{\partial y}\Gamma^\nu_{\lambda\nu}\right)_{,\mu},
\end{aligned}
$$

that is

$$
\frac{\partial}{\partial y}\mathrm{R}=2\left\{\psi h^{\lambda\mu}\mathrm{R}_{\lambda\mu}+g^{\lambda\mu}(\psi h)_{,\lambda\mu}-(\psi h^{\lambda\mu})_{,\lambda\mu}\right\}. \tag{30}
$$

By (I), (3), we get also

$$
\frac{\partial}{\partial y}\zeta=-\frac{1}{y^2}(h+n\psi\chi)
$$

$$
\begin{aligned}
&+\left(\frac{1}{y}+\frac{\psi h}{n-1}\right)\Big[\frac{n-1}{y}(h+n\psi\chi)+g^{\lambda\mu}\psi_{,\lambda\mu}-n\psi\frac{\partial\chi}{\partial y}\\
&+\frac{\psi}{2(n-1)}\left\{(n-2)(h^2-\mathrm{R})+nh^{\mu}_{\lambda}h^{\lambda}_{\mu}\right\}\Big]\\
&+\frac{n}{y}\frac{\partial}{\partial y}(\psi\chi)+\frac{1}{2(n-1)}(h^2-h^{\mu}_{\lambda}h^{\lambda}_{\mu}-\mathrm{R})\frac{\partial\psi}{\partial y}\\
&-\frac{\psi}{n-1}h^{\mu}_{\lambda}\Big[\frac{n-1}{y}(h^{\lambda}_{\mu}+\psi\chi\delta^{\lambda}_{\mu})+\psi(h\,h^{\lambda}_{\mu}-\mathrm{R}^{\lambda}_{\mu})\\
&-\frac{\psi}{2(n-1)}\delta^{\lambda}_{\mu}(h^2-h^{\rho}_{\nu}h^{\nu}_{\rho}-\mathrm{R})+g^{\lambda\mu}\psi_{,\mu\nu}-\psi\delta^{\lambda}_{\mu}\frac{\partial\chi}{\partial y}\Big]\\
&-\frac{\psi}{n-1}\Big[\psi h^{\mu}_{\lambda}\mathrm{R}^{\lambda}_{\mu}+g^{\lambda\mu}\psi_{,\lambda\mu}h+2g^{\lambda\mu}\psi_{,\lambda}h_{,\mu}+\psi g^{\lambda\mu}h_{,\lambda\mu}\\
&-\psi_{,\lambda\mu}h^{\lambda\mu}-2\psi_{,\lambda}h^{\lambda\mu}{}_{,\mu}-\psi h^{\lambda\mu}{}_{,\lambda\mu}\Big]\\
=\;&\frac{1}{y}\left(\frac{n-2}{y}+\psi h\right)(h+n\psi\chi)\\
&+\frac{\psi}{2(n-1)}\left(\frac{1}{y}+\frac{\psi h}{n-1}\right)\left\{(n-2)(h^2-\mathrm{R})+n\,h^{\mu}_{\lambda}h^{\lambda}_{\mu}\right\}\\
&+\frac{1}{2(n-1)}\left(\frac{\partial\psi}{\partial y}+\frac{\psi^2h}{n-1}\right)(h^2-h^{\mu}_{\lambda}h^{\lambda}_{\mu}-\mathrm{R})\\
&+\frac{1}{y}g^{\lambda\mu}\psi_{,\lambda\mu}-\psi^2h\frac{\partial\chi}{\partial y}+\frac{n}{y}\chi\frac{\partial\psi}{\partial y}\\
&-\psi\left(\frac{1}{y}+\frac{\psi h}{n-1}\right)h^{\mu}_{\lambda}h^{\lambda}_{\mu}-\frac{1}{y}\psi^2h\,\chi\\
&-\frac{\psi}{n-1}(2g^{\lambda\mu}\psi_{,\lambda}V_{\mu}+\psi g^{\lambda\mu}V_{\lambda,\mu})\\
=\;&\frac{1}{y}\left(\frac{n-2}{y}+\psi h\right)(h+n\psi\chi)\\
&+\frac{1}{2(n-1)}\left\{\frac{\partial\psi}{\partial y}+\frac{\psi^2h}{n-1}+(n-2)\psi\left(\frac{1}{y}+\frac{\psi h}{n-1}\right)\right\}(h^2-h^{\mu}_{\lambda}h^{\lambda}_{\mu}-\mathrm{R})\\
&+\frac{1}{y}g^{\lambda\mu}\psi_{,\lambda\mu}-\psi^2h\frac{\partial\chi}{\partial y}+\frac{n}{y}\chi\frac{\partial\psi}{\partial y}-\frac{1}{y}\psi^2h\,\chi\\
&-\frac{\psi}{n-1}(2g^{\lambda\mu}\psi_{,\lambda}V_{\mu}+\psi\,g^{\lambda\mu}V_{\lambda,\mu}).
\end{aligned}
$$

Making use of ξ_a, ζ, the last equation becomes

$$
\begin{aligned}
\frac{\partial}{\partial y}\zeta=\;&\frac{1}{y}\left(\frac{n-2}{y}+\psi h\right)(h+n\psi\chi)\\
&+\left(\frac{\partial}{\partial y}\log\psi+\psi h+\frac{n-2}{y}\right)\left\{\zeta-\frac{1}{y}(h+n\psi\chi)\right\}\\
&+\frac{1}{y}g^{\lambda\mu}\psi_{,\lambda\mu}-\psi^2h\frac{\partial\chi}{\partial y}+\frac{n}{y}\chi\frac{\partial\psi}{\partial y}-\frac{1}{y}\psi^2h\chi
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\psi}{n-1}\Big[2g^{\lambda\mu}\,\psi_{,\lambda}\left\{\xi_{,\mu}-(n-1)\psi\,\zeta_{,\mu}\right\}\\
&+\psi\,g^{\lambda\mu}\left\{\xi_{\lambda,\mu}-(n-1)\left(\psi_{,\lambda}\zeta_{,\mu}+\psi\,\zeta_{,\lambda\mu}\right)\right\}\Big]\\
&=\left(\frac{n-2}{y}+\psi h+\frac{\partial}{\partial y}\log\psi\right)\zeta+\frac{\psi^2}{y}\left(\zeta+y\frac{\partial\zeta}{\partial y}\right)(h+n\psi\zeta)\\
&-\psi^3 g^{\lambda\mu}\left(\zeta_{,\lambda\mu}-3y\,\psi^2\zeta_{,\lambda}\zeta_{,\mu}\right)-\psi^2 h\frac{\partial\zeta}{\partial y}\\
&-\frac{n}{y}\psi^3\zeta\left(\zeta+y\frac{\partial\zeta}{\partial y}\right)-\frac{1}{y}\psi^2 h\,\zeta\\
&-\frac{\psi}{n-1}\Big\{2g^{\lambda\mu}\psi_{,\lambda}\,\xi_\mu+\psi\,g^{\lambda\mu}\xi_{\lambda,\mu}+3(n-1)y\,\psi^4 g^{\lambda\mu}\zeta_{,\lambda}\zeta_{,\mu}\\
&-(n-1)\;\psi^2 g^{\lambda\mu}\zeta_{,\lambda\mu}\Big\},
\end{aligned}
$$

that is

$$
\frac{\partial}{\partial y}\zeta=\left(\frac{n-2}{y}+\psi h+\frac{\partial}{\partial y}\log\psi\right)\zeta-\frac{\psi}{n-1}g^{\lambda\mu}\left(2\psi_{,\lambda}\,\xi_\mu+\psi\xi_{\lambda,\mu}\right). \tag{31}
$$

2. The regularization of the system (I). Let us now proceed to the problem to solve system (I) under (II) and the initial conditions. From now on, we shall replace the derivatives and the covariant derivatives of ψ with respects y and Γ^a_{bb} of $V_n(y)$ by those of ζ by means of (26), (27), (28). Notice that the (covariant) derivatives of ψ are polynomials of those of ζ, ζ and ψ. There exist terms with $1/y$ as a factor on the right hand sides of (I_2) and the left hand side of (II_2). We shall endeavor to take off this irregularity of the system of differential equations.

In the first place, according to the course stated above, let us write (I_2) in the following form:

$$
\begin{aligned}
\frac{\partial}{\partial y}h^a_b&=\frac{n-1}{y}\left(h^a_b+\psi\zeta\delta^a_b\right)+\psi\left(h\,h^a_b-R^a_b\right)\\
&-\frac{\psi}{2(n-1)}\delta^a_b\left(h^2-h^\mu_\lambda h^\lambda_\mu-R\right)-\psi\,\delta^a_b\frac{\partial\zeta}{\partial y}\\
&-y\,\psi^3 g^{a\lambda}\zeta_{,b\lambda}+3y^2\psi^5 g^{a\lambda}\zeta_{,b}\zeta_{,\lambda}.
\end{aligned}
\tag{I_2''}
$$

Putting

$$
h^a_b=-\psi\,\zeta\delta^a_b+\sum_{i=1}^{n-1}y^i\underset{(i)}{H}{}^a_b \tag{32}
$$

let us *determine* $\underset{(i)}{H}{}^a_b$ ($i=1, 2, \ldots, n-2$) from the last relation, *so that these* *quantities are polynomials of* $g^{\lambda\mu}$, χ, ψ *and derivatives of* χ, $g^{\lambda\mu}$, *and the differential equations with respect to the unknown quantities* $\underset{(n-1)}{H}{}^a_b$ *replaced for* (I_2) *become regular forms as much as possible.*

For convenience, let us put

$$\underset{(0)}{H}{}^a_b = -\psi\chi\delta^a_b. \tag{33}$$

Substituting (32) in (I''_2) and using (26), (27), (28), we get

$$\begin{aligned} \frac{\partial}{\partial y} h^a_b &= -\delta^a_b\left\{\psi\frac{\partial\chi}{\partial y} - \psi^3\left(\chi^2 + y\chi\frac{\partial\chi}{\partial y}\right)\right\} + \underset{(1)}{H}{}^a_b \\ &+ \sum_{i=1}^{n-2} y^i\left\{(i+1)\underset{(i+1)}{H}{}^a_b + \frac{\partial}{\partial y}\underset{(i)}{H}{}^a_b\right\} + y^{n-1}\frac{\partial}{\partial y}\underset{(n-1)}{H}{}^a_b \\ &= (n-1)\sum_{i=0}^{n-2} y^i \underset{(i+1)}{H}{}^a_b - \psi\left(R^a_b - \frac{R}{2(n-1)}\delta^a_b + \frac{\partial\chi}{\partial y}\delta^a_b\right) \\ &+ \frac{n}{2}\psi^3\chi^2\delta^a_b - y\psi^3 g^{a\lambda}\chi_{,b\lambda} + 3y^2\psi^5 g^{a\lambda}\chi_{,b}\chi_{,\lambda} \\ &+ \psi\sum_{k=1}^{2n-2} y^k \sum_{i+j=k}^{i,j\geqq 0}\left\{\underset{(i)}{H}\underset{(j)}{H}{}^b_c + \frac{1}{2(n-1)}\delta^a_b\left(\underset{(i)}{H}{}^\mu_\lambda\underset{(j)}{H}{}^\lambda_\mu - \underset{(i)}{H}\underset{(j)}{H}\right)\right\}, \end{aligned} \tag{34}$$

where $\underset{(i)}{H} = \underset{(i)}{H}{}^\lambda_\lambda$. Furtheremore, let us put

$$\frac{\partial}{\partial y}\underset{(i)}{H}{}^a_b = \sum_{j=0}^{2n-2-i} y^j \underset{(i,j)}{K}{}^a_b \qquad (i = 0, 1, 2, \ldots, n-2) \tag{35}$$

and *determine* $\underset{(i,j)}{K}{}^a_b$ $(i, j = 0, 1, 2, \cdots, n-3;\ i+j \leqq n-3)$ *as polynomials of* $g^{\lambda\mu}$, χ, ψ *and derivatives of* $g_{\lambda\mu}$, χ *according to the same principle of determining* $\underset{(i)}{H}{}^a_b$ *and do this simultaneously with those of* $\underset{(i)}{H}{}^a_b$.

Now, the constant terms with respect to y on both sides of (34) cancel out with each other when we define $\underset{(1)}{H}{}^a_b$ by the relation

$$\psi^3\chi^2\delta^a_b + \underset{(1)}{H}{}^a_b = (n-1)\underset{(1)}{H}{}^a_b - \psi\left(R^a_b - \frac{R}{2(n-1)}\delta^a_b\right) + \frac{n}{2}\psi^3\chi^2\delta^a_b,$$

that is

$$\underset{(1)}{H}{}^a_b = \frac{\psi}{n-2}\left(R^a_b - \frac{R}{2(n-1)}\delta^a_b\right) - \frac{1}{2}\psi^3\chi^2\delta^a_b. \tag{36}$$

From the last equation we get at once

(36′) $$\underset{(1)}{H} = \underset{(1)}{H}{}^{\lambda}_{\lambda} = \frac{\psi \mathrm{R}}{2(n-1)} - \frac{n}{2}\psi^3 \zeta^2.$$

Form (33), (26), we obtain easily the following relations:

(37) $$\underset{(0,0)}{K}{}^{a}_{b} = -\left(\psi \frac{\partial \zeta}{\partial y} - \psi^3 \zeta^2\right)\delta^a_b,$$

$$\underset{(0,1)}{K}{}^{a}_{b} = \psi^3 \zeta \frac{\partial \zeta}{\partial y}\delta^a_b, \quad \underset{(0,j)}{K}{}^{a}_{b} = 0 \qquad (i > 1).$$

Now, comparing the coefficients of y on the left hand side of (3) with those of the right hand side, let us define $\underset{(2)}{H}{}^a_b$ by the relation

(38) $$\psi^3 \zeta \frac{\partial \zeta}{\partial y}\delta^a_b + 2\underset{(2)}{H}{}^a_b + \underset{(1,0)}{K}{}^{a}_{b} = (n-1)\underset{(2)}{H}{}^a_b - \psi^3 g^{a\lambda}\zeta_{,b\lambda}$$
$$+ \psi\left\{\underset{(1)}{H}\underset{(0)}{H}{}^a_b + \underset{(0)}{H}\underset{(1)}{H}{}^a_b + \frac{1}{2(n-1)}\delta^a_b(\underset{(0)}{H}{}^{\mu}_{\lambda}\underset{(1)}{H}{}^{\lambda}_{\mu} - \underset{(0)}{H}\underset{(1)}{H})\right\},$$

whose right hand side becomes by virtue of (33)

$$= (n-1)\underset{(2)}{H}{}^a_b - \psi^3 g^{a\lambda}\zeta_{,b\lambda} - n\,\psi^2 \zeta \underset{(1)}{H}{}^a_b.$$

Accordingly, we have

$$(n-3)\underset{(2)}{H}{}^a_b = \underset{(1,0)}{K}{}^{a}_{b} + \psi^3\left(\zeta \frac{\partial \zeta}{\partial y}\delta^a_b + g^{a\lambda}\zeta_{,b\lambda}\right) + n\,\psi^2 \zeta \underset{(1)}{H}{}^a_b.$$

In the last equation we have assumed that $\underset{(1,0)}{K}{}^{a}_{b}$ has already been determined for $n > 3$.

On the other hand, if we make use of the geodesic coordinates of the Riemannian space V_n (y) with line element

$$ds^2 = g_{\lambda\mu}(x, y)\, dx^\lambda dx^\mu,$$

we get easily by (10), (I_1)

$$\frac{\partial}{\partial y}\mathrm{R}^a_b = \frac{\partial}{\partial y}\mathrm{R}_b{}^{\lambda a}{}_{\lambda} = \frac{\partial}{\partial y}(g^{a\mu}\mathrm{R}_b{}^{\lambda}{}_{\mu\lambda})$$
$$= 2\psi h^{a\mu}\mathrm{R}_{b\mu}$$
$$+ g^{a\mu}\frac{\partial}{\partial y}\left\{\frac{\partial \Gamma^{\lambda}_{b\mu}}{\partial x^\lambda} - \frac{\partial \Gamma^{\lambda}_{b\lambda}}{\partial x^\mu} + \Gamma^{\rho}_{b\mu}\Gamma^{\lambda}_{\rho\lambda} - \Gamma^{\rho}_{b\lambda}\Gamma^{\lambda}_{\rho\mu}\right\}$$
$$= 2\psi h^a_\lambda \mathrm{R}^\lambda_b + g^{a\mu}\left(\frac{\partial}{\partial y}\Gamma^{\lambda}_{b\mu}\right)_{,\lambda} - g^{a\mu}\left(\frac{\partial}{\partial y}\Gamma^{\lambda}_{b\lambda}\right)_{,\mu}.$$

Putting (24) into the last relation, we obtain

$$
\frac{\partial}{\partial y} R^a_b = 2\psi h^a_\lambda R^\lambda_b + g^{\lambda\mu}(\psi h^a_b)_{,\lambda\mu} - g^{a\mu}(\psi h^\lambda_b)_{,\mu\lambda} - (\psi h^{a\lambda})_{,b\lambda} + g^{a\lambda}(\psi h)_{,b\lambda}. \tag{39}
$$

From (39) we get also (30) by contraction.

