SOME PROPERTIES OF A RIEMANNIAN SPACE ADMITTING A SIMPLY TRANSITIVE GROUP OF TRANSLATIONS*)

BY

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In the present paper we first consider (§1 and §2) some properties of a simply transitive group of translations, or of transformations whose constants of structure are skew-symmetric in three indices or can be made so by a suitable choice of fundamental vectors $\underset{a}{\xi}^{\lambda}$. In later sections we then consider some properties of a Riemannian space admitting a simply transitive group of translations and state for example the following theorem: A Riemannian space V_n which admits a semi-simple simply transitive group of translations G_n admits also a group of motions G_n' which is commutative with G_n. It should be mentioned that we are dealing with local properties only.

§1. The simply transitive group of translations

Let us consider an n-dimensional manifold V and a simply transitive group G_n on V whose fundamental vectors are denoted by $\underset{a}{\xi}^{\lambda}$. For convenience we shall use small Latin letters for the indices of vectors and tensors in the vector space associated with the group and Greek ones for the indices in V. Both letters take on the values 1, 2, $\cdots$, n.

The vectors $\underset{a}{\xi}^{\lambda}$ satisfy equations[1]

$$(1) \qquad \underset{a}{\xi}^{\mu}\underset{b}{\xi}^{\lambda}{}_{,\mu} - \underset{b}{\xi}^{\mu}\underset{a}{\xi}^{\lambda}{}_{,\mu} = C_{ab}^{\cdot\cdot x}\underset{x}{\xi}^{\lambda}$$

where $C_{ab}^{\cdot\cdot d}$ are the constants of structure.

A Riemannian metric will be introduced in the manifold V by putting

$$(2) \qquad g_{\lambda\mu} = \overset{a}{\xi}_{\lambda}\overset{b}{\xi}_{\mu} C_{ab},$$

where $\overset{a}{\xi}_{\lambda}$ are defined by

$$(3) \qquad \underset{a}{\xi}^{\lambda}\overset{a}{\xi}_{\mu} = \delta^{\lambda}_{\mu} \qquad \text{(hence } \underset{a}{\xi}^{\lambda}\overset{b}{\xi}_{\lambda} = \delta^{b}_{a}\text{)}$$

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*) Received in revised form, November 22, 1950

1) We adopt summation convention.

and C_{ab} is a symmetric tensor of rank n in the vector space. The contravariant fundamental tensor $g^{\lambda\mu}$ is then given by

$$(4) \qquad g^{\lambda\mu} = \xi^{\lambda}_{a} \xi^{\mu}_{b} C^{ab}$$

with C^{ab} satisfying $C_{ab} C^{bd} = \delta^{d}_{a}$.

As the numbers C^{ab} are constants, we can get the following theorem by making the Lie derivative[2] of (4) (see [3]):

THEOREM I. *The necessary and sufficient condition that a simply transitive group becomes a group of translations is that we can find out a symmetric matrix* $\|C^{ab}\|$ *of rank n which satisfies the equations*

$$(5) \qquad C_{ab}^{\cdot\cdot x} C^{ay} + C_{ab}^{\cdot\cdot y} C^{ax} = 0.$$

In this theorem we may replace (5) by an equivalent condition

$$(5') \qquad C_{ad}^{\cdot\cdot x} C_{xb} + C_{bd}^{\cdot\cdot x} C_{xa} = 0.$$

Now we can replace the vectors ξ^{λ}_{a} by any linear combinations of them with constant coefficients, and to do so is the same as to make a linear transformation in the vector space associated with the group. If the new constants of structure satisfy $C_{ab}^{\cdot\cdot d} + C_{ad}^{\cdot\cdot b} = 0$ after such transformation, we write them as C_{abd} and say that they are skew-symmetric[3]. Evidently, the constants of structure of any simply transitive group of translations can be made skew-symmetric by a suitable transformation, for we need only to bring C_{ab} to the canonical form δ_{ab}.

With the skew-symmetric C_{abd} we get

$$\begin{aligned} & C_{adx} C_{xyz} C_{bzy} \\ & \quad = - C_{dyx} C_{xaz} C_{bzy} - C_{yax} C_{xdz} C_{bzy} \\ & \quad = 2\ C_{axy} C_{byz} C_{dzx} \end{aligned}$$

by using Jacobi's relation. We see that this expression is skew-symmetric with respect to a and b and hence get the relation

$$(6) \qquad C_{adx} G_{xb} + C_{bdx} G_{xa} = 0$$

where G_{ab} is defined by

2) See reference [4].

