ON WALSH-FOURIER SERIES

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1. Introduction. Let the Rademacher functions be defined by

$$\begin{aligned}(1.1)\qquad &\varphi_0(x) = 1(0 \leqq x < 1/2),\ \varphi_0(x) = -1(1/2 \leqq x < 1),\\ &\varphi_0(x+1) = \varphi_0(x),\ \varphi_n(x) = \varphi_0(2^n x) \qquad (n = 1, 2, \cdots).\end{aligned}$$

Then the Walsh functions are given by

$$(1.2)\qquad \psi_0(x) \equiv 1, \psi_n(x) = \varphi_{n_1}(x)\varphi_{n_2}(x)\cdots\varphi_{n_r}(x)$$

for $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$, where the integers n_i are uniquely determined by $n_{i+1} < n_i$. As is well known, $\{\psi_n(x)\}$ form a complete orthonormal set, and every periodic function $f(x)$ which is integrable on (0, 1) can be expanded into a Walsh-Fourier series

$$(1.3)\qquad f(x) \sim c_0 + c_1\psi_1(x) + c_2\psi_2(x) + \cdots,$$

where the coefficients are given by

$$(1.4)\qquad c_n = \int_0^1 \psi_n(x)f(x)dx \qquad (n = 0, 1, 2\cdots).$$

Recently N. J. Fine [1] has introduced the notion of "dyadic group" and shown that the Walsh functions $\{\psi_n(x)\}$ reduce to the character group of this group. Basing on this fact, he has succeeded in developing the theory of Walsh-Fourier series analogously to that of trigonometric-Fourier series. In the present paper we shall deal with the certain theorems on WFS[2], concerning the Cesàro summability, convergency, special series and the convergence factors. The results obtained here are completely analogous to those in the case of TFS.

Our proofs mostly depend on the fundamental results obtained by N. J. Fine [1], so we shall set up his results which are needed in the sequel.

1°. *The "dyadic group"*. The dyadic group G may be defined as the denumerable direct product of the group with elements 0 and 1, in which the group operation is addition modulo 2. Thus the dyadic group G is the set of all 0, 1 sequences in which the group operation, which we shall denote by $\dotplus$, is addition modulo 2 for each element.

Let $\bar{x}$ be an element of G, $\bar{x} = \{x_1, x_2, \cdots\}$, $x_n = 0, 1$. We define the function

$$(1.5)\qquad \lambda(\bar{x}) = \sum_{n=1}^{\infty} 2^{-n}x_n.$$

The function λ, which maps G on to the closed interval [0, 1], does not

1) In what follows, the periodicity with period 1 is assumed for any function.

2) we shall abbreviate "Walsh-Fourier series" as WFS and "trigonometric Fourier series" as TFS.

have a single-valued inverse on the dyadic rationals; we shall agree to take the finite expansion in that case. Thus for all real x, if we write the inverse as μ,

(1.6) $$\lambda(\mu(x)) = x - [x].$$

If $\bar{x} = \{x_n\}$ and $\bar{y} = \{y_n\}$ are the elements of G, we have

(1.7) $$\bar{x} \dotplus \bar{y} = \{|x_n - y_n|\}.$$

We shall abbreviate $\lambda(\mu(x) \dotplus \mu(y))$ as $x \dotplus y$ for any real x and y. Then if $x = \sum_{n=1}^{\infty} 2^{-n}x_n$, $y = \sum_{n=1}^{\infty} 2^{-n}y_n$, x_n and $y_n = 0, 1$, we have by (1.6) and (1.7)

(1.8) $$x \dotplus y = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|.$$

If $0 \leqq x < 1, 0 \leqq h < 1$, then we have

(1.9) $$|(x \dotplus h) - x| \leqq h.$$

2°. For each fixed x and for all y outside a certain denumerable set, the equation

(1.10) $$\psi_n(x \dotplus y) = \psi_n(x)\psi_n(y)$$

is valid.