Now, in order to determine $\underset{(1,j)}{K}{}^a_b$, we put (26), (27), (28) into the right hand side of (39), and get

$$
\begin{aligned}
\frac{\partial}{\partial y} R^a_b = {} & 2\psi h^a_\lambda R^\lambda_b + g^{\lambda\mu}\psi_{,\lambda\mu} h^a_b + 2g^{\lambda\mu}\psi_{,\lambda} h^a{}_{b,\mu} + \psi g^{\lambda\mu} h^a_{b,\lambda\mu} \\
& - g^{a\mu}\psi_{,\mu\lambda} h^\lambda_b - g^{a\mu}\psi_{,\mu\lambda} h^\lambda_{b,\lambda} - g^{a\mu}\psi_{,\lambda} h^\lambda_{b,\mu} - \psi g^{a\mu} h^\lambda_{b,\mu\lambda} \\
& - \psi_{,b\lambda} h^{a\lambda} - \psi_{,b} h^{a\lambda}{}_{,\lambda} - \psi_{,\lambda} h^{a\lambda}{}_{,b} - \psi h^{a\lambda}{}_{,b\lambda} \\
& + g^{a\lambda}\psi_{,b\lambda} h + g^{a\lambda}\psi_{,b} h_{,\lambda} + g^{a\lambda}\psi_{,\lambda} h_{,b} + \psi g^{a\lambda} h_{,b\lambda} \\
\sim {} & -2\psi^2 \zeta R^a_b + 2y\psi \underset{(1)}{H}{}^a_\lambda R^\lambda_b \\
& + y\psi^4 g^{\lambda\mu}\zeta_{,\lambda\mu}\zeta \delta^a_b + 2y\psi^4 g^{\lambda\mu}\zeta_{,\lambda}\zeta_{,\mu}\delta^a_b \\
& + \psi g^{\lambda\mu}\Big[\Big\{-\psi\zeta_{,\lambda\mu} + y\psi^3(\zeta\zeta_{,\lambda\mu} + 2\zeta_{,\lambda}\zeta_{,\mu})\Big\}\delta^a_b + y\underset{(1)}{H}{}^a_{b,\lambda\mu}\Big] \\
& - y\psi^4 g^{a\mu}\zeta_{,\mu b}\zeta - 2y\psi^4 g^{a\mu}\zeta_{,\mu}\zeta_{,b} \\
& - \psi g^{a\mu}\Big[\Big\{-\psi\zeta_{,\mu b} + y\psi^3(\zeta\zeta_{,\mu b} + 2\zeta_{,\mu}\zeta_{,b})\Big\} + y\underset{(1)}{H}{}^\lambda_{b,\mu\lambda}\Big] \\
& - y\psi^4 g^{a\lambda}\zeta_{,b\lambda}\zeta - 2y\psi^4 g^{a\lambda}\zeta_{,\lambda}\zeta_{,b} \\
& - \psi\Big[g^{a\lambda}\Big\{-\psi\zeta_{,b\lambda} + y\psi^3(\zeta\zeta_{,b\lambda} + 2\zeta_{,b}\zeta_{,\lambda})\Big\} + y\underset{(1)}{H}{}^{a\lambda}{}_{,b\lambda}\Big] \\
& + ny\psi^4 g^{a\lambda}\zeta_{,b\lambda}\zeta + 2ny\psi^4 g^{a\lambda}\zeta_{,b}\zeta_{,\lambda} \\
& + \psi g^{a\lambda}\Big[n\Big\{-\psi\zeta_{,b\lambda} + y\psi^3(\zeta\zeta_{,b\lambda} + 2\zeta_{,b}\zeta_{,\lambda})\Big\} + y\underset{(1)}{H}{}_{,b\lambda}\Big],
\end{aligned}
$$

where $\sim$ denotes an equality within terms of the second orders with respect to y. From the last relation we get

$$
\begin{aligned}
\frac{\partial}{\partial y} R^a_b \sim {} & \psi^3\Big[-2\zeta R^a_b - \delta^a_b \Delta_2(\zeta) - (n-2) g^{ab}\zeta_{,\lambda b}\Big] \\
& + y\Big[2\psi \underset{(1)}{H}{}^a_\lambda R^\lambda_b + 2\psi^4\zeta\Big\{\Delta_2(\zeta)\delta^a_b + (n-2)g^{a\lambda}\zeta_{,\lambda b}\Big\} \\
& + 4\psi^4\Big\{\Delta_1(\zeta)\delta^a_b + (n-2)g^{a\lambda}\zeta_{,\lambda}\zeta_{,b}\Big\} \\
& + \psi\Big\{g^{\lambda\mu}\underset{(1)}{H}{}^a_{b;\lambda\mu} - g^{a\mu}\underset{(1)}{H}{}^\lambda_{b;\mu\lambda} - \underset{(1)}{H}{}^{a\lambda}{}_{;b\lambda} + g^{a\lambda}\underset{(1)}{H}{}_{;b\lambda}\Big\}\Big],
\end{aligned} \tag{39'}
$$

where Δ_1 and Δ_2 denote the Bertrami's differential parameters of the first order and the second order:

$$\Delta_1(z) = g^{\lambda\mu} z_{,\lambda} z_{,\mu}$$
$$\Delta_2(z) = g^{\lambda\mu} z_{,\lambda\mu}.$$

and *semicolons denote the covariant differentiation for quantities depending explicitly on* ψ *regarding it as a constant.* From (39′) we get

$$\begin{aligned}\frac{\partial}{\partial y} R \sim & - \psi^2 \left\{2z R + 2(n-1)\Delta_2(z)\right\} \\ (30') \qquad & + y\Big[2\psi \underset{(1)}{H^\mu_\lambda} R^\lambda_\mu + 4(n-1)\psi^4 z \Delta_2(z) \\ & + 8(n-1)\psi^4 \Delta_1(z) + 2\psi \left\{g^{\lambda\mu} \underset{(1)}{H}_{;\lambda\mu} - \underset{(1)}{H}^{\lambda\mu}{}_{;\lambda\mu}\right\}\Big].\end{aligned}$$

After these preparations, we get by (36)

$$\begin{aligned}\frac{\partial}{\partial y} \underset{(1)}{H^a_b} = & \frac{\psi}{n-2}\left(\frac{\partial}{\partial y} R^a_b - \frac{1}{2(n-1)}\delta^a_b \frac{\partial}{\partial y} R\right) - \psi^3 z \frac{\partial z}{\partial y}\delta_a \\ & - \psi^3\left\{\frac{1}{n-2}\left(R^a_b - \frac{R}{2(n-1)}\delta^a_b\right) - \frac{3}{2}\psi^2 z^2 \delta^a_b\right\}\left(z + y\frac{\partial z}{\partial y}\right) \\ \sim & \frac{\psi^3}{n-2}\left[-2z\left(R^a_b - \frac{R}{2(n-1)}\delta^a_b\right) - (n-2) g^{a\lambda} z_{,\lambda b}\right] \\ & + \frac{y\psi}{n-2}\Big[2\psi\left(\underset{(1)}{H^a_\lambda} R^\lambda_b - \frac{1}{2(n-1)} \underset{(1)}{H^\mu_\lambda} R^\lambda_\mu \delta^a_b\right) \\ & \qquad + 2(n-2)\psi^4 g^{a\lambda}(z z_{,\lambda b} + 2 z_{,\lambda} z_{,b}) \\ & \qquad + \psi\left(g^{\lambda\mu} \underset{(1)}{H^a_{b;\lambda\mu}} - g^{a\mu} \underset{(1)}{H^\lambda_{b;\mu\lambda}} - \underset{(1)}{H}^{a\lambda}{}_{;b\lambda} + g^{a\lambda} \underset{(1)}{H}_{;b\lambda}\right) \\ & \qquad - \frac{\psi}{n-1}\delta^a_b\left(g^{\lambda\mu} \underset{(1)}{H}_{;\lambda\mu} - \underset{(1)}{H}^{\lambda\mu}{}_{;\lambda\mu}\right)\Big] \\ & - \psi^3 z \frac{\partial z}{\partial y}\delta^a_b \\ & - \psi^3 z \left\{\frac{1}{n-2}\left(R^a_b - \frac{R}{2(n-1)}\delta^a_b\right) - \frac{3}{2}\psi^2 z^2 \delta^a\right\} \\ & - y\psi^3 \frac{\partial z}{\partial y}\left\{\frac{1}{n-2}\left(R^a_b - \frac{R}{2(n-1)}\delta^a_b\right) - \frac{3}{2}\psi^2 z^2 \delta^a_b\right\}.\end{aligned}$$

Hence, let us put

$$\begin{aligned}\underset{(1,0)}{K^a_b} = & -\frac{3}{n-2}\psi^3 z\left(R^a_b - \frac{R}{2(n-1)}\delta^a_b\right) \\ (40) \qquad & - \psi^3\left(g^{a\lambda} z_{,\lambda b} + z\frac{\partial z}{\partial y}\delta^a_b - \frac{3}{2}\psi^2 z^3 \delta^a_b\right) \\ = & -3\psi^2 z \underset{(1)}{H^a_b} - \psi^3\left(g^{a\lambda} z_{,\lambda b} + z\frac{\partial z}{\partial y}\delta^a_b\right),\end{aligned}$$

$$
\begin{aligned}
(41)\qquad \underset{(1,1)}{K}{}^{a}_{b} = {} & \frac{2\psi^2}{n-2}\Big(\underset{(1)}{H}{}^{a}_{\lambda} R^{\lambda}_{b} - \frac{1}{2(n-1)}\,\delta^a_b\, \underset{(1)}{H}{}^{\mu}_{\lambda} R^{\lambda}_{\mu}\Big) \\
& + 2\psi^5 g^{a\lambda}(\zeta\,\zeta_{,\lambda b} + 2\zeta_{,\lambda}\,\zeta_{,b}) \\
& + \frac{\psi^2}{n-2}\Big\{g^{\lambda\mu}\underset{(1)}{H}{}^{a}_{b;\lambda\mu} - g^{a\mu}\underset{(1)}{H}{}^{\lambda}_{b;\mu\lambda} - \underset{(1)}{H}{}^{a\lambda}{}_{;b\lambda} + g^{a\lambda}\underset{(1)}{H}{}_{;b\lambda} \\
& \qquad - \frac{1}{n-1}\,\delta^a_b\Big(g^{\lambda\mu}\underset{(1)}{H}{}_{;\lambda\mu} - \underset{(1)}{H}{}^{\lambda\mu}{}_{;\lambda\mu}\Big)\Big\} \\
& - \psi^3\frac{\partial\zeta}{\partial y}\Big\{\frac{1}{n-2}\Big(R^a_b - \frac{R}{2(n-1)}\,\delta^a_b\Big) - \frac{3}{2}\,\psi^2\zeta^2\delta^a_b\Big\}.
\end{aligned}
$$

Then, by means of (40) we have $\underset{(2)}{H}{}^a_b$ as follows:

$$
\begin{aligned}
(n-3)\,\underset{(2)}{H}{}^a_b = {} & -3\psi^2\zeta\,\underset{(1)}{H}{}^a_b - \psi^3\Big(g^{a\lambda}\zeta_{,\lambda b} + \zeta\frac{\partial\zeta}{\partial y}\,\delta^a_b\Big) \\
& + \psi^3\Big(\zeta\frac{\partial\zeta}{\partial y}\,\delta^a_b + g^{a\lambda}\zeta_{,b\lambda}\Big) + n\,\psi^2\zeta\,\underset{(1)}{H}{}^a_b \\
= {} & (n-3)\,\psi^2\zeta\,\underset{(1)}{H}{}^a_b.
\end{aligned}
$$

Accordingly, if $n > 3$, we define $\underset{(2)}{H}{}^a_b$ by

$$
(42)\qquad \underset{(2)}{H}{}^a_b = \psi^2\zeta\,\underset{(1)}{H}{}^a_b.
$$

Now, from the last relation we get

$$
\frac{\partial}{\partial y}\underset{(2)}{H}{}^a_b = \psi^2\zeta\frac{\partial}{\partial y}\underset{(1)}{H}{}^a_b + \psi^2\,\underset{(1)}{H}{}^a_b\frac{\partial\zeta}{\partial y} + 2\psi\zeta\,\underset{(1)}{H}{}^a_b\frac{\partial\psi}{\partial y}.
$$

By means of (29), let us put

$$
\begin{aligned}
(43)\qquad \underset{(2,0)}{K}{}^a_b &= \psi^2\Big(\frac{\partial\zeta}{\partial y} - 2\psi^2\zeta^2\Big)\underset{(1)}{H}{}^a_b + \psi^2\zeta\,\underset{(1,0)}{K}{}^a_b \\
&= \psi^2\Big(\frac{\partial\zeta}{\partial y} - 5\psi^2\zeta^2\Big)\underset{(1)}{H}{}^a_b - \psi^5\zeta\Big(g^{a\lambda}\zeta_{,\lambda b} + \zeta\frac{\partial\zeta}{\partial y}\,\delta^a_b\Big).
\end{aligned}
$$

Lastly, comparing the coefficients of y^2 on both sides of (34) with each other, we have the relation

$$
\begin{aligned}
3\,\underset{(3)}{H}{}^a_b + \underset{(2,0)}{K}{}^a_b + \underset{(1,1)}{K}{}^a_b = {} & (n-1)\,\underset{(3)}{H}{}^a_b + 3\psi^5 g^{a\lambda}\zeta_{,\lambda}\zeta_{,b} \\
& + \psi\sum_{i+j=2}\Big\{\underset{(i)}{H}\,\underset{(j)}{H}{}^a_b + \frac{1}{2(n-1)}\,\delta^a_b\,(\underset{(i)}{H}{}^{\mu}_{\lambda}\underset{(j)}{H}{}^{\lambda}_{\mu} - \underset{(i)}{H}\,\underset{(j)}{H})\Big\}.
\end{aligned}
$$

Accordingly, if $n > 4$, by means of (36), (41), (42), (43), we define $\underset{(3)}{H}{}^a_b$ by the relation

$$\text{(44)} \qquad \underset{(3)}{H}{}^a_b = \frac{1}{n-4}\Big[\underset{(2,0)}{K}{}^a_b + \underset{(1,1)}{K}{}^a_b - 3\psi^5 g^{a\lambda} z_{,\lambda} z_{,b} - \psi \sum_{i+j=2} \Big\{\underset{(i)}{H}\underset{(j)}{H}{}^a_b + \frac{1}{2(n-1)} \delta^a_b (\underset{(i)}{H}{}^\mu_\lambda \underset{(j)}{H}{}^\lambda_\mu - \underset{(i)}{H}\underset{(j)}{H})\Big\}\Big].$$

3. Tensor L^a_b. According to the results of the last section, suppose inductively that we have been able to define $\underset{(i)}{H}{}^a_b$, $\underset{(s,j-s)}{K}{}^a_b$ for $p \leq n-3$ so that

$$\text{(45)} \qquad \underset{(i)}{H}{}^a_b = \underset{(i)}{H}{}^a_b\Big(g^{\lambda\mu};\ R^\lambda_\mu;\ \ldots;\ R^\lambda_{\mu,\rho_1\cdots\rho_{i-1}};\ \psi;\ z;\ \ldots;\ \Big(\frac{\partial^k z}{\partial y^k}\Big)_{,\rho_1\cdots\rho_h};\ \ldots\Big)$$
$$(i = 1,\ 2,\ \ldots,\ p;\ k+h \leq i-1)$$

and

$$\text{(46)} \qquad \underset{(s,j-s)}{K}{}^a_b = \underset{(s,j-s)}{K}{}^a_b\Big(g^{\lambda\mu};\ R^\lambda_\mu;\ \ldots;\ R^\lambda_{\mu,\rho_1\cdots\rho_{i-1}};\ \psi;\ z;\ \ldots;\ \Big(\frac{\partial^k z}{\partial y^k}\Big)_{,\rho_1\cdots\rho_h};\ \ldots\Big)$$
$$(i = 1,\ 2,\ \ldots,\ p-1;\ s=1,\ 2,\ \ldots,\ j;\ k+h \leq j)$$

which are polynomials of the quantities enclosed in round brackets as shown above, and the coefficients of y^i $(i = 1, 2, \ldots, p-1)$ on both sides of (34) are equal to each other. We may suppose here $p \geq 3$. Then, comparing the coefficients of y^p on both sides of (34) with each other, we define $\underset{(p+1)}{H}{}^a_b$ by the equation

$$(p+1)\underset{(p+1)}{H}{}^a_b + \sum_{s=1}^{p} \underset{(s,p-s)}{K}{}^a_b = (n-1)\underset{(p+1)}{H}{}^a_b + \psi \sum_{i=0}^{p} \Big\{\underset{(i)}{H}\underset{(p-i)}{H}{}^a_b + \frac{1}{2(n-1)} \delta^a_b (\underset{(i)}{H}{}^\mu_\lambda \underset{(p-i)}{H}{}^\lambda_\mu - \underset{(i)}{H}\underset{(p-i)}{H})\Big\},$$

that is

$$\text{(47)} \qquad (n-p-2)\underset{(p+1)}{H}{}^a_b = \sum_{s=1}^{p} \underset{(s,p-s)}{K}{}^a_b - \psi \sum_{i=0}^{p} \Big[\underset{(i)}{H}\underset{(p-i)}{H}{}^a_b + \frac{1}{2(n-1)} \delta^a_b (\underset{(i)}{H}{}^\mu_\lambda \underset{(p-i)}{H}{}^\lambda_\mu - \underset{(i)}{H}\underset{(p-i)}{H})\Big],$$

since $n-p-2 > 0$. In the last equation, the quantities enclosed in square brackets have been alreaddy defined by the hypothesis of induction. We assume that $\underset{(s,p-s)}{K}{}^{a}_{b}$ have been defined too.

On the other hand, since $\underset{(s)}{H}{}^{a}_{b}$ $(s=1, 2, \dots, p)$ are known quantities, we have

$$\begin{aligned}
\frac{\partial}{\partial y}\underset{(s)}{H}{}^{a}_{b} &= \sum_{\lambda\leq\mu} 2\psi\, h^{\lambda\mu}\left(\partial \underset{(s)}{H}{}^{a}_{b} \quad \partial g^{\lambda\mu}\right) \\
&+ \sum_{\substack{\lambda\ \mu \\ 0\leq k<s\leq 1}} \left(\partial \underset{(s)}{H}{}^{a}_{b} \Big/ \partial R^{\lambda}_{\mu,\rho_1\cdots\rho_k}\right)\frac{\partial}{\partial y} R^{\lambda}_{\mu\,\rho_1\cdots\rho_k} \\
&+ \sum_{j+t\leq s-1}\left(\partial \underset{(s)}{H}{}^{a}_{b} \quad \partial\left\{\left(\frac{\partial^j z}{\partial y^j}\right)_{,\rho_1\cdots\rho_t}\right\}\right)\frac{\partial}{\partial y}\left\{\left(\frac{\partial z}{\partial y^j}\right)_{,\rho_1\cdots\rho_t}\right\} \\
&- \psi^3\left(z + y\frac{\partial z}{\partial y}\right)\partial \underset{(s)}{H}{}^{a}_{b} \Big/ \partial\psi.
\end{aligned} \tag{48}$$

However, by (24), (39) and the relation

$$\begin{aligned}
\frac{\partial}{\partial y} R^{\lambda}_{\mu,\,\rho_1\cdots\rho_k} &= \left(\frac{\partial}{\partial y} R^{\lambda}_{\mu\,,\rho_1\cdots\rho_{k-1}}\right)_{,\rho_k} + \left(\frac{\partial}{\partial y}\Gamma^{\lambda}_{\nu\,\rho_k}\right) R^{\nu}_{\lambda,\,\rho_1\cdots\rho_{k-1}} \\
&- \left(\frac{\partial}{\partial y}\Gamma^{\nu}_{\mu\rho_k}\right) R^{\lambda}_{\nu,\rho_1\cdots\rho_{k-1}} - \sum_{h=1}^{k-1}\left(\frac{\partial}{\partial y}\Gamma^{\nu}_{\rho_h\,\rho_k}\right) R^{\lambda}_{\mu,\,\rho_1\cdots\rho_{k-1}\,\nu\,\rho_{h+1}\cdots\rho_{k-1}},
\end{aligned}$$

we can easily see by induction that $\frac{\partial}{\partial y} R^{\lambda}_{\mu,\rho_1\cdots\rho_k}$ is a linear form of h^{τ}_{ν}; h^{τ}_{ν,α_1}; $\dots$; $h^{\tau}_{\nu,\alpha_1\cdots\alpha_{k+2}}$ whose coefficients are polynomials of $g^{\tau\nu}$; R^{τ}_{ν}; $\dots$; $R^{\tau}_{\nu,\alpha_1\cdots\alpha_k}$; ψ; $\psi_{,\alpha_1}$; $\dots$; $\psi_{,\alpha_1\cdots\alpha_{k+2}}$. While, by means of analogous relations as (26), (27), (28) derived from (4), the coefficients stated above are polynomials of $g^{\tau\nu}$; R^{τ}_{ν}; $\dots$; $R^{\tau}_{\nu,\alpha_1\cdots\alpha_k}$; z; $z_{,\alpha_1}$; $\dots$; $z_{,\alpha_1\cdots\alpha_{k+2}}$; y.