3) By C_{abc} we do not mean the quantity $C_{ab}^{\cdot\cdot x} g_{xc}$ with $g_{ab} = - C_{ax}^{\cdot\cdot y} C_{by}^{\cdot\cdot x}$. But on the other hand we see that we can always take skew-symmetric constants of structure for a semi-simple group.

$$(7) \qquad G_{ab} = - C_{axy} C_{byx}.$$

If the group is semi-simple, the rank of G_{ab} is n and we can take G_{ab} as C_{ab} in (2), for (6) is then equivalent to (5). Then we get

$$(8) \qquad R_{\mu\nu} = \frac{1}{4} G_{ab} \xi^a_\mu \xi^b_\nu$$

by calculating the curvature tensor. Thus we have the

THEOREM II[4]. *If a space V admits a simply transitive group of transformations which is semi-simple and hence has skew-symmetric constants of structure (or has constants of structure which can be made skew-symmetric by a suitable transformation in the vector space associated with the group), then we can find a Riemannian metric such that the group becomes the group of translations and the space is an Einstein space.*

§ 2. Decomposition of the group

Now let us consider an orthogonal transformation in the vector space

$$(9) \qquad \eta^\lambda_a = P^x_a \xi^\lambda_x, \qquad \eta^b_\mu = Q^b_x \xi^x_\mu,$$

where the coefficients satisfy $P^x_a Q^b_x = \delta^b_a$, $P^x_a Q^a_y = \delta^x_y$ and $P^x_a = Q^a_x$. Then the constants of structure are transformed into $K_{ab}^{\cdot\cdot d} = C_{xy}^{\cdot\cdot z} P^x_a P^y_b Q^d_z$, which will be easily found to be skew-symmetric with respect to a, b and d.

Let

$$(10) \qquad g_{\lambda\mu} = C_{ab} \xi^a_\lambda \xi^b_\mu$$

be a tensor such that G_n is the group of translations with respect to this metric. Then we get (5′) which will become by the orthogonal transformation (9)

$$(11) \qquad K_{adx} K_{xb} + K_{bdx} K_{xa} = 0$$

with $K_{ab} = P^x_a P^y_b C_{xy}$. The indices are lowered, for the constants of structure are assumed to be skew-symmetric and the transformation (9) is orthogonal.

We can find an orthogonal transformation that makes the matrix $\|K_{ab}\|$ diagonal, that is,

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4) See [3]. See also the related theorem of Cartan and Schouten, [1], [2] p.206, [4] p. 24.

(12) $$K_{ab} = K_a \delta_{ab},$$

and in this case we get from (11)

(13) $$K_{abd}(K_b - K_a) = 0.$$

If the eigenvalues K_a are not all the same, we may put

$$K_1 = K_2 = \cdots = K_p,\ K_p \neq K_{p+1},\ K_{p+2},\ \cdots,\ K_n.$$

Then we get from (13)

(14) $$K_{APd} = K_{ABP} = K_{PQA} = 0$$

where the indices run as follows: $A, B, \cdots = 1, 2, \cdots, p$; $P, Q, \cdots = p+1, p+2, \cdots, n$. (14) means that the group G_n is not simple. Hence if the group is simple we get

$$C_{ab} = C\delta_{ab}$$

as long as C_{ab} satisfies (5′). As G_{ab} satisfies (6) we get $G_{ab} = G\delta_{ab}$. Accordingly we have the following theorem:

THEOREM III. *If a simply transitive group G_n is simple, there is essentially only one Riemannian metric with respect to which the group is a group of translations. The space is then an Einstein space.*

When the constants of structure are skew-symmetric it is evident that $C_{ab} = \delta_{ab}$ satisfies (5). Hence the group is a group of translations with

(15) $$g_{\lambda\mu} = C\Sigma_x \xi^x_\lambda \xi^x_\mu$$

as the fundamental tensor. Let us consider that the group is a group of translations with respect to the metric