Then, for $j+t\leq s-1$, we have

$$\begin{aligned}
\frac{\partial}{\partial y}\left\{\left(\frac{\partial^j z}{\partial y^j}\right)_{,\rho_1\cdots\rho_t}\right\} &= \left[\frac{\partial}{\partial y}\left\{\left(\frac{\partial^j z}{\partial y^j}\right)_{,\rho_1\cdots\rho_{t-1}}\right\}\right]_{,\rho_t} \\
&- \sum_{h=1}^{t-1}\left(\frac{\partial}{\partial y}\Gamma^{\nu}_{\rho_h\,\rho_t}\right)\left(\frac{\partial^j z}{\partial y^j}\right)_{,\rho_1\cdots\rho_{h-1}\,\nu\,\rho_{n+1}\cdots\rho_{t+1}}
\end{aligned}$$

and from this we can easily see that

$$\frac{\partial}{\partial y}\left\{\left(\frac{\partial^j z}{\partial y^z}\right)_{,\rho_1\cdots\rho_t}\right\} = \left(\frac{\partial^{j+1} z}{\partial y_{j+1}}\right)_{,\rho_1\cdots\rho_t} +$$

a linear form of $\left(\frac{\partial^j z}{\partial y^j}\right)_{,\alpha_1}$; $\dots$; $\left(\frac{\partial^j z}{\partial y^j}\right)_{,\alpha_1\cdots\alpha_{t-1}}$ whose coefficients are linear forms of $h^{\lambda}_{\mu}, \ldots, h^{\lambda}_{\mu,\alpha_1\cdots\alpha_{t-1}}$ with polynomials of ψ; z; $z_{,\alpha_1}$; $\ldots$; $z_{,\alpha_1\cdots\alpha_{t-1}}$; y as their coefficients.

After these preparations, let us consider the question whether we can determine $\underset{(s,\ p-s)}{K}{}^a_b$ or not, so that it has the properties shown in (46), provided that p is replaced by $p+$

Now, in the terms included in the first Σ on the right hand side of (48), $\underset{(h-s)}{H}{}^{\lambda}_{\mu}$ are noteworthy and other quantities are supposed to be known by the hypothesis of induction. In the terms included in the second Σ the quantities to be noticed are $\underset{(p-s)}{H}{}^{\tau}_{\nu;\alpha_1\cdots\alpha_{s+1}}$ and the orders of differentiations of R^{μ}_{λ}, z included in these quantities are clearly $\leq (p-s-1)+(s+1)=p$ by (45). Regarding the terms included in the third Σ, the quantities $\underset{(p-s)}{H}{}^{\lambda}_{\mu;\alpha_1\cdots\alpha_{t-1}}$ are to be noticed, but the orders of differentiations of R^{τ}_{ν}, z included in these quantities are $(p-s-1)+(t-1)=p+t-s-2<p$ since $0\leq t\leq s-1$. Therefore, we see that $\underset{(s,\ p-s)}{K}{}^a_b$ can be defined by means of the quantities already known by induction according to (48) and formula (35). Hence, we see also that $\underset{(p+1)}{H}{}^a_b$ can be defined by (47) so that it has the properties shown in (45), putting $i\leq p+1$. Thus, we have proved inductively that (45) and (46) hold good for $p=n-2$. In other words, we can difine successively $\underset{(1)}{H}{}^a_b$, $\underset{(2)}{H}{}^a_b$, $\ldots$, $\underset{(n-2)}{H}{}^a_b$ so that the apparent cofficients of $y, y^2, \ldots, y^{n-3}$ in (34) cancel out with each other respectively and they have the properties shown in (45), putting $i=1, 2, \ldots, n-2$.

Lastly, if $n>2$, the coefficient of y^{n-2} reduced on the left hand side of (34) is

$$(n-1)\underset{(n-1)}{H}{}^a_b+\sum_{s=1}^{n-2}\underset{(s,\ n-2-s)}{K}{}^a_b+\epsilon_{n3}\psi^3 z\frac{\delta z}{\delta y}\delta^a_b$$

and the one reduced on the right hand side is

$$(n-1)\underset{(n-1)}{H}{}^a_b+\psi\sum_{i=0}^{n-2}\left\{\underset{(i)}{H}\ \underset{(n-2-i)}{H}{}^a_b+\frac{1}{2(n-1)}\delta^a_b\left(\underset{(i)}{H}{}^{\mu}_{\lambda}\ \underset{(n-2-i)}{H}{}^{\lambda}_{\mu}-\underset{(i)}{H}\ \underset{(n-2-i)}{H}\right)\right\}$$
$$-\epsilon_{n3}\psi^3 g^{a\lambda}z_{,b\lambda}+3\epsilon_{n4}\psi^5 g^{a\lambda}z_{,b}z_{,\lambda},$$

where $\epsilon_{ij}=1\ (i=j)$, $\epsilon_{ij}=0\ (i\neq j)$. Hence, subtracting these quantities, let us define an important tensor by means of the quantities already known as follows:

$$L^a_b=-\epsilon_{n3}\psi^3\left(z\frac{\partial z}{\partial y}\delta^a_b+g^{a\lambda}z_{,b\lambda}\right)+3\epsilon_{n4}\psi^5 g^{a\lambda}z_{,\lambda}z_{,b}-\sum_{s=1}^{n-2}\underset{(s,\ n-2-s)}{K}{}^a_b \tag{49}$$

$$+\psi\sum_{i=0}^{n-2}\left\{\underset{(i)}{H}\,\underset{(n-2-i)}{H}{}^{a}_{b}+\frac{1}{2(n-1)}\,\delta^a_b\,(\underset{(i)}{H}{}^{\mu}_{\lambda}\,\underset{(n-2-i)}{H}{}^{\lambda}_{\mu}-\underset{(i)}{H}\,\underset{(n-2-i)}{H})\right\}.$$

By virtue of (4,5), (4,6) $L^{a}_{b}{}'_{s}$ are polynomials of $g^{\lambda\mu}$; R^{λ}_{μ}; ...; $R^{\lambda}_{\mu,\rho_1\cdots\rho_{n-2}}$; ψ; z; ...; $\left(\frac{\partial^k z}{\partial y^k}\right)_{,\rho_1\cdots\rho_h}$; $(k+h\leq n-2)$. We shall show in future that this tensor plays an important rôle to solve the problem stated in introduction.

4. The regularization of the system (II), (1). Making use of the results of the last sections, (34) becomes

$$\frac{\partial}{\partial y}\underset{(n-1)}{H}{}^{a}_{b}=\frac{1}{y}L^{a}_{b}+\psi\sum_{s=0}^{n-1}y^s\Big[-\sum_{i+j=s+n-1}^{1\leq i\leq n-2}\underset{(i,j)}{K}{}^{a}_{b}$$
$$+\sum_{i+j=s+n-1}^{0\leq i\leq n-1}\left\{\underset{(i)}{H}\,\underset{(j)}{H}{}^{a}_{b}+\frac{1}{2(n-1)}\,\delta^a_b\,(\underset{(i)}{H}{}^{\mu}_{\lambda}\underset{(j)}{H}{}^{\lambda}_{\mu}-\underset{(i)}{H}\,\underset{(j)}{H})\right\}\Big], \tag{50}$$

where $\underset{(i,j)}{K}{}^{a}_{b}$ on the right hand side are linear forms of $\underset{(n-1)}{H}{}^{\lambda}_{\mu}$ and its covariant derivatives determined by the methods in the last section, as is easily shown.

Now, we have by (32)

$$\xi_a\equiv(h_{,a}-h^{\lambda}_{a,\lambda})+(n-1)\,\psi z_{,a}$$
$$=\sum_{i=0}^{n=1}y^i\,(\underset{(i)}{H}_{,a}-\underset{(i)}{H}{}^{\lambda}_{a,\lambda})+(n-1)\,\psi z_{,a}.$$

Since we get by (27)

$$\underset{(i)}{H}{}^{a}_{b,c}=\underset{(i)}{H}{}^{a}_{b;c}-y\psi^3 z_{,c}\frac{\partial}{\partial\psi}\underset{(i)}{H}{}^{a}_{b}\qquad (i=0,\,1,\,2,\,\ldots,\,n-2),$$

let us put

$$\xi_a\equiv\sum_{i=0}^{n-1}y^i\,\underset{(i)}{\xi}_a, \tag{51}$$

where

$$\underset{(i)}{\xi}_a\equiv\underset{(i)}{H}_{;a}-\underset{(i)}{H}{}^{\lambda}_{a;\lambda}-\psi^3\left(z_{,a}\frac{\partial}{\partial\psi}\underset{(i-1)}{H}-z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(i-1)}{H}{}^{\lambda}_{a}\right) \tag{52}$$
$$(i=0,\,1,\,2,\,\ldots,\,n-1)$$

and

$$\underset{(n-1)}{H}{}_{a,c}^{\ b} \equiv \underset{(n-1)}{H}{}_{a;c}^{\ b}.$$

We can easily see by virtue of (33) that

$$(53) \qquad \underset{(0)}{\xi}_a \equiv \underset{(0)}{H}_{;a} - \underset{(0)}{H}{}_{a;\lambda}^{\lambda} + (n-1)\,\psi\chi_{,a} = 0.$$

Now, since we have $\underset{(0)}{H} = -n\psi\chi$, we can put by (32)

$$(54) \qquad \zeta \equiv \frac{1}{y}(h + n\psi\chi) + \frac{\psi}{2(n-1)}(h^2 - h_\lambda^\mu h_\mu^\lambda - \mathrm{R})$$
$$= \sum_{i=0}^{n-1} y^i \underset{(i)}{\zeta},$$

where

$$(55) \qquad \underset{(i)}{\zeta} \equiv \underset{(i+1)}{H} + \frac{\psi}{2(n-1)} \sum_{s=0}^{i} (\underset{(s)}{H}\underset{(i-s)}{H} - \underset{(s)}{H}{}_\lambda^\mu \underset{(i-s)}{H}{}_\mu^\lambda) - \frac{\psi \mathrm{R}\epsilon_{i0}}{2(n-1)}$$
$$(i = 0, 1, 2, \ldots, n-2).$$

We can easily see by (33) and (36) that

$$(56) \qquad \begin{aligned} \underset{(0)}{\zeta} &\equiv \underset{(1)}{H} + \frac{\psi}{2(n-1)}(\underset{(0)}{H}\underset{(0)}{H} - \underset{(0)}{H}{}_\lambda^\mu \underset{(0)}{H}{}_\mu^\lambda - \mathrm{R}) \\ &= \underset{(1)}{H} + \frac{\psi}{2(n-1)}(n(n-1)\psi^2\chi^2 - \mathrm{R}) = 0. \end{aligned}$$

Then, we get by (36) and (52)

$$\begin{aligned} \underset{(1)}{\xi}_a &= \underset{(1)}{H}_{;a} - \underset{(1)}{H}{}_{a;\lambda}^{\lambda} - \psi^3\left(\chi_{,a}\frac{\partial}{\partial\psi}\underset{(0)}{H} - \chi_{,\lambda}\frac{\partial}{\partial\psi}\underset{(0)}{H}{}_a^\lambda\right) \\ &= \frac{\psi}{2(n-1)}\mathrm{R}_{,a} - n\,\psi^3\chi\chi_{,a} \\ &\quad - \frac{\psi}{n-2}\left(\mathrm{R}_{a,\lambda}^{\lambda} - \frac{1}{2(n-1)}\mathrm{R}_{,a}\right) + \psi^3\chi\chi_{,a} + (n-1)\,\psi^3\chi\chi_{,a} \\ &= \frac{\psi}{2(n-2)}(\mathrm{R}_{,a} - 2\mathrm{R}_{a,\lambda}^{\lambda}) = 0. \end{aligned}$$

By means of the relation above and (42) we get likewise

$$\begin{aligned} \underset{(2)}{\xi}_a &= \underset{(2)}{H}_{;a} - \underset{(2)}{H}{}_{a;\lambda}^{\lambda} - \psi^3\left(\chi_{,a}\frac{\partial}{\partial\psi}\underset{(1)}{H} - \chi_{,\lambda}\frac{\partial}{\partial\psi}\underset{(1)}{H}{}_a^\lambda\right) \\ &= \psi^2\chi(\underset{(1)}{H}_{;a} - \underset{(1)}{H}{}_{a;\lambda}^{\lambda}) + \psi^2(\chi_{,a}\underset{(1)}{H} - \chi_{,\lambda}\underset{(1)}{H}{}_a^\lambda) \end{aligned}$$

$$- \psi^2 \left\{\chi_{,a} (\underset{(1)}{H} - n\psi^3\chi^2) - \chi_{,\lambda} (\underset{(1)}{H}{}^{\lambda}_{a} - \psi^3\chi^2\delta^{\lambda}_{a})\right\}$$
$$= \psi^2\chi \left\{\underset{(1)}{H}_{;a} - \underset{(1)}{H}{}^{\lambda}_{a;\lambda} - \psi^3 (-n\chi\chi_{,a} + \chi_{,\lambda}\chi\delta^{\lambda}_{a})\right\}$$
$$= \psi^2\chi\underset{(1)}{\xi}_a = 0.$$

Furthermore, by (33), (36), (42) and (55), we get

$$\underset{(1)}{\zeta} = \underset{(2)}{H} + \frac{\psi}{n-1}(\underset{(0)}{H}\underset{(1)}{H} - \underset{(0)}{H}{}^{\mu}_{\lambda}\underset{(1)}{H}{}^{\lambda}_{\mu})$$
$$= \psi^2\chi\underset{(1)}{H} + \frac{\psi}{n-1}(-n\psi\chi\underset{(1)}{H} + \psi\chi\delta^{\mu}_{\lambda}\underset{(1)}{H}{}^{\lambda}_{\mu}) = 0.$$

Thus we get the relations

$$\underset{(0)}{\xi}_a = \underset{(1)}{\xi}_a = \underset{(2)}{\xi}_a = 0, \quad \underset{(0)}{\zeta} = \underset{(1)}{\zeta} = 0. \tag{57}$$

5. The regularization of the system (II), (2). According to the results of the last section, let us prove by induction the following relations:

$$\underset{(0)}{\xi}_a = \underset{(1)}{\xi}_a = \underset{(2)}{\xi}_a = \dots = \underset{(n-2)}{\xi}_a = 0,$$
$$\underset{(0)}{\zeta} = \underset{(1)}{\zeta} = \underset{(2)}{\zeta} = \dots = \underset{(n-3)}{\zeta} = 0.$$

First, we suppose that the relations

$$\begin{cases}\underset{(0)}{\xi}_a = \underset{(1)}{\xi}_a = \dots = \underset{(p)}{\xi}_a = 0, \\ \underset{(0)}{\zeta} = \underset{(1)}{\zeta} = \dots = \underset{(p-1)}{\zeta} = 0\end{cases} \qquad (2 \leq p < n-2) \tag{58}$$

hold good, and we shall prove that $\underset{(p+1)}{\xi}_a = 0$, $\underset{(p)}{\zeta} = 0$. From (44) and (47) we get

$$\begin{aligned}(n-p-2)\underset{(p+1)}{H}{}^{a}_{b} &= \sum_{s=1}^{p}\underset{(s,p-s)}{K}{}^{a}_{b} - 3\epsilon_{2p}\psi^5 g^{a\lambda}\chi_{,\lambda}\chi_{,b} \\ &- \psi\sum_{s=0}^{p}\left\{\underset{(s)}{H}\underset{(p-s)}{H}{}^{a}_{b} + \frac{1}{2(n-1)}\delta^a_b(\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\lambda}_{\mu} - \underset{(s)}{H}\underset{(p-s)}{H})\right\},\end{aligned} \tag{47'}$$

and hence

$$\begin{aligned}(n-p-2)\underset{(p+1)}{H} &= \sum_{s=1}^{p}\underset{(s,p-s)}{K} - 3\epsilon_{2p}\psi^5\Delta_1(\chi) \\ &- \frac{\psi}{2(n-1)}\sum_{s=0}^{p}\left\{n\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\lambda}_{\mu} + (n-2)\underset{(s)}{H}\underset{(p-s)}{H}\right\}.\end{aligned}$$

Accordingly, we get from these relations

$$(n-p-2)\underset{(p+1)}{H}{}_{;a} = \sum_{s=1}^{p} \underset{(s,p-s)}{K}{}_{;a} \quad 3\epsilon_{2p}\psi^5(\Delta_1(z))_{,a}$$

$$-\frac{\psi}{n-1}\sum_{s=0}^{p}\left\{n\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\lambda}_{\mu;a} + (n-2)\underset{(s)}{H}\underset{(p-s)}{H}{}_{;a}\right\},$$

$$(n-p-2)\underset{(p+1)}{H}{}^{\lambda}_{a;\lambda} = \sum_{s=1}^{p}\underset{(s,p-s)}{K}{}^{\lambda}_{a;\lambda}$$

$$-3\epsilon_{2p}\psi^5(z_{,a}\Delta_2(z) + g^{\lambda\mu}z_{,\lambda}z_{,a\mu})$$

$$-\psi\sum_{s=0}^{p}\left\{\underset{(s)}{H}\underset{(p-s)}{H}{}^{\lambda}_{a;\lambda} + \underset{(s)}{H}{}_{;\lambda}\underset{(p-s)}{H}{}^{\lambda}_{a}\right.$$

$$\left.+\frac{1}{n-1}(\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\mu}_{\lambda;a} - \underset{(s)}{H}\underset{(p-s)}{H}{}_{;a})\right\}$$

and subtracting the latter from the former, we get

$$(n-p-2)\,(\underset{(p+1)}{H}{}_{;a} - \underset{(p+1)}{H}{}^{\lambda}_{a;\lambda}) = \sum_{s=1}^{p}(\underset{(s,p-s)}{K}{}_{;a} - \underset{(s,p-s)}{K}{}^{\lambda}_{a;\lambda})$$

$$-\psi\sum_{s=0}^{p}(\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\lambda}_{\mu;a} + \underset{(s)}{H}\underset{(p-s)}{H}{}_{;a} - \underset{(s)}{H}\underset{(p-s)}{H}{}^{\lambda}_{a;\lambda} - \underset{(s)}{H}{}_{;\lambda}\underset{(p-s)}{H}{}^{\lambda}_{a})$$

$$+3\epsilon_{2p}\psi^5(z_{,a}\Delta_2(z) - g^{\lambda\mu}z_{,\lambda}z_{,\mu a}),$$

The last equation is reduced by (52), (53) and the assumption (58) to

$$(n-p-2)\,(\underset{(p+1)}{H}{}_{;a} - \underset{(p+1)}{H}{}^{\lambda}_{a;\lambda}) = \sum_{s=1}^{p}(\underset{(s,p-s)}{K}{}_{;a} - \underset{(s,p-s)}{K}{}^{\lambda}_{a;\lambda})$$