(16) $$\overline{g}_{\lambda\mu} = \overline{C}_{ab}\xi^a_\lambda \xi^b_\mu$$

too. Then after an orthogonal transformation that makes $\overline{C}_{ab}$ diagonal, we get the equations having the same form as (13). Hence if (16) is essentially different from (15), we must have (14). We can now use again the letters C and ξ instead of K and η and write (14) as

(17) $$C_{APd} = C_{ABP} = C_{PQA} = 0.$$

It will be easily found that on account of (17) we can find out a coordinate system satisfying the condition

$$(18) \qquad \begin{aligned} &\xi_A^\alpha = \xi_A^\alpha(x^1, \cdots, x^p), &&\xi_A^\pi = 0, \\ &\xi_P^\alpha = 0, &&\xi_P^\pi = \xi_P^\pi(x^{p+1}, \cdots, x^n) \end{aligned}$$

where the indices α and π are used for the manifold V and take on the values $\alpha = 1, \cdots, p$; $\pi = p+1, \cdots, n$. (18) means that the group and the space are decomposed simultaneously. Hence we get the

THEOREM IV. *If a simply transitive group of transformations G_n with skew-symmetric constants of structure admits two or more essentially different Riemannian metrics with respect to which G_n becomes a group of translations, then the group and the space are simultaneously decomposed into $G_p \times G_{n-p}$ and $V_p \times V_{n-p}$.*

If the group is not semi-simple, one of the eigenvalues of G_{ab} is equal to zero. If $G_{ab} = 0$ we get $C_{abd} = 0$ from $\Sigma_{x,y} C_{axy} C_{axy} = 0$. Hence a simply transitive group which is neither Abelian nor semi-simple and has skew-symmetric constants of structure has the matrix $\|G_{ab}\|$ with eigenvalues not all the same. On the other hand the equations

$$C_{adx} X_{xb} + C_{bdx} X_{xa} = 0$$

are satisfied with $X_{ab} = \delta_{ab}$ and $X_{ab} = G_{ab}$, and this fact leads to the consequence that G_n is decomposable into $G_p \times G_{n-p}$.

Performing such decomposition as far as possible we get the

THEOREM V. *The Riemannian metric that makes a simply transitive group G_n with skew-symmetric constants of structure a group of translations makes the space V an Einstein space or a product of Einstein spaces. The group G_n is a simple group or a direct product of simple groups.*

The latter part of the theorem follows from the fact that a semi-simple group is a simple group or a direct product of simple groups.

§ 3. A one-parameter group of motions in a Riemannian space admitting a simply transitive group of translations

In preceding sections we considered some properties of a Riemannian space admitting a simply transitive group of translations G_n. In deriving the first theorem we used the property of a group of translations that the quantities defined by

$$g_{ab} = g_{\lambda\mu} \xi_a^\lambda \xi_b^\mu$$

are constant, see [2] p. 212, [4] p. 32. As we consider that the group is simply transitive we can find the vectors ξ^{a}_{λ} and a tensor g^{ab} in the vector space associated with the group such that $\xi^{\lambda}_{a}\xi^{b}_{\lambda} = \delta^{a}_{b}$, $g^{ab} = \xi^{a}_{\lambda}\xi^{b}_{\mu}g^{\lambda\mu}$, $g^{ab}g_{bd} = \delta^{a}_{d}$ and $\xi^{a}_{\lambda} = g^{ab}g_{\lambda\mu}\xi^{\mu}_{b}$.

We can add here some relations which can be obtained from (1) and the fact that a translation is a kind of motions, that is, ξ^{λ}_{a} satisfy Killing's equations[5]

$$g_{\lambda\mu}\xi^{\lambda}_{a;\nu} + g_{\lambda\nu}\xi^{\lambda}_{a;\mu} = 0,$$

which give on account of preceding relations

$$\xi^{\mu}_{a}\xi^{\lambda}_{b;\mu} + \xi^{\mu}_{b}\xi^{\lambda}_{a;\mu} = 0.$$

Then we get the equations

$$\xi^{\mu}_{a}\xi^{\lambda}_{b;\mu} = \frac{1}{2}C_{ab}^{\cdot\cdot x}\xi^{\lambda}_{x}, \tag{19}$$

which are the bases for deriving the curvature property of the space, see [3].

Now, let us consider that the vector

$$\xi^{\lambda}_{0} = h^{x}\xi^{\lambda}_{x} \tag{20}$$

defines a motion in the space. We get from Killing's equations the equations

$$(h^{x}{}_{,\mu}\xi^{y}_{\lambda} + h^{x}{}_{,\lambda}\xi^{y}_{\mu})g_{xy} = 0. \tag{21}$$

If we further assume that the vector ξ^{λ}_{0} and the vectors ξ^{λ}_{a} conjointly are the fundamental vectors of an $(n+1)$-parameter group G_{n+1}, we get from Jacobi's relations

$$C_{ab}^{\cdot\cdot 0}C_{c0}^{\cdot\cdot d} + C_{bc}^{\cdot\cdot 0}C_{a0}^{\cdot\cdot d} + C_{ca}^{\cdot\cdot 0}C_{b0}^{\cdot\cdot d} = 0,$$
$$C_{ab}^{\cdot\cdot x}C_{cx}^{\cdot\cdot 0} + C_{bc}^{\cdot\cdot x}C_{ax}^{\cdot\cdot 0} + C_{ca}^{\cdot\cdot x}C_{bx}^{\cdot\cdot 0} + C_{ab}^{\cdot\cdot 0}C_{c0}^{\cdot\cdot 0} + C_{bc}^{\cdot\cdot 0}C_{a0}^{\cdot\cdot 0} + C_{ca}^{\cdot\cdot 0}C_{b0}^{\cdot\cdot 0} = 0,$$
$$C_{ab}^{\cdot\cdot x}C_{0x}^{\cdot\cdot d} + C_{b0}^{\cdot\cdot x}C_{ax}^{\cdot\cdot d} + C_{0a}^{\cdot\cdot x}C_{bx}^{\cdot\cdot d} + C_{b0}^{\cdot\cdot 0}C_{a0}^{\cdot\cdot d} + C_{0a}^{\cdot\cdot 0}C_{b0}^{\cdot\cdot d} = 0,$$
$$C_{ab}^{\cdot\cdot x}C_{0x}^{\cdot\cdot 0} + C_{b0}^{\cdot\cdot x}C_{ax}^{\cdot\cdot 0} + C_{0a}^{\cdot\cdot x}C_{bx}^{\cdot\cdot 0} = 0.$$

We must put $C_{ab}^{\cdot\cdot 0} = 0$ for G_{n} is a subgroup of G_{n+1}. Then a solution is obtained by putting

5) A semi-colon means covariant derivation with respect to the metric $g_{\lambda\mu}$.

(22) $$C_{ab}^{\cdot\cdot 0} = C_{0b}^{\cdot\cdot 0} = C_{0a}^{\cdot\cdot b} = 0$$

which means that the group G_1' generated by ξ_0^λ and the group G_n are commutative.

The Lie derivatives of ξ_0^λ with respect to the group G_{n+1} are given by

$$X_a \xi_0^\lambda = C_{a0}^{\cdot\cdot 0} \xi_0^\lambda + C_{a0}^{\cdot\cdot x} \xi_x^\lambda$$

which vanish on account of (22). On the other hand we get from (20)

$$X_a \xi_0^\lambda = (X_a h^x) \xi_x^\lambda + h^x X_a \xi_x^\lambda$$
$$= (X_a h^b + C_{ax}^{\cdot\cdot b} h^x) \xi_b^\lambda.$$

Hence h^a must satisfy the differential equations

(23) $$X_a h^b = - C_{ax}^{\cdot\cdot b} h^x.$$

It will be easily understood that (23) is just the necessary and sufficient condition that ξ_0^λ define a group of motions and that this group and G_n are commutative. For if we multiply the left hand side of (21) by $\xi_a^\lambda \xi_b^\mu$ and contract we get $X_b h^x g_{ax} + X_a h^x g_{xb}$ which vanishes on account of (23) and (5').