(59)
$$+\psi\sum_{s=0}^{p}(\underset{(s)}{H}{}_{;\lambda}\underset{(p-s)}{H}{}^{\lambda}_{a} - \underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\lambda}_{\mu;a})$$

$$-\psi^4\sum_{s=0}^{p-1}\underset{(s)}{H}\left(z_{,a}\frac{\partial}{\partial\psi}\underset{(p-s-1)}{H} - z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(p-s-1)}{H}{}^{\lambda}_{a}\right)$$

$$+(n-1)\psi^2 z_{,a}\underset{(p)}{H} + 3\epsilon_{2p}\psi^5(z_{,a}\Delta_2(z) - g^{\lambda\mu}z_{,\lambda}z_{,\mu a}).$$

Now, we get by (27) and (35)

$$\left(\frac{\partial}{\partial v}\underset{(s)}{H}{}^{b}_{a}\right)_{,c} = \left(\sum_{j=0}^{2n-2-s} v^j\underset{(s,j)}{K}{}^{b}_{a}\right)_{,c}$$

$$= \sum_{j\geq 0} v^j\left(\underset{(s,j)}{K}{}^{b}_{a;c} - v\psi^3 z_{,c}\frac{\partial}{\partial\psi}\underset{(s,j)}{K}{}^{b}_{a}\right).$$

Hence, we have

$$(60)\qquad \Big(\frac{\partial}{\partial y}\underset{(s)}{H}\Big)_{,a} - \Big(\frac{\partial}{\partial y}\underset{(s)}{H}{}^{\lambda}_{a}\Big)_{,\lambda} = \sum_{j\geqq 0} y^j\Big\{(\underset{(s,j)}{K}{}_{;a} - \underset{(s,j)}{K}{}^{\lambda}_{a;\lambda}) - y\psi^3\Big(z_{,a}\frac{\partial}{\partial\psi}\underset{(s,j)}{K} - z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(s,j)}{K}{}^{\lambda}_{a}\Big)\Big\}.$$

On the other hand, we get by (24)

$$\Big(\frac{\partial}{\partial y}\underset{(s)}{H}\Big)_{,a} - \Big(\frac{\partial}{\partial y}\underset{(s)}{H}{}^{\lambda}_{a}\Big)_{,\lambda} = \frac{\partial}{\partial y}(\underset{(s)}{H}{}_{,a} - \underset{(s)}{H}{}^{\lambda}_{a,\lambda}) + \underset{(s)}{H}{}^{\lambda}_{a}\frac{\partial}{\partial y}\Gamma^{\mu}_{\lambda\mu} - \underset{(s)}{H}{}^{\mu}_{\lambda}\frac{\partial}{\partial y}\Gamma^{\lambda}_{a\mu}$$
$$= \frac{\partial}{\partial y}(\underset{(s)}{H}{}_{,a} - \underset{(s)}{H}{}^{\lambda}_{a,\lambda}) - \underset{(s)}{H}{}^{\lambda}_{a}(\psi h)_{,\lambda} + \underset{(s)}{H}{}^{\mu}_{\lambda}(\psi h^{\lambda}_{\mu})_{,a},$$

that is

$$(61)\qquad \Big(\frac{\partial}{\partial y}\underset{(s)}{H}\Big)_{,a} - \Big(\frac{\partial}{\partial y}\underset{(s)}{H}{}^{\lambda}_{a}\Big)_{,\lambda} = \frac{\partial}{\partial y}(\underset{(s)}{H}{}_{,a} - \underset{(s)}{H}{}^{\lambda}_{a,\lambda}) + \psi(\underset{(s)}{H}{}^{\mu}_{\lambda}h^{\lambda}_{\mu,a} - \underset{(s)}{H}{}^{\lambda}_{a}h_{,\lambda}) - y\psi^3(z_{,a}\underset{(s)}{H}{}^{\mu}_{\lambda}h^{\lambda}_{\mu} - z_{,\lambda}\underset{(s)}{H}{}^{\lambda}_{a}h).$$

Then, by (52) and the assumption (58), we have the relation

$$\underset{(s)}{H}{}_{,a} - \underset{(s)}{H}{}^{\lambda}_{a,\lambda} = \underset{(s)}{H}{}_{;a} - \underset{(s)}{H}{}^{\lambda}_{a;\lambda} - y\psi^3\Big(z_{,a}\frac{\partial}{\partial\psi}\underset{(s)}{H} - z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(s)}{H}{}^{\lambda}_{a}\Big)$$
$$= \psi^3\Big(z_{,a}\frac{\partial}{\partial\psi}\underset{(s-1)}{H} - z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(s-1)}{H}{}^{\lambda}_{a}\Big) - y\psi^3\Big(z_{,a}\frac{\partial}{\partial\psi}\underset{(s)}{H} - z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(s)}{H}{}^{\lambda}_{a}\Big)\quad (s = 1, 2, \cdots, p).$$

If we introduce as (35) quantities $\underset{(s,j)}{M}{}^{a}_{b}$ analogous to $\underset{(s,j)}{K}{}^{a}_{b}$ by the relation

$$(62)\qquad \frac{\partial}{\partial y}\Big(\frac{\partial}{\partial\psi}\underset{(s)}{H}{}^{a}_{b}\Big) = \sum_{j\geqq 0} y^j\underset{(s,j)}{M}{}^{a}_{b},$$

we can see that $\underset{(s,j)}{M}{}^{a}_{b}$ can be determined successively so that they have the same properties as (46). Hence, making use of these quantities, we have

$$\frac{\partial}{\partial y}(\underset{(s)}{H}{}_{,a} - \underset{(s)}{H}{}^{\lambda}_{a,\lambda})$$

$$
\begin{aligned}
&= \psi^3\Big[\Big(\frac{\partial z}{\partial y}\Big)_{,a}\Big(\frac{\partial}{\partial\psi}\underset{(s-1)}{H} - y\frac{\partial}{\partial\psi}\underset{(s)}{H}\Big) - \Big(\frac{\partial z}{\partial y}\Big)_{,\lambda}\Big(\frac{\partial}{\partial\psi}\underset{(s-1)}{H}{}^{\lambda}_{a} - y\frac{\partial}{\partial\psi}\underset{(s)}{H}{}^{\lambda}_{a}\Big)\Big] \\
&\quad - \psi^3 z_{,a}\Big[3\psi^2\Big(z + y\frac{\partial z}{\partial y}\Big)\Big(\frac{\partial}{\partial\psi}\underset{(s-1)}{H} - y\frac{\partial}{\partial\psi}\underset{(s)}{H}\Big) + \frac{\partial}{\partial\psi}\underset{(s)}{H} \\
(63)\qquad &\quad - \sum_{j\geqq 0} y^j \underset{(s-1,j)}{M} + \sum_{j\geqq 0} y^{j+1}\underset{(s,j)}{M}\Big] \\
&\quad + \psi^3 z_{,\lambda}\Big[3\psi^2\Big(z + y\frac{\partial z}{\partial y}\Big)\Big(\frac{\partial}{\partial\psi}\underset{(s-1)}{H}{}^{\lambda}_{a} - y\frac{\partial}{\partial\psi}\underset{(s)}{H}{}^{\lambda}_{a}\Big) + \frac{\partial}{\partial\psi}\underset{(s)}{H}{}^{\lambda}_{a} \\
&\quad - \sum_{j\geqq 0} y^j \underset{(s-1,j)}{M}{}^{\lambda}_{a} + \sum_{j\geqq 0} y^{j+1}\underset{(s,j)}{M}{}^{\lambda}_{a}\Big].
\end{aligned}
$$

Accordingly, comparing the coefficients of y^{p-s} of the terms on the terms on the right hand sides of (60), (61) with each other and making use of (63), we get the following relations:

$$
\begin{aligned}
&\sum_{s=1}^{p}\Big(\underset{(s,p-s)}{K}{}_{;a} - \underset{(s,p-s)}{K}{}^{\lambda}_{a;\lambda}\Big) \\
&= \psi^3\sum_{s=1}^{p-1}\Big(z_{,a}\frac{\partial}{\partial\psi}\underset{(s,p-s-1)}{K} - z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(s,p-s-1)}{K}{}^{\lambda}_{a}\Big) \\
&\quad + \psi^3\Big[z_{,a}\sum_{s=1}^{p}\Big(\underset{(s-1,p-s)}{M} - \underset{(s,p-s-1)}{M}\Big) + z_{,\lambda}\sum_{s=1}^{p}\Big(\underset{(s-1,p-s)}{M}{}^{\lambda}_{a} - \underset{(s,p-s-1)}{M}{}^{\lambda}_{a}\Big)\Big] \\
&\quad - \psi^3\Big[z_{,a}\sum_{s=1}^{p-1}\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - z_{,\lambda}\sum_{s=1}^{p-1}\underset{(s)}{H}{}^{\lambda}_{a}\underset{(p-s-1)}{H}\Big] \\
&\quad + \psi\sum_{s=1}^{p}\Big(\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\lambda}_{\mu;a} - \underset{(s)}{H}{}^{\lambda}_{a}\underset{(p-s)}{H}{}_{;\lambda}\Big) \\
&\quad - \psi^4\sum_{s=1}^{p-1}\Big(z_{,a}\underset{(s)}{H}{}^{\mu}_{\lambda}\frac{\partial}{\partial\psi}\underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - z_{,\lambda}\underset{(s)}{H}{}^{\lambda}_{a}\frac{\partial}{\partial\psi}\underset{(p-s-1)}{H}\Big) \\
&\quad + \psi^3\Big\{\Big(\frac{\partial z}{\partial y}\Big)_{,a}\frac{\partial}{\partial\psi}\underset{(p-1)}{H} - \Big(\frac{\partial z}{\partial y}\Big)_{,\lambda}\frac{\partial}{\partial\psi}\underset{(p-1)}{H}{}^{\lambda}_{a}\Big\} \\
&\quad - 3\psi^5 z\Big(z_{,a}\frac{\partial}{\partial\psi}\underset{(p-1)}{H} - z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(p-1)}{H}{}^{\lambda}_{a}\Big) \\
&\quad - \psi^5\Big(z_{,a}\frac{\partial}{\partial\psi}\underset{(p)}{H} - z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(p)}{H}{}^{\lambda}_{a}\Big) \\
&\quad - \psi^3\Big\{\Big(\frac{\partial z}{\partial y}\Big)_{,a}\frac{\partial}{\partial\psi}\underset{(p-1)}{H} - \Big(\frac{\partial z}{\partial y}\Big)_{,\lambda}\frac{\partial}{\partial\psi}\underset{(p-1)}{H}{}^{\lambda}_{a}\Big\} \\
&\quad + 3\psi^5\Big\{z_{,a}\Big(z\frac{\partial}{\partial\psi}\underset{(p-1)}{H} - \frac{\partial z}{\partial y}\frac{\partial}{\partial\psi}\underset{(p-2)}{H}\Big)
\end{aligned}
$$

$$- \zeta_{,\lambda}\left(\zeta \frac{\partial}{\partial \psi} \underset{(p-1)}{H}{}^{\lambda}_{a} - \frac{\partial \zeta}{\partial y} \frac{\partial}{\partial \psi} \underset{(p-2)}{H}{}^{a}_{\lambda}\right)\Big\}$$

$$+ 3\psi^5 \frac{\partial \zeta}{\partial y}\left\{\zeta_{,a} \frac{\partial}{\partial \psi} \underset{(p-2)}{H} - \zeta_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p-2)}{H}{}^{\lambda}_{a}\right\}(1 - \epsilon_{2p}),$$

where $\underset{(p,-1)}{M}{}^{a}_{b} = 0$ and we need a factor $(1 - \epsilon_{2p})$ for the last term since $p \geqq 2$, $s = 1, 2, \ldots, p$. It is clear that $\underset{(0,p-1)}{M}{}^{a}_{b} = 0$ from (33). The relation above is readily reduced to

$$\sum_{s=1}^{p}\left(\underset{(s,p-s)}{K}{}_{;a} - \underset{(s,p-s)}{K}{}^{\lambda}_{a;\lambda}\right)$$

$$= \psi^3 \sum_{b=1}^{p}\left(\zeta_{,a} \frac{\partial}{\partial \psi} \underset{(s,p-s-1)}{K} - \zeta_{,\lambda} \frac{\partial}{\partial \psi} \underset{(s,p-s-1)}{K}{}^{\lambda}_{a}\right)$$

$$- \psi^3\left[\zeta_{,a} \sum_{s=1}^{p-1} \underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - \zeta_{,\lambda} \sum_{s=1}^{p-1} \underset{(s)}{H}{}^{\lambda}_{a} \underset{(p-s-1)}{H}\right]$$

$$(64) \qquad + \psi \sum_{s=1}^{p}\left(\underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(p-s)}{H}{}^{\lambda}_{\mu;a} - \underset{(s)}{H}{}^{\lambda}_{a} \underset{(p-s)}{H}{}_{;\lambda}\right)$$

$$- \psi^4 \sum_{s=1}^{p-1}\left(\zeta_{,a} \underset{(s)}{H}{}^{\mu}_{\lambda} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - \zeta_{,\lambda} \underset{(s)}{H}{}^{\lambda}_{a} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H}\right)$$

$$- \psi^3\left(\zeta_{,a} \frac{\partial}{\partial \psi} \underset{(p)}{H} - \zeta_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p)}{H}{}^{\lambda}_{a}\right)$$

$$- 3\epsilon_{2p} \psi^5 \frac{\partial \zeta}{\partial y}\left(\zeta_{,a} \frac{\partial}{\partial \psi} \underset{(p-2)}{H} - \zeta_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p-2)}{H}{}^{\lambda}_{a}\right).$$

Now, substituting (64) in (59) and making use of (47′), (52) and (58), we see that the following relation holds good:

$$(n - p - 2) \underset{(p+1)}{\xi_a} = (n - p - 2)\Big\{\underset{(p+1)}{H}{}_{;a} - \underset{(p+1)}{H}{}^{\lambda}_{a;\lambda}$$

$$- \psi^3\left(\zeta_{,a} \frac{\partial}{\partial \psi} \underset{(p)}{H} - \zeta_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p)}{H}{}^{\lambda}_{a}\right)\Big\}$$

$$= \psi^3 \sum_{s=1}^{p-1}\left(\zeta_{,a} \frac{\partial}{\partial \psi} \underset{(s,p-1)}{K} - \zeta_{,\lambda} \frac{\partial}{\partial \psi} \underset{(s,p-s-1)}{K}{}^{\lambda}_{a}\right)$$

$$- \psi^3\left[\zeta_{,a} \sum_{s=1}^{p-1} \underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - \zeta_{,\lambda} \sum_{s=1}^{p-1} \underset{(s)}{H}{}^{\lambda}_{a} \underset{(p-s-1)}{H}\right]$$

$$
\begin{aligned}
&- \psi \left(\underset{(0)}{H}{}^{\mu}_{\lambda} \underset{(p)}{H}{}^{\lambda}_{\mu;a} - \underset{(0)}{H}{}^{\lambda}_{a} \underset{(p)}{H}{}_{;\lambda} \right) \\
&- \psi^4 \sum_{s=1}^{p-1} \left(\mathfrak{z}_{,a} \underset{(s)}{H}{}^{\mu}_{\lambda} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - \mathfrak{z}_{,\lambda} \underset{(s)}{H}{}^{\lambda}_{a} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H} \right) \\
&- \psi^3 \left(\mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(p)}{H} - \mathfrak{z}_{,\lambda} \underset{(p)}{H}{}^{\lambda}_{a} \right) \\
&- 3\epsilon_{2p} \psi^5 \frac{\partial \mathfrak{z}}{\partial y} \left(\mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(p-2)}{H} - \mathfrak{z}_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p-2)}{H}{}^{\lambda}_{a} \right) \\
&- \psi^4 \sum_{s=0}^{p-1} \underset{(s)}{H} \left(\mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H} - \mathfrak{z}_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H}{}^{\lambda}_{a} \right) + (n-1) \psi^2 \mathfrak{z}_{,a} \underset{(p)}{H} \\
&+ 3\epsilon_{2p} \left(\mathfrak{z}_{,a} \Delta_2(\mathfrak{z}) - g^{\lambda\mu} \mathfrak{z}_{,\lambda} \mathfrak{z}_{,\mu a} \right) \\
&- (n-p-2) \psi^3 \left(\mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(p)}{H} - \mathfrak{z}_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p)}{H}{}^{\lambda}_{a} \right) \\
= {} & \psi^3 \mathfrak{z}_{,a} \left[\frac{1}{2(n-1)} \sum_{s=0}^{p-1} \left\{ n \underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} + (n-2) \underset{(s)}{H} \underset{(p-s-1)}{H} \right\} \right. \\
&+ \frac{1}{n-1} \sum_{s=0}^{p-1} \left\{ n \underset{(s)}{H}{}^{\mu}_{\lambda} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} + (n-2) \underset{(s)}{H} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H} \right\} \\
&\left. - 3\epsilon_{2p} \psi^2 \left(n \mathfrak{z} \frac{\partial \mathfrak{z}}{\partial y} + \Delta_2(\mathfrak{z}) \right) + 15 \epsilon_{3p} \psi^4 \Delta_1(\mathfrak{z}) \right] \\
&- \psi^3 \mathfrak{z}_{,\lambda} \left[\sum_{s=0}^{p-1} \left\{ \underset{(s)}{H} \underset{(p-s-1)}{H}{}^{\lambda}_{a} + \frac{1}{2(n-1)} \delta^{\lambda}_{a} \left(\underset{(s)}{H}{}^{\rho}_{\mu} \underset{(p-s-1)}{H}{}^{\mu}_{\rho} - \underset{(s)}{H} \underset{(p-s-1)}{H} \right) \right\} \right. \\
&+ \psi \sum_{s=0}^{p-1} \left\{ \underset{(s)}{H} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H}{}^{\lambda}_{a} + \underset{(p-s-1)}{H}{}^{\lambda}_{a} \frac{\partial}{\partial \psi} \underset{(s)}{H} \right. \\
&\left. + \frac{1}{(n-1)} \delta^{\lambda}_{a} \left(\underset{(s)}{H}{}^{\rho}_{\mu} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H}{}^{\mu}_{\rho} - \underset{(s)}{H} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H} \right) \right\} \\
&\left. - 3\epsilon_{2p} \psi^2 \left(\mathfrak{z} \frac{\partial \mathfrak{z}}{\partial y} \delta^{\lambda}_{a} + g^{\lambda\mu} \mathfrak{z}_{,a\mu} \right) + 15 \epsilon_{3p} \psi^4 g^{\lambda\mu} \mathfrak{z}_{,\mu} \mathfrak{z}_{,a} \right] \\
&- \psi^3 \left[\mathfrak{z}_{,a} \sum_{s=1}^{p-1} \underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - \mathfrak{z}_{,\lambda} \sum_{s=1}^{p-1} \underset{(s)}{H}{}^{\lambda}_{a} \underset{(p-s-1)}{H} \right] \\
&- \psi^4 \sum_{s=1}^{p-1} \left(\mathfrak{z}_{,a} \underset{(s)}{H}{}^{\mu}_{\lambda} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - \mathfrak{z}_{,\lambda} \underset{(s)}{H}{}^{\lambda}_{a} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H} \right) \\
&- 3\epsilon_{2p} \psi^5 \frac{\partial \mathfrak{z}}{\partial y} \left(\mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(p-2)}{H} - \mathfrak{z}_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p-2)}{H}{}^{\lambda}_{a} \right)
\end{aligned}
$$

$$
\begin{aligned}
&- \psi^4 \sum_{s=0}^{p-1} \underset{(s)}{H}\left(\mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H} - \mathfrak{z}_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p-s-1)}{H}{}^{\lambda}_{a}\right) \\
&+ 3\epsilon_{2p}\, \psi^5 \left(\mathfrak{z}_{,a} \Delta_2(\mathfrak{z}) - g^{\lambda\mu} \mathfrak{z}_{,\lambda} \mathfrak{z}_{,\mu a}\right) + (n-1)\psi^2 \mathfrak{z}_{,a} \underset{(p)}{H} \\
= &- \frac{\psi^3}{2} \mathfrak{z}_{,a} \sum_{s=0}^{p-1} \left(\underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - \underset{(s)}{H} \underset{(p-s-1)}{H} \right) + \psi^3 \mathfrak{z}_{,a} \underset{(0)}{H}{}^{\mu}_{\lambda} \underset{(p-1)}{H}{}^{\lambda}_{\mu} \\
&+ \psi^4 \mathfrak{z}_{,a} \underset{(0)}{H}{}^{\mu}_{\lambda} \frac{\partial}{\partial \psi} \underset{(p-1)}{H}{}^{\lambda}_{\mu} - \psi^3 \mathfrak{z}_{,\lambda} \underset{(0)}{H}{}^{\lambda}_{a} \underset{(p-1)}{H} \\
&- \psi^4 \mathfrak{z}_{,\lambda} \underset{(0)}{H}{}^{\lambda}_{a} \frac{\partial}{\partial \psi} \underset{(p-1)}{H} + (n-1)\psi^2 \mathfrak{z}_{,a} \underset{(p)}{H} \\
&- 3\epsilon_{2p}(n-1)\psi^5 \mathfrak{z} \frac{\partial \mathfrak{z}}{\partial y} \mathfrak{z}_{,a} \\
&- 3\epsilon_{2p}\, \psi^5 \frac{\partial \mathfrak{z}}{\partial y} \left(\mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(p-2)}{H} - \mathfrak{z}_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p-2)}{H}{}^{\lambda}_{a} \right) \\
= &- \frac{\psi^3}{2} \mathfrak{z}_{,a} \sum_{s=0}^{p-1} \left(\underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} - \underset{(s)}{H} \underset{(p-s-1)}{H} \right) + (n-1)\psi^3 \mathfrak{z}_{,a} \underset{(p)}{H} \\
&- 3\epsilon_{2p}(n-1)\psi^5 \mathfrak{z} \frac{\partial \mathfrak{z}}{\partial y} \mathfrak{z}_{,a} \\
&- 3\epsilon_{2p}\, \psi^5 \frac{\partial \mathfrak{z}}{\partial y} \left(\mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(p-2)}{H} - \mathfrak{z}_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p-2)}{H}{}^{\lambda}_{a} \right) \\
= &(n-1)\psi^2 \mathfrak{z}_{,a} \left[\underset{(p)}{H} + \frac{\psi}{2(n-1)} \sum_{s=0}^{p-1} \left(\underset{(s)}{H} \underset{(p-s-1)}{H} - \underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(p-s-1)}{H}{}^{\lambda}_{\mu} \right) \right] \\
&- 3\epsilon_{2p}\, \psi^5 \frac{\partial \mathfrak{z}}{\partial y} \left[(n-1)\mathfrak{z}\mathfrak{z}_{,a} + \mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(p-2)}{H} - \mathfrak{z}_{,\lambda} \frac{\partial}{\partial \psi} \underset{(p-2)}{H}{}^{\lambda}_{a} \right].
\end{aligned}
$$

Since we have from (33)

$$
(n-1)\mathfrak{z}\mathfrak{z}_{,a} + \mathfrak{z}_{,a} \frac{\partial}{\partial \psi} \underset{(0)}{H} - \mathfrak{z}_{,\lambda} \frac{\partial}{\partial \psi} \underset{(0)}{H}{}^{\lambda}_{a} = 0,
$$

we obtain from the relations above by means of the defintion of $\underset{(p-1)}{\zeta}$ (55)

$$
(n-p-2) \underset{(p+1)}{\xi_a} = (n-1)\psi^2 \mathfrak{z}_{,a} \underset{(p-1)}{\zeta}, \tag{65}
$$

from which, by means of the assumption (58), we get lastly the relation

$$
\underset{(p+1)}{\xi_a} = 0.
$$

6. The regularization of the system (II), (3). In this section we shall prove that $\underset{(p)}{\zeta}=0$. We get by (47′), (55) the relation

$$(n-p-2)\underset{(p)}{\zeta}=\sum_{s=1}^{p}\underset{(s,\ p-s)}{K}$$
$$-\frac{\psi}{2(n-1)}\sum_{s=0}^{p}\left\{n\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\lambda}_{\mu}+(n-2)\underset{(s)}{H}\underset{(p-s)}{H}\right\}$$
$$-3\epsilon_{2p}\psi^{5}\Delta_{1}(z),$$
$$+\frac{(n-p-2)}{2(n-1)}\psi\sum_{s=0}^{p}\left(\underset{(s)}{H}\underset{(p-s)}{H}-\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\lambda}_{\mu}\right)$$

that is

$$(66)\qquad (n-p-2)\underset{(p)}{\zeta}=\sum_{s=1}^{p}\underset{(s,\ p-s)}{K}-3\epsilon_{2p}\psi^{5}\Delta_{1}(z)$$
$$-\frac{\psi}{2(n-1)}\sum_{s=0}^{p}\left\{(2n-p-2)\underset{(s)}{H}{}^{\mu}_{\lambda}\underset{(p-s)}{H}{}^{\lambda}_{\mu}+p\underset{(s)}{H}\underset{(p-s)}{H}\right\}.$$

On the other hand, by (55) and the assumption (58), we have for $s=2,3,\ldots,p$

$$\frac{\partial}{\partial y}\underset{(s)}{H}=\frac{\psi}{n-1}\sum_{t=0}^{s-1}\left\{\left(\frac{\partial}{\partial y}\underset{(t)}{H}{}^{\mu}_{\lambda}\right)\underset{(s-t-1)}{H}{}^{\lambda}_{\mu}-\left(\frac{\partial}{\partial y}\underset{(t)}{H}\right)\underset{(s-t-1)}{H}\right\}$$
$$-\frac{\psi^{3}}{2(n-1)}\left(z+y\frac{\partial z}{\partial y}\right)\sum_{t=0}^{s-1}\left(\underset{(t)}{H}{}^{\mu}_{\lambda}\underset{(s-t-1)}{H}{}^{\lambda}_{\mu}-\underset{(t)}{H}\underset{(s-t-1)}{H}\right),$$

hence we get the relation

$$(67)\qquad \sum_{s=2}^{p}\underset{(s\ p-s)}{K}=\frac{\psi}{n-1}\sum_{s=2}^{p}\sum_{t=0}^{s-1}\left(\underset{(t,\ p-s)}{K}{}^{\mu}_{\lambda}\underset{(s-t-1)}{H}{}^{\lambda}_{\mu}-\underset{(t,\ p-s)}{K}\underset{(s-t-1)}{H}\right)$$
$$-\frac{\psi^{3}z}{2(n-1)}\sum_{t=0}^{p-1}\left(\underset{(t)}{H}{}^{\mu}_{\lambda}\underset{(p-t-1)}{H}{}^{\lambda}_{\mu}-\underset{(t)}{H}\underset{(p-t-1)}{H}\right)$$
$$-\frac{\psi^{3}}{2(n-1)}\frac{\partial z}{\partial y}\sum_{t=0}^{p-2}\left(\underset{(t)}{H}{}^{\mu}_{\lambda}\underset{(p-t-2)}{H}{}_{\mu}-\underset{(t)}{H}\underset{(p-t-2)}{H}\right).$$

In the next place, let us consider $\underset{(1,\ p-1)}{K}$. Making use of (II_1) and the assumption (58), we get

$$
\begin{aligned}
&g^{\lambda\mu}\left\{(\psi h)_{,\lambda\mu}-(\psi h^\rho_\lambda)_{,\rho\mu}\right\}\\
&\quad= g^{\lambda\mu}\left[\psi(h_{,\lambda}-h^\rho_{\lambda,\rho})-y\psi^3(\zeta_{,\lambda}h-\zeta_{,\rho}h^\rho_\lambda)\right]_{,\mu}\\
&\quad= g^{\lambda\mu}\left[\psi\left\{\xi_\lambda-(n-1)\psi\zeta_{,\lambda}\right\}-y\psi^3(\zeta_{,\lambda}h-\zeta_{,\rho}h^\rho_\lambda)\right]_{,\mu}\\
&\quad= g^{\lambda\mu}\left[\psi\sum_{j\geqq 0}y^j\underset{(j)}{\xi_\lambda}-(n-1)\psi^2\zeta_{,\lambda}-y\psi^3(\zeta_{,\lambda}h-\zeta_{,\rho}h^\rho_\lambda)\right]_{,\mu}\\
&\quad= g^{\lambda\mu}\Big(\psi\sum_{j>p}y^j\underset{(j)}{\xi_\lambda}\Big)_{,\mu}-(n-1)\psi^2\Delta_2(\zeta)+2(n-1)y\psi^4\Delta_1(\zeta)\\
&\qquad+3y^2\psi^5(\Delta_1(\zeta)h-\zeta_{,\lambda}\zeta_{,\mu}h^{\lambda\mu})\\
&\qquad-y\psi^3(\Delta_2(\zeta)h-\zeta_{,\lambda\mu}h^{\lambda\mu})\\
&\qquad-y\psi^3(g^{\lambda\mu}\zeta_{,\lambda}h_{,\mu}-\zeta_{,\mu}h^{\mu\lambda}{}_{,\lambda}).
\end{aligned}
$$

Now, from (36′), (47′) we get

$$
\begin{aligned}
\frac{\partial}{\partial y}\underset{(1)}{H}&=-\psi^3\Big(\zeta+y\frac{\partial\zeta}{\partial y}\Big)\Big(\frac{R}{2(n-1)}-\frac{3n}{2}\psi^2\zeta^2\Big)\\
&\quad+\frac{\psi}{n-1}\left[\psi h^\mu_\lambda R^\lambda_\mu+g^{\lambda\mu}\left\{(\psi h)_{,\lambda\mu}-(\psi h^\rho_\lambda)_{,\rho\mu}\right\}\right]\\
&\quad-n\psi^3\zeta\frac{\partial\zeta}{\partial y}.
\end{aligned}
$$

If we put the relation above into the last equation, we get the relation

$$
\begin{aligned}
\underset{(1,p-1)}{K}&=\frac{\psi^2}{n-1}R^\mu_\lambda\underset{(p-1)}{H}{}^\lambda_\mu\\
&\quad+\frac{3}{n-1}(1-\epsilon_{2p})\psi^6\Big(\Delta_1(\zeta)\underset{(p-3)}{H}-\zeta_{,\lambda}\zeta_{,\mu}\underset{(p-3)}{H}{}^{\lambda\mu}\Big)\\
&\quad-\frac{\psi^4}{n-1}\Big(\Delta_2(\zeta)\underset{(p-2)}{H}-\zeta_{,\lambda\mu}\underset{(p-2)}{H}{}^{\lambda\mu}\Big)\\
&\quad-\frac{\psi^4}{n-1}g^{\lambda\mu}\zeta_{,\lambda}\Big(\underset{(p-2)}{H}{}_{;\mu}-\underset{(p-2)}{H}{}^\rho_{\mu;\rho}\Big)\\
&\quad+\frac{1}{n-1}(1-\epsilon_{2p})\psi^7\Big(\Delta_1(\zeta)\frac{\partial}{\partial\psi}\underset{(p-3)}{H}-\zeta_{,\lambda}\zeta_{,\mu}\frac{\partial}{\partial\psi}\underset{(p-3)}{H}{}^{\lambda\mu}\Big)\\
&\quad+\epsilon_{2p}\Big[-\psi^2\frac{\partial\zeta}{\partial y}\Big(\underset{(1)}{H}-n\psi^3\zeta^2\Big)+2\psi^5\Delta_1(\zeta)\Big].
\end{aligned}
$$

On the other hand, we get by (36), (36′)

$$\frac{\psi^2}{n-1} R^\mu_\lambda \underset{(p-1)}{H}{}^\lambda_\mu = \frac{n-2}{n-1}\psi\left(\underset{(1)}{H}{}^\mu_\lambda + \frac{\psi R}{2(n-1)(n-2)}\delta^\mu_\lambda + \frac{1}{2}\psi^3 \mathfrak{z}^2 \delta^\mu_\lambda\right)\underset{(p-1)}{H}{}^\lambda_\mu$$

$$= \frac{n-2}{n-1}\psi \underset{(1)}{H}{}^\mu_\lambda \underset{(p-1)}{H}{}^\lambda_\mu + \frac{\psi}{n-1}\left(\underset{(1)}{H} + \frac{n}{2}\psi^3\mathfrak{z}^2\right)\underset{(p-1)}{H} + \frac{n-2}{2(n-1)}\psi^4 \mathfrak{z}^2 \underset{(p-1)}{H}$$

$$= \frac{\psi}{n-1}\left\{(n-2)\underset{(1)}{H}{}^\mu_\lambda \underset{(p-1)}{H}{}^\lambda_\mu + \underset{(1)}{H}\underset{(p-1)}{H}\right\} + \psi^4 \mathfrak{z}^2 \underset{(p-1)}{H}.$$

Putting the relation into the left hand side of the above one for $\underset{(1,\, p-1)}{K}$, we have

$$\underset{(1,\, p-1)}{K} = \frac{\psi}{n-1}\left\{(n-2)\underset{(1)}{H}{}^\mu_\lambda \underset{(p-1)}{H}{}^\lambda_\mu + \underset{(1)}{H}\underset{(p-1)}{H}\right\} + \psi^4 \mathfrak{z}^2 \underset{(p-1)}{H}$$
$$+ \frac{3}{n-1}(1-\epsilon_{2p})\psi^6\left(\Delta_1(\mathfrak{z})\underset{(p-3)}{H} - \mathfrak{z}_{,\lambda}\mathfrak{z}_{,\mu}\underset{(p-3)}{H}{}^{\lambda\mu}\right)$$
$$- \frac{\psi^4}{n-1}\left(\Delta_2(\mathfrak{z})\underset{(p-2)}{H} - \mathfrak{z}_{,\lambda\mu}\underset{(p-2)}{H}{}^{\lambda\mu}\right)$$
$$- \frac{1}{n-1}(1-\epsilon_{2p})\psi^7 g^{\lambda\mu}\mathfrak{z}_{,\lambda}\left(\mathfrak{z}_{,\mu}\frac{\partial}{\partial\psi}\underset{(p-3)}{H} - \mathfrak{z}_{,\nu}\frac{\partial}{\partial\psi}\underset{(p-3)}{H}{}^\nu_\mu\right)$$
$$+ \epsilon_{2p}\psi^5\Delta_1(\mathfrak{z})$$
$$+ \frac{1}{n-1}(1-\epsilon_{2p})\psi^7\left(\Delta_1(\mathfrak{z})\frac{\partial}{\partial\psi}\underset{(p-3)}{H} - \mathfrak{z}_{,\lambda}\mathfrak{z}_{,\mu}\frac{\partial}{\partial\psi}\underset{(p-3)}{H}{}^{\lambda\mu}\right)$$
$$+ \epsilon_{2p}\left\{-\psi^2\frac{\partial\mathfrak{z}}{\partial y}\left(\underset{(1)}{H} - n\psi^3\mathfrak{z}^2\right) + 2\psi^5\Delta_1(\mathfrak{z})\right\},$$

that is

$$\underset{(1,\, p-1)}{K} = \frac{\psi}{n-1}\left\{(n-2)\underset{(1)}{H}{}^\mu_\lambda \underset{(p-1)}{H}{}^\lambda_\mu + \underset{(1)}{H}\underset{(p-1)}{H}\right\} + \psi^2 \mathfrak{z}^2 \underset{(p-1)}{H}$$
$$+ \frac{3}{n-1}\psi^6\left(\Delta_1(\mathfrak{z})\underset{(p-3)}{H} - \mathfrak{z}_{,\lambda}\mathfrak{z}_{,\mu}\underset{(p-3)}{H}{}^{\lambda\mu}\right)$$
(68)
$$- \frac{\psi^4}{n-1}\left(\Delta_2(\mathfrak{z})\underset{(p-2)}{H} - \mathfrak{z}_{,\lambda\mu}\underset{(p-2)}{H}{}^{\lambda\mu}\right)$$

$$+ \epsilon_2 \psi_p^2 \left\{ - \frac{\partial z}{\partial y} \left(\underset{(1)}{H} - n\psi^3 z^2 \right) + 3\psi^3 \Delta_1(z) \right\}.$$

In the computation above, we have assumed that $\underset{p-3}{H}{}^{\mu}_{\lambda} = 0$ for $p =$

Now, we start with the verification of $\underset{(2)}{\zeta} = 0$. By means of (66), (68) and (43), we get the following relation:

$$\begin{aligned}
(n-4)\underset{(2)}{\zeta} &= \underset{(1,1)}{K} + \underset{(2,0)}{K} - 3\psi^5 \Delta_1(z) \\
&\quad - \frac{1}{n-1} \sum_{s=0}^{2} \left\{ (n-2) \underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(2-s)}{H}{}^{\lambda}_{\mu} + \underset{(s)}{H}\underset{(2-s)}{H} \right\} \\
&= \frac{\psi}{n-1} \left\{ (n-2) \underset{(1)}{H}{}^{\mu}_{\lambda} \underset{(1)}{H}{}^{\lambda}_{\mu} + \underset{(1)}{H}\underset{(1)}{H} \right\} + \psi^2 z^2 \underset{(1)}{H} \\
&\quad - \frac{\psi^4}{n-1} \left(\Delta_2(\underset{(0)}{H}) - z_{,\lambda\mu} \underset{(0)}{H}{}^{\lambda\mu} \right) \\
&\quad - \psi^2 \frac{\partial z}{\partial y} \left(\underset{(1)}{H} - n\psi^3 z^2 \right) + 3\psi^5 \Delta_1(z) \\
&\quad + \psi^2 \left(\frac{\partial z}{\partial y} - 5\psi^2 z^2 \right) \underset{(1)}{H} - \psi^5 z \Delta_2(z) - n\psi^5 z^2 \frac{\partial z}{\partial y} \\
&\quad - 3\psi^5 \Delta_1(z) \\
&\quad - \frac{\psi}{n-1} \sum_{s=0}^{2} \left\{ (n-2) \underset{(s)}{H}{}^{\mu}_{\lambda} \underset{(2-s)}{H}{}^{\lambda}_{\mu} + \underset{(s)}{H}\underset{(2-s)}{H} \right\} \\
&= -4\psi^4 z^2 \underset{(1)}{H} - \frac{2\psi}{n-1} \left\{ (n-2) \underset{(0)}{H}{}^{\mu}_{\lambda} \underset{(2)}{H}{}^{\lambda}_{\mu} + \underset{(0)}{H}\underset{(2)}{H} \right\} \\
&= -4\psi^4 z^2 \underset{(1)}{H} + 4\psi^2 z \underset{(2)}{H} \\
&= 4\psi^2 z (\underset{(2)}{H} - \psi^2 z \underset{(1)}{H}).
\end{aligned}$$