ξ_0^λ can not essentially generate a group of translations, for we get from $(g_{\lambda\mu} \xi_0^\lambda \xi_a^\mu)_{;\nu} \xi_b^\nu = 0$ and (20)

$$X_b (g_{\lambda\mu} \xi_x^\lambda \xi_a^\mu h^x) = X^b (g_{ax} h^x) = g_{ax} X_b h^x = 0$$

which imply that h^a be constant. Thus we have the

THEOREM VI. *A Riemannian space V_n admitting a non-Abelian simply transitive group of translations G_n admits also a one-parameter group of motions G'_1 such that G_n and G'_1 are subgroups of a group $G_{n+1} = G'_1 \times G_n$.*

§ 4. The group of motions containing the group of translations

Now let us assume that the rank of the matrix $\| C_{ax}^{\cdot\cdot b} \|$ where x denotes the rows and a and b the columns is p. Then the set of equations

(24) $$C_{ax}^{\cdot\cdot b} u^x = 0$$

has $n-p$ independent solutions

(25) $$u^x = C_P^x \qquad (P = p+1, \cdots, n)$$

where C's are constants, and we can find out p independent non-constant solutions of (23),

$$h^x_A. \qquad (A = 1, 2, \cdots, p) \tag{26}$$

If we make new symbols

$$Y_A = h^x_A X_x \tag{27}$$

we get

$$[Y_A, X_a] \equiv Y_A X_a - X_a Y_A = 0 \tag{28}$$

on account of (23). Furthermore we can obtain

$$\begin{aligned} [Y_A, Y_B] &= h^a_A h^b_B C^{\cdot\cdot d}_{ab} X_d + (h^a_A X_a h^b_B) X_b \\ &\quad - (h^b_B X_b h^a_A) X_a \\ &= - h^a_A h^b_B C^{\cdot\cdot d}_{ab} X_d, \end{aligned} \tag{29}$$

where

$$h^a_A h^b_B C^{\cdot\cdot d}_{ab} = h^d_{AB} \tag{30}$$

is again a solution of (23). Hence we can put

$$h^a_{AB} = - D^{\cdot\cdot x}_{AB} h^a_x = - D^{\cdot\cdot D}_{AB} h^a_D - D^{\cdot\cdot \rho}_{AB} C^a_\rho \tag{31}$$

with constant D's, and get

$$[Y_A, Y_B] = D^{\cdot\cdot D}_{AB} Y_D + D^{\cdot\cdot \rho}_{AB} C^a_\rho X_a. \tag{32}$$

For symbols X_a and Y_A together Jacobi identities are satisfied and they make a group. If the group G_n is semi-simple (24) has no non-zero solution as we get

$$C^{\cdot\cdot d}_{ax} u^x C^{\cdot\cdot a}_{db} = - g_{bx} u^x = 0.$$

Besides we get $[Y_A, Y_B] = D^{\cdot\cdot D}_{AB} Y_D$. Thus we have the

THEOREM VII. *A Riemannian space admitting a simply transitive group of translations G_n admits also a group of motions G_{n+p}, and G_n is an invariant subgroup of G_{n+p}. p is such a number that $n-p$ is the number of independent solutions of* (24). *Especially when G_n is semi-simple, $p=n$ and the group G_{n+n} is the direct product of G_n (translations) and G_n' (motions).*

Next, let us consider that G_n is not semi-simple. It was stated in §1 that we can choose the fundamental vectors so that the constants of structure are skew-symmetric in three indices. Then the set of equations

$$C_{adx} X_{xb} + C_{bdx} X_{xa} = 0$$

has two essentially different solutions $X_{ab} = \delta_{ab}$ and $X_{ab} = G_{ab}$, and the group is decomposed. After performing the decomposition as far as possible, we find that G_n is the product of several semi-simple (simple) groups of parameters at least three and $n-p$ one-parameter groups. The product of the former groups is also a semi-simple group and the product of the latter is an Abelian group. As the space V_n is also simultaneously decomposed we have the

THEOREM VIII. *A Riemannian space V_n admitting a simply transitive group of translations G_n admits also a group of motions G'_n commutative with G_n if G_n is semi-simple, or if V_n is not a direct product of a one-dimensional Riemannian space and an $n-1$-dimensional Riemannian space. If V_n is a direct product of $n-p$ one-dimensional spaces and a V_p which can not be decomposed into a one-dimensional space and a V_{p-1}, then it admits a group of motions G'_p commutative with G_n.*

REFERENCES

[1] Cartan, E. and J. A. Schouten: Proc. Akad. van Wetens. Amsterdam, **29** (1926), 803-815.

[2] Eisenhart, L. P. : *Continuous Groups of Transformations* (1933).

[3] Tashiro, Y.: to be published shortly.

[4] Yano, K.: *Groups of Transformations in Generalized Spaces* (1949).

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