Hence, if $n - 4 \neq 0$, we get by (42) the relation

$$\underset{(2)}{\zeta} = 0.$$

In the next place, if $p > 2$, making use of (37), (40), (41) and (47′), we obtain the following relation

$$\sum_{s=2}^{p} \sum_{t=0}^{s-1} \left(\underset{(t\ p-s)}{K}{}^{\mu}_{\lambda} \cdot \underset{(s-t-1)}{H}{}^{\lambda}_{\mu} - \underset{(t,\ p-s)}{K} \underset{(s-t-1)}{H} \right)$$

$$
\begin{aligned}
&= \sum_{m=0}^{p-2}\sum_{t=1}^{p-m-1}\Big(\underset{(t,p-m-1-t)}{K}{}^{\mu}{}_{\lambda}\underset{(m)}{H}{}^{\lambda}{}_{\mu} - \underset{(t,p-m-1-t)}{K}\underset{(m)}{H}\Big) \\
&\quad + \sum_{s=2}^{p}\Big(\underset{(0,p-s)}{K}{}^{\mu}{}_{\lambda}\underset{(s-1)}{H}{}^{\lambda}{}_{\mu} - \underset{(0,p-s)}{K}\underset{(s-1)}{H}\Big) \\
&= \sum_{m=0}^{p-3}\underset{(m)}{H}{}^{\mu}{}_{\lambda}\Big[(n-p+m-1)\underset{(p-m)}{H}{}^{\lambda}{}_{\mu} + \psi\sum_{i=0}^{p-m-1}\Big\{\underset{(i)}{H}\underset{(p-m-1-i)}{H}{}^{\lambda}{}_{\mu} \\
&\quad + \frac{1}{2(n-1)}\delta^{\lambda}_{\mu}\Big(\underset{(i)}{H}{}^{\rho}{}_{\nu}\underset{(p-m-1-i)}{H}{}^{\nu}{}_{\rho} - \underset{(i)}{H}\underset{(p-m-1-i)}{H}\Big)\Big\}\Big] \\
&\quad - \sum_{m=0}^{p-3}\underset{(m)}{H}\Big[(n-p+m-1)\underset{(p-m)}{H} \\
&\quad + \frac{\psi}{2(n-1)}\sum_{i=0}^{p-m-1}\Big(n\underset{(i)}{H}{}^{\mu}{}_{\lambda}\underset{(p-m-1-i)}{H}{}^{\lambda}{}_{\mu} + (n-2)\underset{(i)}{H}\underset{(p-m-1-i)}{H}\Big)\Big] \\
&\quad + 3\psi^5\big(z_{,\lambda}z_{,\mu}\underset{(p-3)}{H}{}^{\lambda\mu} - \Delta_1(z)\underset{(p-3)}{H}\big) \\
&\quad - \Big\{3\psi^2 z\underset{(1)}{H}{}^{\mu}{}_{\lambda} + \psi^3 g^{\mu\nu}\Big(z_{,\lambda\nu} + z\frac{\partial z}{\partial y}g_{\lambda\nu}\Big)\Big\}\underset{(p-2)}{H}{}^{\lambda}{}_{\mu} \\
&\quad + \Big\{3\psi^2 z\underset{(1)}{H} + \psi^3\Big(\Delta_2(z) + nz\frac{\partial z}{\partial y}\Big)\Big\}\underset{(p-2)}{H} \\
&\quad - (n-1)\psi^3 z\frac{\partial z}{\partial y}\underset{(p-2)}{H} + (n-1)\Big(\psi\frac{\partial z}{\partial y} - \psi^3 z^2\Big)\underset{(p-1)}{H}.
\end{aligned}
$$

By means of (67), (68) and the relation above, we get the following relation

$$
\begin{aligned}
\sum_{s=1}^{p}\underset{(s,\,p-s)}{K} &= \frac{\psi}{n-1}\Big[\sum_{m=0}^{p-3}(n-p+m-1)\Big(\underset{(m)}{H}{}^{\mu}{}_{\lambda}\underset{(p-m)}{H}{}^{\lambda}{}_{\mu} - \underset{(m)}{H}\underset{(p-m)}{H}\Big) \\
&\quad + \psi\sum_{m=0}^{p-3}\sum_{i=0}^{p-m-1}\Big\{\underset{(i)}{H}\underset{(m)}{H}{}^{\mu}{}_{\lambda}\underset{(p-m-1-i)}{H}{}^{\lambda}{}_{\mu} \\
&\quad - \frac{1}{2}\underset{(m)}{H}\Big(\underset{(i)}{H}{}^{\mu}{}_{\lambda}\underset{(p-m-1-i)}{H}{}^{\lambda}{}_{\mu} + \underset{(i)}{H}\underset{(p-m-1-i)}{H}\Big)\Big\} \\
&\quad + 3\psi^5\Big(z_{,\lambda}z_{,\mu}\underset{(p-3)}{H}{}^{\lambda}{}_{\mu} - \Delta_1(z)\underset{(p-3)}{H}\Big) \\
&\quad - 3\psi^2 z\Big(\underset{(1)}{H}{}^{\mu}{}_{\lambda}\underset{(p-2)}{H}{}^{\lambda}{}_{\mu} - \underset{(1)}{H}\underset{(p-2)}{H}\Big) - \psi^3\Big(z_{,\lambda\mu}\underset{(p-2)}{H}{}^{\lambda\mu} - \Delta_2(z)\underset{(p-2)}{H}\Big)
\end{aligned}
$$

$$+ (n-1)\Big(\psi \frac{\partial z}{\partial y} - \psi^3 z^2\Big) \underset{(p-1)}{H}\Big]$$

$$+ \frac{\psi}{n-1}\Big\{(n-2) \underset{(1)}{H^\mu_\lambda} \underset{(p-2)}{H^\lambda_\mu} - \underset{(1)}{H} \underset{(p-2)}{H}\Big\} + \psi^4 z^2 \underset{(p-1)}{H}$$

$$+ \frac{3}{n-1} \psi^6 \Big(\Delta_1(z) \underset{(p-3)}{H} - z_{,\lambda} z_{,\mu} \underset{(p-3)}{H^{\lambda\mu}}\Big)$$

$$- \frac{\psi^4}{n-1} \Big(\Delta_2(z) \underset{(p-2)}{H} - z_{,\lambda\mu} \underset{(p-2)}{H^{\lambda\mu}}\Big)$$

$$- \frac{1}{2(n-1)} \psi^3 z \sum_{t=0}^{p-1} \Big(\underset{(t)}{H^\mu_\lambda} \underset{(p-t-1)}{H^\lambda_\mu} - \underset{(t)}{H} \underset{(p-t-1)}{H}\Big)$$

$$- \frac{1}{2(n-1)} \psi^3 \frac{\partial z}{\partial y} \sum_{t=0}^{p-2} \Big(\underset{(t)}{H^\mu_\lambda} \underset{(p-t-2)}{H^\lambda_\mu} - \underset{(t)}{H} \underset{(p-t-2)}{H}\Big)$$

$$= \frac{\psi}{n-1} \Big[\sum_{m=0}^{p-2} (n-p+m-1) \Big(\underset{(m)}{H^\mu_\lambda} \underset{(p-m)}{H^\lambda_\mu} - \underset{(m)}{H} \underset{(p-m)}{H}\Big)$$

$$+ \psi \sum_{m=0}^{p-3} \sum_{i=0}^{p-m-1} \Big\{\underset{(i)}{H} \underset{(m)}{H^\mu_\lambda} \underset{(p-m-1-i)}{H^\lambda_\mu}$$

$$- \frac{1}{2} \underset{(m)}{H} \Big(\underset{(i)}{H^\mu_\lambda} \underset{(p-m-1-i)}{H^\lambda_\mu} + \underset{(i)}{H} \underset{(p-m-1-i)}{H}\Big)\Big\}$$

$$- n \Big(\underset{(2)}{H^\mu_\lambda} \underset{(p-2)}{H^\lambda_\mu} - \underset{(2)}{H} \underset{(p-2)}{H}\Big)\Big]$$

$$+ \frac{\psi}{n-1} \Big\{(n-2) \underset{(1)}{H^\mu_\lambda} \underset{(p-1)}{H^\lambda_\mu} + \underset{(1)}{H} \underset{(p-1)}{H}\Big\}$$

$$- \frac{1}{2(n-1)} \psi^3 z \sum_{t=0}^{p-1} \Big(\underset{(t)}{H^\mu_\lambda} \underset{(p-t-1)}{H^\lambda_\mu} - \underset{(t)}{H} \underset{(p-t-1)}{H}\Big)$$

$$+ \psi^2 \frac{\partial z}{\partial y} \Big[\underset{(p-1)}{H} + \frac{\psi}{2(n-1)} \sum_{t=0}^{p-2} \Big(\underset{(t)}{H} \underset{(p-t-2)}{H} - \underset{(t)}{H^\mu_\lambda} \underset{(p-t-2)}{H^\lambda_\mu}\Big)\Big].$$

On the other hand, we can see that

$$\sum_{m=0}^{p-3} \sum_{i=0}^{p-m-1} \Big(\underset{(i)}{H} \underset{(m)}{H^\mu_\lambda} \underset{(p-m-1-i)}{H^\lambda_\mu} - \frac{1}{2} \underset{(m)}{H} \underset{(i)}{H^\mu_\lambda} \underset{(p-m-1-i)}{H^\lambda_\mu}\Big)$$

$$= \frac{1}{2} \sum_{m=0}^{p-1} \underset{(m)}{H} \sum_{i=0}^{p-m-1} \underset{(i)}{H^\mu_\lambda} \underset{(p-m-i)}{H^\lambda_\mu}$$

$$- \underset{(0)}{H}\left(\underset{(p-2)}{H}{}^{\mu}_{\lambda} \underset{(1)}{H}{}^{\lambda}_{\mu} + \underset{(p-1)}{H}{}^{\mu}_{\lambda} \underset{(0)}{H}{}^{\lambda}_{\mu}\right) - \underset{(1)}{H}\,\underset{(p-2)}{H}{}^{\mu}_{\lambda} \underset{(0)}{H}{}^{\lambda}_{\mu}$$

$$+ \frac{1}{2}\left(\underset{(p-2)}{H}\,\underset{(0)}{H}{}^{\mu}_{\lambda} \underset{(1)}{H}{}^{\lambda}_{\mu} + \underset{(p-1)}{H}\,\underset{(0)}{H}{}^{\mu}_{\lambda} \underset{(0)}{H}{}^{\lambda}_{\mu} + \underset{(p-2)}{H}\,\underset{(1)}{H}{}^{\mu}_{\lambda} \underset{(0)}{H}{}^{\lambda}_{\mu}\right)$$

$$= \frac{1}{2}\sum_{m=0}^{p-1} \underset{(m)}{H} \sum_{i=0}^{p-m-1} \underset{(i)}{H}{}^{\mu}_{\lambda} \underset{(p-m-1-i)}{H}{}^{\lambda}_{\mu} + n\psi\zeta\, \underset{(1)}{H}{}^{\mu}_{\lambda} \underset{(p-2)}{H}{}^{\lambda}_{\mu}$$

$$- \frac{n}{2}\psi^2 \zeta^2 \underset{(p-1)}{H}.$$

Hence, making use of the relation $\underset{(p-2)}{\zeta} = 0$, the relation above becomes

$$\sum_{s=1}^{p} \underset{(s,p-s)}{K} = \frac{\psi}{n-1}\Bigg[\sum_{m=0}^{p-2}(n-p+m-1)\left(\underset{(m)}{H}{}^{\mu}_{\lambda} \underset{(p-m)}{H}{}^{\lambda}_{\mu} - \underset{(m)}{H}\,\underset{(p-m)}{H}\right)$$

$$+ \frac{\psi}{2}\sum_{m=0}^{p-1} \underset{(m)}{H} \sum_{i=0}^{p-m-1}\left(\underset{(i)}{H}{}^{\mu}_{\lambda} \underset{(p-m-i-1)}{H}{}^{\lambda}_{\mu} - \underset{(i)}{H}\,\underset{(p-m-i-1)}{H}\right)$$

$$+ n\psi^2\zeta\, \underset{(1)}{H}{}^{\mu}_{\lambda} \underset{(p-2)}{H}{}^{\lambda}_{\mu} - \frac{n}{2}\psi^3\zeta^2 \underset{(p-1)}{H}$$

$$- \frac{\psi}{2}\left(2n\psi\zeta\, \underset{(1)}{H}\,\underset{(p-2)}{H} - n^2\psi^2\zeta^2 \underset{(p-1)}{H} - n\left(\underset{(2)}{H}{}^{\mu}_{\lambda} \underset{(p-2)}{H}{}^{\lambda}_{\mu} - \underset{(2)}{H}\,\underset{(p-2)}{H}\right)\right)\Bigg]$$

$$+ \frac{\psi}{n-1}\left\{(n-2)\underset{(1)}{H}{}^{\mu}_{\lambda} \underset{(p-1)}{H}{}^{\lambda}_{\mu} + \underset{(1)}{H}\,\underset{(p-1)}{H}\right\}$$

$$- \frac{1}{2(n-1)}\psi^3\zeta \sum_{t=0}^{p-1}\left(\underset{(t)}{H}{}^{\mu}_{\lambda} \underset{(p-t-1)}{H}{}^{\lambda}_{\mu} - \underset{(t)}{H}\,\underset{(p-t-1)}{H}\right).$$

Furthermore, by means of (55) and the assumption (58), the relation becomes

$$\sum_{s=1}^{p} \underset{(s,\,p-s)}{K} = \frac{\psi}{n-1}\Bigg[\sum_{m=0}^{p-2}(n-p+m-1)\left(\underset{(m)}{H}{}^{\mu}_{\lambda} \underset{(p-m)}{H}{}^{\lambda}_{\mu} - \underset{(m)}{H}\,\underset{(p-m)}{H}\right)$$

$$+ (n-1)\sum_{m=0}^{p-2} \underset{(m)}{H}\,\underset{(p-m)}{H} + \frac{\psi}{2}\underset{(p-1)}{H}\left(\underset{(0)}{H}{}^{\mu}_{\lambda} \underset{(0)}{H}{}^{\lambda}_{\mu} - \underset{(0)}{H}\,\underset{(0)}{H}\right)$$

$$+ \frac{1}{2}(n^2-n)\psi^3\zeta^2 \underset{(p-1)}{H}$$

$$+ (n-2)\underset{(1)}{H}{}^{\mu}_{\lambda} \underset{(p-1)}{H}{}^{\lambda}_{\mu} + \underset{(1)}{H}\,\underset{(1)}{H}\Bigg] - \psi^2\zeta \underset{(p)}{H}$$

$$= \frac{\psi}{n-1}\Bigg[\Bigg\{\sum_{m=0}^{p}(n-p+m-1)\Big(\underset{(m)}{H}{}^{\mu}_{\lambda}\underset{(p-m)}{H}{}^{\lambda}_{\mu} - \underset{(m)}{H}\,\underset{(p-m)}{H}\Big)$$

$$-(n-2)\Big(\underset{(p-1)}{H}{}^{\mu}_{\lambda}\underset{(1)}{H}{}^{\lambda}_{\mu} - \underset{(p-1)}{H}\,\underset{(1)}{H}\Big) - (n-1)^2\psi\zeta\underset{(p)}{H}\Bigg\}$$

$$+(n-1)\Bigg\{\sum_{m=0}^{p}\underset{(m)}{H}\,\underset{(p-m)}{H} - \underset{(p-1)}{H}\,\underset{(1)}{H} + n\psi\zeta\underset{(p)}{H}\Bigg\}$$

$$+(n-2)\underset{(1)}{H}{}^{\mu}_{\lambda}\underset{(p-1)}{H}{}^{\lambda}_{\mu} + \underset{(1)}{H}\,\underset{(p-1)}{H}\Bigg] - \psi^2\zeta\underset{(p)}{H},$$

that is

$$(69)\qquad \sum_{s=1}^{p}\underset{(s,\,p-s)}{K} = \frac{\psi}{n-1}\sum_{m=}^{p}\Big\{(n-p+m-1)\underset{(m)}{H}{}^{\mu}_{\lambda}\underset{(p-m)}{H}{}^{\lambda}_{\mu} + (p-m)\underset{(m)}{H}\,\underset{(p-m)}{H}\Big\}.$$

If we put $p-m=m'$, the last relation is also represented as

$$\sum_{s=1}^{p}\underset{(s,\,p-s)}{K} = \frac{\psi}{n-1}\sum_{m=0}^{p}\Big\{(n-m-1)\underset{(m)}{H}{}^{\mu}_{\lambda}\underset{(p-m)}{H}{}^{\lambda}_{\mu} + m\underset{(m)}{H}\,\underset{(p-m)}{H}\Big\},$$

hence we obtain from the two relations the following one

$$(69')\qquad 2\sum_{s=1}^{p}\underset{(s,\,p-s)}{K} = \frac{\psi}{n-1}\sum_{m=}^{p}\Big\{(2n-p-2)\underset{(m)}{H}{}^{\mu}_{\lambda}\underset{(p-m)}{H}{}^{\lambda}_{\mu} + p\underset{(m)}{H}\,\underset{(p-m)}{H}\Big\}.$$

Now, if we substitute (69′) into (66), we obtain

$$(n-p-2)\underset{(p)}{\zeta} = 0,$$

accordingly, we get

$$\underset{(p)}{\zeta} = 0,$$

since $n-p-2>0$ by the assumption (58).

Thus we have proved that we obtain

$$\underset{(p+1)}{\xi_a} = 0,\ \underset{(p)}{\zeta} = 0$$

from (58). Accordingly, we see that the following relations hold good:

$$(70)\qquad \begin{cases} \underset{(0)}{\xi_a} = \underset{(1)}{\xi_a} = \cdots = \underset{(n-2)}{\xi_a} = 0, \\ \underset{(0)}{\zeta} = \underset{(1)}{\zeta} = \cdots = \underset{(n-3)}{\zeta} = 0. \end{cases}$$

7. The imbedding theorem. Owing to the result obtained in the previous sections, if we put

$$(71)\qquad \xi_a = y^{n-1}\eta_a, \quad \zeta = y^{n-2}\tau, \quad \underset{(n-1)}{H}{}^a_b = H^a_b,$$

we get by (51), (52), (54), (55) and (56)

$$(72)\qquad \eta_a \equiv H_{,a} H^\lambda_{a,\lambda} - \psi^3\left(z_{,a}\frac{\partial}{\partial\psi}\underset{(n-2)}{H} - z_{,\lambda}\frac{\partial}{\partial\psi}\underset{(n-2)}{H}{}^\lambda_a\right),$$

$$\tau \equiv H + \frac{\psi}{2(n-1)}\Big[\sum_{s=0}^{n-2}\left(\underset{(s)}{H}\underset{(n-2-s)}{H} - \underset{(s)}{H}{}^\mu_\lambda \underset{(n-2-s)}{H}{}^\lambda_\mu\right)$$

$$(73)\qquad + \sum_{i=n-1}^{2n-3} y^{i-n+2}\Big\{2\left(\underset{(i-n+1)}{H} H - \underset{(i-n+1)}{H}{}^\mu_\lambda H^\lambda_\mu\right) + \sum_{s=i-n+2}^{n-2}\left(\underset{(s)}{H}\underset{(i-s)}{H} - \underset{(s)}{H}{}^\mu_\lambda \underset{(i-s)}{H}{}^\lambda_\mu\right)\Big\}$$

$$+ 2y^n\left(HH - H^\mu_\lambda H^\lambda_\mu\right)\Big].$$

Now, if we put (71) into (29) and (31), we obtain respectively the following relations:

$$(29')\qquad \frac{\partial}{\partial y}\eta_a = \psi h\,\eta_a - (n-1)\psi^2 z_{,a}\tau,$$

$$(31')\qquad \frac{\partial}{\partial y}\tau = \left\{\psi h - \psi^2\left(z + y\frac{\partial z}{\partial y}\right)\right\}\tau + \frac{2}{n-1}y^2\psi^4 g^{\lambda\mu} z_{,\lambda}\eta_\mu - \frac{1}{n-1}y\psi^2 g^{\lambda\mu}\eta_{\lambda,\mu}.$$

We notice that these equations are linear with respect to η_a, τ and the coefficients of the terms on the right hand sides are regular with respect to y near $y = 0$.

Thus, we see that the condition stated in section 1, § 3 in order that our problem can be solved is replaced by the following one. We can solve the system of differential equations for the unknown quantities g_{ab}, H_{ab}

$$(I_1^*)\qquad \frac{\partial}{\partial y} g_{ab} = -2\psi\left(\sum_{i=0}^{n-2} y^i \underset{(i)}{H}_{ab} + y^{n-1} H_{ab}\right),$$

$$(\mathrm{I}_2^*) \qquad \frac{\partial}{\partial y} H^b_a = \frac{1}{y} L^a_b + \psi \sum_{s=0}^{n-1} y^s \Big[- \sum_{\substack{i+j=s+n-1 \\ 1 \leqq i \leqq n-2}} \underset{(i,j)}{K}{}^a_b$$

$$+ \sum_{\substack{i+j=s+n-1 \\ 0 \leqq i \leqq n-1}} \Big\{ \underset{(i)}{H} \underset{(j)}{H}{}^a_b + \frac{1}{2(n-1)} \delta^a_b \Big(\underset{(i)}{H}{}^\mu_\lambda \underset{(j)}{H}{}^\lambda_\mu - \underset{(i)}{H} \underset{(j)}{H} \Big) \Big\} \Big]$$

under the conditions

$$(\mathrm{II}_1^*) \qquad \eta_a = 0,$$

$$(\mathrm{II}_2^*) \qquad \tau = 0,$$

and the initial conditions

$$\Big[g_{ab}(x, y) \Big]_{y=0} = g_{ab}(x),$$

where $\underset{(n-1)}{H}{}^\lambda_\mu = H^\lambda_\mu$.

Accordingly, by virtue of (29′) and (31′), we see that the condition above is reduced to the following condition: We can determine a tensor $H_{ab}(x)$ and a scalar $\chi(x, y)$ so that

$$\Big[H_{ab}(x, y) \Big]_{y=0} = H_{ab}(x)$$

and

$$\eta_a = 0, \quad \tau = 0, \quad L^a_b = 0, \text{ for } y = 0$$

are satisfied.

On the other hand, L^a_b is function of $g^{\lambda\mu}$; R^λ_μ; R^λ_{μ,ρ_1}; $\cdots$; $R^\lambda_{\mu,\rho_1\rho_2\cdots\rho_{n-2}}$; ψ; χ; $\ldots$; $\left(\frac{\partial^k \chi}{\partial y^k}\right)_{,\rho_1\cdots\rho_h}$; $\ldots$ $(k + h \leqq n - 2)$ as shown in (49). Hence, the conditions $\Big[L^a_b(n, y) \Big]_{y=0} = 0$ may be regarded as equations of the given space V_n with unknown quantitities $\Big[\left(\frac{\partial^k \chi}{\partial y^k}\right)_{,\rho_1\cdots\rho_h} \Big]_{y=0}$ $(k + h \geqq n - 2)$. We notice that ψ becomes 1 for $y = 0$ by (4). If $\Big[L^a_b(x, y) \Big]_{y=0}$ can be solved with respect to $\Big[\left(\frac{\partial^k \chi}{\partial y^k}\right)_{,\rho_1\cdots\rho_h} \Big]_{y=0}$, we can easily see that the equations

$$[\eta_a]_{y=0} = H_{,a} - H^\lambda_{a,\lambda} - \Big[\chi_{,a} \frac{\partial}{\partial \psi} \underset{(n-2)}{H} - \chi_{,\lambda} \frac{\partial}{\partial \psi} \underset{(n-2)}{H}{}^\lambda_a \Big]_{y=0} = 0,$$

$$[\tau]_{y=0} = H + \frac{1}{2(n-1)} \Big[\sum_{s=0}^{n-2} \Big(\underset{(s)}{H} \underset{(n-2-s)}{H} - \underset{(s)}{H}{}^\mu_\lambda \underset{(n-2-s)}{H}{}^\lambda_\mu \Big) \Big]_{y=0} = 0$$

are solvable with respect to H^a_b.

Consequently we obtain the following theorem:

THEOREM 2. *A necessary and sufficient condition that a given Riemannian space* $V_n (n > 2)$ *with line element* $ds^2 = g_{\lambda\mu}(x)\,dx^\lambda dx^\mu$ *can be imbedded in a space* V_{n+1} *as a hypersurface which is the image of a hypersphere invariant under the group of holonomy of the space with a normal conformal connexion corresponding to* V_{n+1} *is that the system of equations*

$$L^b_a\left(g^{\lambda\mu};\ \mathrm{R}^\lambda_\mu;\ \ldots,\ \mathrm{R}^\lambda_{\mu,\rho_1\cdots\rho_{n-2}};\ \psi;\ \zeta;\ \cdots;\left(\frac{\partial^k\zeta}{\partial y^k}\right)_{,\rho_1\cdots\rho_h};\ \cdots\right) = 0$$
$$(k + h \leq n - 2,\ \psi = 1)$$

is solvable with respect to $\zeta,\ \dfrac{\partial\zeta}{\partial y},\ \ldots,\ \dfrac{\partial^{n-2}\zeta}{\partial y^{n-2}}$ *regarding them as unknown quantities.*

REMARK. If a Riemannian space V_n can be imbedded in a V_{n+1} as stated above, we see from (32) that the relation

$$h_{ab} = -\psi\zeta g_{ab}.$$

holds good on the hyperesurface $\mathfrak{F}^n$.

Hence, we see that *if* $\zeta(x, 0) \neq 0$, $\mathfrak{F}_n$ *is a totally umbilical hypersurface of* V_{n+1} *and if* $\zeta(x, 0) = 0$, $\mathfrak{F}_n$ *is a tatally geodesic hypersurface of* V_{n+1}.

§ 4. The invariant hypersphere and an imbedding problem (Continued).

1. The imbedding problem for V_2. In the last paragraph we have excluded the case of two-dimensional Riemannsan space as an exception. We shall investigate this case, starting with the fundamental system of equations (I) and (II).

Let K be the Gaussian total curvature of a Riemannian space V_2, then we have

$$\mathrm{R}^a_b = K\,\delta^a_b,\quad \mathrm{R} = 2K$$

as is well-known. As in § 3, if we put

$$h^a_b = -\psi\zeta\delta^a_b + y\,H^a_b$$

and substitute it into (I_2), we get by means of

$$R^a_b - \frac{1}{2} R\, \delta^a_b = 0,$$

the following equation:

$$
\begin{aligned}
& y\,\psi^3 z \frac{\partial z}{\partial y} \delta^a_b + y \frac{\partial}{\partial y} H^a_b \\
&= -y\,\psi^3 g^{a\lambda} z_{,b\lambda} + 3y^2 \psi^5 g^{a\lambda} z_{,b} z_{,\lambda} \\
&\quad + y\psi \left\{ \underset{(0)}{H} H^a_b + H \underset{(0)}{H}{}^a_b + \delta^a_b \left(\underset{(0)}{H}{}^\mu_\lambda H^\lambda_\mu - \underset{(0)}{H} H \right) \right\} \\
&\quad + y^2 \psi \left\{ H H^a_b + \frac{1}{2} \delta^a_b \left(H^\mu_\lambda H^\mu_\lambda - HH \right) \right\} \\
&= -y\,\psi^3 g^{a\lambda} z_{,b\lambda} + 3y^2 \psi^5 g^{a\lambda} z_{,b} z_{,\lambda} \\
&\quad - 2y\,\psi^2 z H^a_b + y^2 \psi \left(H H^a_b - \delta^a_b \,|H^\lambda_\mu| \right),
\end{aligned}
$$

that is

$$
\text{(I}_2''\text{)} \qquad
\begin{aligned}
\frac{\partial}{\partial y} H^a_b &= -\psi^3 \left(z \frac{\partial z}{\partial y} \delta^a_b + g^{a\lambda} z_{,b\lambda} \right) - 2\psi^2 z H^a_b \\
&\quad + y\psi \left(3\psi^4 g^{a\lambda} z_{,b} z_{,\lambda} + H H^a_b - \delta^a_b \,|H^\lambda_\mu| \right).
\end{aligned}
$$

(I$_1$) becomes

$$
\text{(I}_1''\text{)} \qquad \frac{\partial}{\partial y} g_{ab} = 2\psi\,(\psi z\, g_{ab} - H_{ab}).
$$

Then, if we put $\xi_a = y\eta_a$, (II$_1$) is replaced by

$$
\text{(II}_1''\text{)} \qquad \eta_a \equiv H_{,a} - H^\lambda_{a,\lambda} + \psi^3 z\, z_{,a} = 0.
$$

Regarding (II$_2$), we get the relation

$$
\begin{aligned}
\zeta &\equiv H + \frac{\psi}{2} \Big\{ (-2\psi z + y H)^2 \\
&\qquad - \left(-\psi z\, \delta^\mu_\lambda + y H^\mu_\lambda \right) \left(-\psi z\, \delta^\lambda_\mu + y H^\lambda_\mu \right) - R \Big\} \\
&= H + \frac{\psi}{2} \left\{ 2\psi^2 z^2 - 2y\,\psi z H + y^2 \left(HH - H^\mu_\lambda H^\lambda_\mu \right) - R \right\},
\end{aligned}
$$

hence we have

$$
\text{(II}_2''\text{)} \qquad \zeta \equiv H + \psi\,(\psi^2 z^2 - y\psi z H + y^2 \,|H^\lambda_\mu| - K) = 0.
$$

In the next place, (29) is replaced by

$$\frac{\partial}{\partial y}\eta_a = \psi(-2\psi\chi + yH)\eta_a - \psi^2\chi_{,a}\zeta, \tag{29''}$$

and (31) becomes by means of (26)

$$\begin{aligned}\frac{\partial}{\partial y}\zeta &= \left(-2\psi^2\chi + y\psi H + \frac{\partial}{\partial y}\log\psi\right)\zeta \\ &\quad - \psi g^{\lambda\mu}(-2y^2\psi^3\chi_{,\lambda}\eta_\mu + y\psi\eta_{\lambda,\mu}) \\ &= \left(-3\psi^2\chi + y\psi H - y\psi^2\frac{\partial\chi}{\partial y}\right)\zeta + 2y^2\psi^4 g^{\lambda\mu}\chi_{,\lambda}\eta_\mu - y\psi^2 g^{\lambda\mu}\eta_{\lambda,\mu},\end{aligned}$$

that is

$$\begin{aligned}\frac{\partial}{\partial y}\zeta &= \left(-3\psi^2\chi + y\psi H - y\psi^2\frac{\partial\chi}{\partial y}\right)\zeta + 2y^2\psi^4 g^{\lambda\mu}\chi_{,\lambda}\eta_\mu \\ &\quad - y\psi^2 g^{\lambda\mu}\eta_{\lambda,\mu}.\end{aligned} \tag{31''}$$

Hence, we see that if the quantities η_a, ζ calculated from a solution of (I_1''), (I_2'') satisfy the relations

$$\eta_a = 0, \quad \zeta = 0$$

at $y = 0$, on account of (29″) and (31″), these relations also hold good for any y near zero. Now, (II″) becomes at $y = 0$

$$\begin{cases} H_{,a} - H^\lambda_{a,\lambda} + \chi\chi_{,a} = 0, \\ H + \chi^2 - K = 0. \end{cases} \tag{75}$$

The last equations are clearly solvable with respect to H^λ_μ. Hereby $\chi(x,y)$ may be considered as an arbitrary given quantits. Thus, we get the following theorem;

THEOREM 3. *Any Riemannian space V_2 can be imbedded in a Riemannian space V_3 as a surface which is the image of a sphere invariant under the group of holonomy of the space with a normal conformal connexion corresponding to the V_3. Then the surface is totally umbilical or totally geodesic and the principal curvature may be taken arbitraily.*

2. CASE $\chi(x, y) \equiv 0$.[6] In this section, we shall investigate especially the case such that $\chi(x, y) \equiv 0$ and point out an essential difference between even dimensional Riemannian spaces and odd dimensional ones through properties

6) See [I].

of L^a_b.

Let us put $n > 2$, $\chi(x, y) \equiv 0$ in the theory of § 3. Then, we have $\psi \equiv 1$ by (4) and we get easily from (33), (36), (37), (40), (41), (42) and (43) the following relations:

$$\underset{(0)}{H}{}^a_b = 0,\ \underset{(1)}{H}{}^a_b = \frac{1}{n-2}\left(R^a_b - \frac{R}{2(n-1)}\delta^a_b\right),\ \underset{(2)}{H}{}^a_b = 0,$$

$$\underset{(0,i)}{K}{}^a_b = 0 \quad (i \geqq 0),\ \underset{(1,0)}{K}{}^a_b = 0,$$

$$\underset{(1,1)}{K}{}^a_b = \frac{1}{(n-2)^2}\Big\{2R^a_\lambda R^\lambda_b - \frac{1}{n-1} R R^a_b + \frac{1}{2(n-1)} g^{a\lambda} R_{,\lambda b}$$

$$+ g^{\lambda\mu} R^a_{b,\lambda\mu} + \frac{1}{2} g^{a\lambda} R_{,b\lambda} - g^{a\lambda} R^\mu_{b,\lambda\mu} - R^{a\lambda}{}_{,b\lambda}\Big\}$$

$$- \frac{1}{(n-1)(n-2)^2}\delta^a_b\Big\{R^\mu_\lambda R^\lambda_\mu - \frac{R^2}{2(n-1)} - R^{\lambda\mu}{}_{,\lambda\mu} + \Delta_2(R)\Big\},$$

$$\underset{(2,i)}{K} = 0 \qquad (i \geqq 0).$$

Then, making use of the relations above, we get from (44)

$$\underset{(3)}{H}{}^a_b = \frac{1}{n-4}\Big\{\underset{(1,1)}{K}{}^a_b + \underset{(1)}{H}\underset{(1)}{H}{}^a_b - \frac{1}{2(n-1)}\delta^a_b\Big(\underset{(1)}{H}{}^\mu_\lambda \underset{(1)}{H}{}^\lambda_\mu - \underset{(1)}{H}\underset{(1)}{H}\Big)\Big\}$$

$$= \frac{1}{(n-2)^2(n-4)}\Big\{2R^\lambda_b R^a_\lambda - \frac{n}{2(n-1)} R R^a_b + g^{\lambda\mu} R^a_{b,\lambda\mu}$$

$$+ \frac{n}{2(n-1)} g^{a\lambda} R_{,\lambda b} - g^{a\lambda} R^\mu_{b,\lambda\mu} - R^{a\lambda}{}_{,b\lambda}\Big\}$$

$$- \frac{1}{2(n-1)(n-2)^2(n-4)}\delta^a_b\Big\{3R^\mu_\lambda R^\lambda_\mu - \frac{3n}{4(n-1)} R^2 + \Delta_2(R)\Big\}.$$

Now, we see from (45), (46) that $\underset{(i)}{H}{}^a_b$, $\underset{(s,j-s)}{K}{}^a_b$ are determined successively as follows:

$$\underset{(i)}{H}{}^a_b = \underset{(i)}{H}{}^a_b\Big(g^{\lambda\mu},\ R^\lambda_\mu;\ \cdots;\ R^\lambda_{\mu,\rho_1\cdots\rho_{i-1}}\Big),$$

$$\underset{(s,j-s)}{K}{}^a_b = \underset{(s,j-s)}{K}{}^a_b\Big(g^{\lambda\mu},\ R^\lambda_\mu;\ \cdots;\ R^\lambda_{\mu,\rho_1\cdots\rho_j}\Big)$$

$$(i = 1, 2, \ldots, n-2;\ j = 1, 2, \ldots, n-3;\ s = 0, 1, 2, \ldots, j).$$

Furthermore, since (48) becomes

$$\frac{\partial}{\partial y} \underset{(s)}{H}{}^a_b = \sum_{\lambda \leqq \mu} 2h^{\lambda\mu}\left(\partial \underset{(s)}{H}{}^a_b \Big/ \partial g^{\lambda\mu}\right)$$

$$+ \sum_{\substack{\lambda, \mu \\ 0 \leqq k \leqq s-1}} \left(\partial \underset{(s)}{H}{}^a_b \Big/ \partial R^{\lambda}_{\mu,\, \rho_1 \cdots \rho_k}\right) \frac{\partial}{\partial y} R^{\lambda}_{\mu,\, \rho_1 \cdots \rho_k},$$

where $\frac{\partial}{\partial y} R^{\lambda}_{\mu,\, \rho_1 \cdots \rho_k}$ is a linear form of h^{τ}_{ν}; $h^{\tau}_{\nu,\, \alpha_1}$; $\cdots$; $h^{\tau}_{\nu,\, \alpha_1 \cdots \alpha_{k+2}}$ with coefficients which are polynomials of $g^{\tau\nu}$, R^{τ}_{ν}; $\cdots$; $R^{\tau}_{\nu,\, \alpha_1 \cdots \alpha_k}$, we see by induction that

$$\underset{(2i)}{H}{}^a_b = 0, \quad \underset{(2i,\, j)}{K}{}^a_b = 0, \quad \underset{(j,\, 2i)}{K}{}^a_b = 0.$$

Hence, if $n = an$ is an odd number, we get by (49) the identity

$$L^a_b(g^{\lambda\mu}, R^{\lambda}_{\mu}; \cdots; R^{\lambda}_{\mu,\, \rho_1 \cdots \rho_{n-2}}) = 0.$$

On the other hand, according to § 1, we see that the point at infinity with respect to the natural frame at any point in the space with a normal conformal connexion corresponding to V_{n+1} whose group of holonomy fixes a real hypersphere is on the hypersphere if $\mathfrak{z} \equiv 0$.

Accordingly, we obtain from Theorem 2 the following theorem:

THEOREM 4. *For $n = 2m + 1$ $(m \geqq 1)$ any Riemannian space V_n can be imbedded in a Riemannian space V_{n+1} as a hypersurface which is the image of a hypersphere $\mathfrak{S}_n$ invariant under the group of holonomy of the space with a normal conformal connexion C_{n+1} corresponding to V_{n+1} so that the point at infinity with respect to the natural frame at any point of C_{n+1} is always on $\mathfrak{S}_n$.*

Analogously, we get the following theorem:

THEOREM 4'. *For $n = 2m$ $(m > 1)$ Theorem 4 holds good if and only if*

$$L^a_b(g^{\lambda\mu}, R^{\lambda}_{\mu}; \cdots; R^{\lambda}_{\mu,\, \rho_1 \cdots \rho_{n-2}}) = 0.$$

3. L^a_b. We have seen in the previous arguments that the tensor L^a_b plays an important rôle for our problem. However, it is generally difficult to represent explicitly the components of the tensor by means of $g^{\lambda\mu}$, R^{λ}_{μ}, ψ, $\mathfrak{z}$. In this section, we shall calculate the components of L^a_b for $n = 3$, 4 and should like to imagine the general case from these examples.

i) Case $n = 3$. We get from (49)

$$L_b^a = -\underset{(1,0)}{K}{}_b^a - \zeta\frac{\partial\zeta}{\partial y}\delta_b^a - g^{a\lambda}\zeta_{,b\lambda}$$

$$+\psi\left\{\underset{(0)}{H}\underset{(1)}{H}{}_b^a + \underset{(1)}{H}\underset{(0)}{H}{}_b^a + \frac{1}{2}\delta^a\left(\underset{(0)}{H}{}_\lambda^\mu\underset{(1)}{H}{}_\mu^\lambda - \underset{(0)}{H}\underset{(1)}{H}\right)\right\},$$

which is reduced by (33), (36) and (40) to

$$L_b^a = 3\psi^2\zeta\underset{(1)}{H}{}_b^a + \psi^3\left(g^{a\lambda}\zeta_{,\lambda b} + \zeta\frac{\partial\zeta}{\partial y}\delta_b^a\right)$$

$$-\zeta\frac{\partial\zeta}{\partial y}\delta_b^a - g^{a\lambda}\zeta_{,b\lambda} - 3\psi^2\zeta\underset{(1)}{H}{}_b^a = 0.$$

Accordingly, we get the following theorem.

THEOREM 5. *For any Riemannian space V_3 the tensor L_b^a vanishes for any ζ.*

ii) Case $n = 4$. We get from (36)

$$\underset{(1)}{H}{}_b^a = \frac{\psi}{2}\left\{R_b^a - \left(\frac{R}{6} + \psi^2\zeta^2\right)\delta_a^b\right\},$$

$$\underset{(1)}{H} = \frac{1}{6}\psi R - 2\psi^3\zeta^2.$$

Then, we get by means of (41), (42), (43) and (49) the relation

$$L_b^a = -\underset{(1,1)}{K}{}_b^a - \underset{(2,0)}{K}{}_b^a + 3\psi^5 g^{a\lambda}\zeta_{,\lambda}\zeta_{,h}$$

$$+\psi\left\{\underset{(0)}{H}\underset{(2)}{H}{}_b^a + \underset{(2)}{H}\underset{(0)}{H}{}_b^a + \underset{(1)}{H}\underset{(1)}{H}{}_b^a + \frac{1}{3}\delta_b^a\left(\underset{(0)}{H}{}_\lambda^\mu\underset{(2)}{H}{}_\mu^\lambda - \underset{(0)}{H}\underset{(2)}{H}\right)\right.$$

$$\left.+\frac{1}{6}\delta_b^a\left(\underset{(1)}{H}{}_\lambda^\mu\underset{(1)}{H}{}_\mu^\lambda - \underset{(1)}{H}\underset{(1)}{H}\right)\right\}$$

$$= -\psi^2\left(\underset{(1)}{H}{}_\lambda^a R_b^\lambda - \frac{1}{6}\delta_b^a\underset{(1)}{H}{}_\lambda^\mu R_\mu^\lambda\right)$$

$$-2\psi^5 g^{a\lambda}(\zeta\zeta_{,\lambda b} + 2\zeta_{,\lambda}\zeta_{,b})$$

$$+\frac{\psi^2}{2}\left\{g^{a\mu}\underset{(1)}{H}{}_{b;\mu\lambda}^\lambda + \underset{(1)}{H}{}^{a\lambda}{}_{;b\lambda} - g^{\lambda\mu}\underset{(1)}{H}{}_{b;\lambda\mu}^a\right.$$

$$\left.-g^{a\lambda}H_{;b\lambda} + \frac{1}{3}\delta_b^a\left(g^{\lambda\mu}\underset{(1)}{H}{}_{;\lambda\mu} - \underset{(1)}{H}{}^{\lambda\mu}{}_{;\lambda\mu}\right)\right\}$$

$$+\frac{\psi^3}{2}\frac{\partial z}{\partial y}\left(R^a_b-\frac{R}{6}\delta^a_b-3\psi^2 z^2\delta^a_b\right)$$

$$-\psi^2\left(\frac{\partial z}{\partial y}-5\psi^2 z^2\right)\underset{(1)}{H}{}^a_b+\psi^5 z\left(g^{a\lambda}z_{,\lambda b}+z\frac{\partial z}{\partial y}\delta^a_b\right)$$

$$+2\psi^5 g^{a\lambda}z_{,\lambda}z_{,b}$$

$$-4\psi^4 z^2\underset{(1)}{H}{}^a_b-\psi^4 z^2\underset{(1)}{H}\delta^a_b+\psi\underset{(1)}{H}\underset{(1)}{H}{}^a_b+\psi^4 z^2\underset{(1)}{H}\delta^a_b$$

$$+\frac{\psi}{6}\delta^a_b\left(\underset{(1)}{H}{}^\mu_\lambda\underset{(1)}{H}{}^\lambda_\mu-\underset{(1)}{H}\underset{(1)}{H}\right)$$

$$=-\frac{\psi^3}{2}\left\{R^a_\lambda R^\lambda_b-\left(\frac{R}{6}+\psi^2 z^2\right)R^a_b\right\}$$

$$+\frac{\psi^3}{12}\delta^a_b\left\{R^\mu_\lambda R^\lambda_\mu-\left(\frac{R}{6}+\psi^2 z^2\right)R\right\}-\psi^5 g^{a\lambda}(z z_{,\lambda b}+z_{,\lambda}z_{,b})$$

$$+\frac{\psi^3}{2}\Big[\frac{1}{2}g^{a\mu}\left(R^\lambda_{b,\mu\lambda}-\frac{1}{6}R_{,\mu b}\right)-\psi^2 g^{a\mu}(z z_{,\mu b}+z_{,\mu}z_{,b})$$

$$+\frac{1}{2}g^{a\mu}\left(R^\lambda_{\mu,b\lambda}-\frac{1}{6}R_{,b\mu}\right)-\psi^2 g^{a\mu}(z z_{,b\mu}+z_{,b}z_{,\mu})$$

$$-\frac{1}{2}g^{\lambda\mu}\ R^a_{b,\lambda\mu}-\frac{1}{6}\delta^a_b R_{,\lambda\mu}\Big)+\psi^2\delta^a_b(z\Delta_2(z)+\Delta_1(z))$$

$$-\frac{1}{6}g^{a\lambda}R_{,b\lambda}+4\psi^2 g^{a\mu}(z z_{,b\mu}+z_{,\mu}z_{,b})$$

$$+\frac{1}{3}\delta^a_b\Big\{\frac{1}{6}\Delta_2(R)-4\psi^2(z\Delta_2(z)+\Delta_1(z)-\frac{1}{2}R^{\lambda\mu}{}_{,\lambda\mu}$$

$$+\frac{1}{12}\Delta_2(R)+\psi^2 z\left(\Delta_2(z)+\Delta_1(z)\right)\Big\}\Big]$$

$$+\frac{1}{2}\psi^5 z^2\left\{R^a_b-\left(\frac{R}{6}+\psi^2 z^2\right)\delta^a_b\right\}$$

$$+\frac{\psi^3}{2}\left\{\left(R^a_b-\frac{R}{6}\delta^a_b\right)\frac{R}{6}-\psi^2 z^2\left(-\frac{R}{6}\delta^a_b+2R^a_b\right)+2\psi^4 z^4\delta^a_b\right\}$$

$$+\frac{\psi^3}{6}\delta^a_b\Big\{\frac{1}{4}\left(R^\mu_\lambda-\frac{R}{6}\delta^\mu_\lambda\right)\left(R^\lambda_\mu-\frac{R}{6}\delta^\lambda_\mu\right)-\frac{1}{6}\psi^2 z^2 R$$

$$+\psi^4 z^4-\frac{R^2}{36}+\frac{2}{3}\psi^2 z^2 R-4\psi^4 z^4\Big\},$$

that is

$$L_b^a = -\psi^3\Big[\frac{1}{2}R_b^\lambda R_\lambda^a - \frac{1}{6}R R_b^a + \frac{1}{4}g^{\lambda\mu}R^a_{.b,\lambda\mu}$$

(76) $$+\frac{1}{6}g^{a\lambda}R_{,\lambda b} - \frac{1}{4}g^{a\lambda}R^\mu_{b,\lambda\mu} - \frac{1}{4}R^{a\lambda}{}_{,b\lambda}$$

$$-\frac{1}{24}\delta_b^a\Big(3R_\lambda^\mu R_\mu^\lambda - R^2 + \Delta_2(R)\Big)\Big].$$

From the relation above we get the following theorem:

THEOREM 6. *Whether we can imbed a given Riemannian space V_4 in a Riemann space V_5 as a hypersurface in the sense stated in Theorem 2 without regard to z, or it is entirely impossible.*

§5. An application. 1. The invariant hypersphere and the Campbell's theorem.

The space V_{n+1} in Theorem 2 is conformal with an Einstein space with a negative scalar curvature.[7] We shall investigate in the last paragraph the problem to imbed a given Riemannian space V_n in an Einstein space A_{n+1} as a hypersurface in the sense stated in Theorem 2.

Making use of (16), (17) and (18), we can prove that *any Riemannian space V_n with line element*

$$ds^2 = g_{\lambda\mu}(x)\,dx^\lambda\,dx^\mu$$

can be imbedded in an Einstein space A_{n+1} with a given scalar curvature $(n+1)k$ as a hypersurface and if the line element of A_{n+1} is $ds^2 = g_{\lambda\mu}(x, y)\,dx^\lambda\,dx^\mu + (\psi(x\ y)\ dy)^2$ $(g_{\lambda\mu}(x, 0) = g_{\lambda\mu}(x))$, the following equations hold good

(I) $$\frac{\partial}{\partial y}g_{ab} = -2\psi h_{ab},$$

(III) $$\frac{\partial}{\partial y}h_{ab} = k\psi g_{ab} + \psi\Big(Q_{ab} - h_a^\lambda h_{b\lambda}\Big) + \psi_{,ab},$$

(IV) $$V_a \equiv h_{,a} - h^a_{a,\lambda} = 0,$$

(V) $$(n-1)k + h^2 - h_\lambda^\mu h_\mu^\lambda - R = (n-1)k + Q = 0,$$

7) See [I], no. 1.

where $Q_{ab} = h\, h_{ab} - h^{\lambda}_{a} h_{b\lambda} - R_{ab}$ and the meanings of notations are similar to those in § 3. This is a generalization of the Campbell's theorem.[8)]

Now, (I'_2) can be written as

$$\frac{\partial}{\partial y} h_{ab} = \frac{n-1}{y}(h_{ab} + \psi\chi g_{ab}) + \psi(Q_{ab} - h^{\lambda}_{a} h_{b\lambda})$$

$$- \frac{\psi Q}{2(n-1)} g_{ab} + \psi_{,ab} - \psi \frac{\partial \chi}{\partial y} g_{ab}.$$

Comparing the right hand side of the relation above with the one of (III), we get the relation

$$\frac{n-1}{y}(h_{ab} + \psi\chi g_{ab}) = \psi g_{ab}\left(\frac{\partial \chi}{\partial y} + k + \frac{Q}{2(n-1)}\right),$$

that is

$$h_{ab} = \psi g_{ab}\left\{-\chi + \frac{y}{n-1}\left(\frac{\partial \chi}{\partial y} + k + \frac{Q}{2(n-1)}\right)\right\}. \tag{77}$$

Then, from (II_1) and (IV) we get

$$\chi_{,a} = 0.$$

Hence, we see that χ must be a function dependent only on y. By virtue of (4), ψ is also so. Putting (V) into (II_2), we get

$$\frac{1}{y}(h + n\psi\chi) - \frac{k}{2}\psi = 0. \tag{78}$$

Putting (V) into (77), we get

$$h_{ab} = \psi g_{ab}\left\{-\chi + \frac{y}{n-1}\left(\frac{\partial \chi}{\partial y} + \frac{k}{2}\right)\right\},$$

hence we get

$$h = n\psi\left\{-\chi + \frac{y}{n-1}\left(\frac{\partial \chi}{\partial y} + \frac{k}{2}\right)\right\}.$$

From the relation and (78), we obtain

8) [I], no. 12 or J.E. Campbell, A course of differential geometry, (1926).

$$\frac{n}{n-1}\left(\frac{\partial z}{\partial y}+\frac{k}{2}\right)-\frac{k}{2}=0,$$

that is

$$\frac{\partial z}{\partial y}=-\frac{k}{2n}.$$

Hence, by integration, we get

$$z=\alpha-\frac{k}{2n}y \qquad (\alpha=\text{constant}). \tag{79}$$

Then, since we have

$$-z+\frac{y}{n-1}\left(\frac{\partial z}{\partial y}+\frac{k}{2}\right)=-\alpha+\frac{k}{2}y,$$

(77) can be replaced by

$$h_{ab}=\psi\left(-\alpha+\frac{k}{n}y\right)g_{ab}. \tag{80}$$

Now by (4) and (79), ψ becomes

$$\psi^2=\frac{1}{1+2yz}=\frac{1}{1+2\alpha y-\frac{k}{n}y^2}. \tag{81}$$

Since we have

$$(n-1)k+h^2-h^{\mu}_{\lambda}h^{\lambda}_{\mu}=(n-1)(k+n\alpha^2)\psi^2,$$

(V) can be written as

$$(n-1)\ (k+n\alpha^2)\psi^2-R=0. \tag{82}$$

Conversely, if we have (79), (80) and (82), then (II_1) (II_2) (IV) and (V) are clearly satisfied.

Lastly, if we put (80) into (III), since we get from both sides the relations

$$\begin{aligned}\frac{\partial}{\partial y}h_{ab}&=-2\psi^2\left(-\alpha+\frac{k}{n}y\right)h_{ab}+\left\{\frac{\partial\psi}{\partial y}\left(-\alpha+\frac{k}{n}y\right)+\psi\frac{k}{n}\right\}g_{ab}\\&=\psi\left\{-2\psi^2\left(-\alpha+\frac{k}{n}y\right)^2+\left(-\alpha+\frac{k}{n}y\right)\frac{\partial}{\partial y}\log\psi+\frac{k}{n}\right\}g_{ab}.\end{aligned}$$

$$k\psi g_{ab} + \psi\left(h\, h_{ab} - 2h^{\lambda}_{a} h_{b\lambda} - R_{ab}\right) + \psi_{,ab}$$

$$= \psi k g_{ab} + \psi^{3}\left(-\alpha + \frac{k}{n}y\right)^{2}(n-2)\, g_{ab} - \psi R_{ab},$$

we have

$$\left\{-\left(-\alpha + \frac{k}{n}y\right)\frac{\partial}{\partial y}\log\psi + \frac{n-1}{n}k + n\psi^{2}\left(-\alpha + \frac{k}{n}y\right)^{2}\right\} g_{ab}$$
$$- R_{ab} = 0,$$

that is

$$\xi_{ab} \equiv \frac{n-1}{n}(k + n\alpha^{2})\,\psi^{2} g_{ab} - R_{ab} = 0. \tag{83}$$

Thus, we see that *a necessary and sufficient condition that a given Riemannian space V_n with line element $ds^2 = g_{\lambda\mu}(x)\, dx^{\lambda} dx^{\mu}$ can be imbedded in an Einstein space with the scalar curvature $(n+1)\, k$ as a hypersurface in the sense stated in the beginning is as follows : the differential equations*

$$\frac{\partial}{\partial y} g_{ab} = -\frac{2\left(-\alpha + \frac{k}{n}y\right)}{1 + 2\alpha y - \frac{k}{n}y^{2}}\, g_{ab} \tag{VI}$$

are solvable under the conditions

$$\xi_{ab} \equiv \frac{(n-1)\,(k + n\alpha^{2})}{n + 2n\alpha y - k y^{2}}\, g_{ab} - R_{ab} = 0 \tag{VII}$$

and the initial conditions

$$\left[g_{ab}(x, y)\right]_{y=0} = g_{ab}(x).$$

2. Solutions of (VI), (VII).

For any solutions of (VI), let us calculate the quantities ξ_{ab}. Then, we get easily from (39) the relation

$$\frac{\partial}{\partial y} R^{b}_{a} = 2\psi\, h^{a}_{\lambda} R^{\lambda}_{b} = 2\psi^{2}\left(-\alpha + \frac{k}{n}y\right) R^{a}_{b}$$

and

$$\frac{\partial}{\partial y}\frac{1}{n+2n\alpha y-ky^2}=2\psi^2\left(-\alpha+\frac{k}{n}\right)\frac{1}{n+2n\alpha y-ky^2}\cdot$$

Accordingly, we have

$$\frac{\partial}{\partial y}\xi^a_b=2\psi^2\left(-\alpha+\frac{k}{n}\right)\xi^a_b. \tag{84}$$

Hence, we see that if $\xi^a_b=0$ at $y=0$, ξ^a_b vanishes near $y=0$. Accordingly, at $y=0$ it must be

$$R_{ab}=\frac{n-1}{n}(k+n\alpha^2)g_{ab}.$$

The last relation showes that V_n must be an Enstein space A_n $(n>2)$ or a surface with a constant curvature.

Thus, we obtain the following theorem:

THEOREM 7. *A necessary and sufficient condition that a given Riemannian space V_n can be imbedded in an Einstein space A_{n+1} as a hypersurface which is the image of a hypersphere invariant under the group of holonomy of the space with a normal conformal connexion corresponding to A_{n+1} is that V_n is an Einstein space $(n>2)$ or a surface with a constant curvature.*

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