3°. Let x be a fixed real number and let y belong to a measurable set A lying in the unit interval. By $T_x(A)$ we shall mean the set $x \dotplus y, y \in A$. Then T_x is a measure preserving transformation, that is $|T_x(A)| = |A|$. Therefore, if $f(x)$ is integrable then for every fixed x

(1.11) $$\int_0^1 f(x \dotplus y)dy = \int_0^1 f(y)dy.$$

From this it follows that, for $f(x) \sim \sum_{n=0}^{\infty} c_n\psi_n(x)$,

(1.12) $$f(x \dotplus h) \sim \sum_{n=0}^{\infty} c_n\psi_n(h)\psi_n(x).$$

4°. *Partial sums and Dirichlet kernel.* If $f(x)$ has (1.3) as its WFS, we shall set

(1.13) $$s_n(x) \equiv s_n(x; f) \equiv \sum_{k=0}^{n-1} c_k\psi_k(x). \qquad (n = 1, 2, \dots).$$

The "*Dirichlet kernel*" of WFS is defined by

(1.14) $$D_n(x) \equiv \psi_0(x) + \psi_1(x) + \cdots + \psi_{n-1}(x),$$

then $s_n(x)$ can be writen as

(1.15) $$s_n(x) = \int_0^1 f(x \dotplus t)D_n(t)dt.$$

The size of $D_n(x)$ is given by

(1.16) $$|D_n(x)| < 2/x \qquad (0 < x < 1).$$

The "Lebesgue constant"

$$L_n = \int_0^1 |D_n(t)|\, dt \tag{1.17}$$

satisfies the relation

$$L_n = O(\log n). \tag{1.18}$$

If we write n in the from $n = p\cdot 2^k + q$, $0 \leqq q \leqq 2^k$, then we have for any x

$$D_n(x) = D_{2^k}(x) D_p(2^k x) + \psi_p(2^k x) D_q(x). \tag{1.19}$$

5°. *"Fejér kernel"*. The kernel for $(C, 1)$ summability (Fejér kernel) is defined by

$$K_n(x) = \frac{1}{n} \sum_{k=1}^{n} D_k(x). \tag{1.20}$$

For this kernel the following relations hold:

$$K_{2^n}(x) \geqq 0 \qquad (n = 0, 1, 2, \cdots, 0 \leqq x < 1). \tag{1.21}$$

$$2^{n+1} K_{2^{n+1}}(x) = (1 + \psi_{2^n}(x)) 2^n K_{2^n}(x) + 2^n D_{2^n}(x). \tag{1.22}$$

Let $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$, $n_1 > n_2 > \cdots > n_r \geqq 0$, and $n' = n - 2^{n_1}$, $n^{(i)} = n^{(i-1)} - 2^{n_i}$, $i = 2, \cdots, r$. Then

$$nK_n(x) = \sum_{i=1}^{r} 2^{n_i} \psi_{n-n^{(i)}}(x)_{2^{n_i}}(x) + \sum_{i=1}^{r} n^{(i)} D_{2^{n_i}}(x). \tag{1.23}$$

For any non-negative integer n and any real $\alpha\ (> -1)$ we shall define

$$A_n^{(\alpha)} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}, \quad A_0^{(\alpha)} = 1. \tag{1.24}$$

Then the kernel for (C, α) summability is defined by

$$K_n^{(\alpha)}(x) = \frac{1}{A_{n-1}^{(\alpha)}} \sum_{k=0}^{n-1} A_{n-k-1}^{(\alpha)} \psi_k(x). \qquad (n = 1, 2, \cdots) \tag{1.25}$$

2. Cesaro summability of negative order. In a preceding paper [8] we have proved that $\{n^{-\alpha}\}$, $0 < \alpha < 1$, is a $(C, -\alpha)$ summability factor for the TFS of the integrable functions. In this section we shall prove that the same holds for the WFS of the integrable functions.

THEOREM 1. *Let $f(x)$ be integrable and its WFS be*

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x). \tag{2.1}$$

Then the series

$$\sum_{n=0}^{\infty} c_n \psi_n(x)/(n+1)^{\alpha} \qquad (0 < \alpha < 1) \tag{2.2}$$

is summable $(C, -\alpha)$ almost everywhere.

The proof based on the following lemmas.

LEMMA 1. *Let* $H_n^{(\alpha)}(x) = \sum_{k=0}^{n-1} \psi_k(x)/(k+1)^{\alpha}$, $0 < \alpha < 1$. *Then*

$$(2.3) \qquad |H_n^{(\alpha)}(x)| < A_\alpha x^{\alpha-1} \qquad (0 < x < 1).$$

PROOF. The proof is quite analogous to that in the case of TFS (see Salem and Zygmund [5]). For $x \leqq 1/n$, we have

$$(2.4) \qquad |H_n^\alpha(x)| \leqq \sum_{k=0}^{n-1} 1/(k+1)^\alpha = O(n^{1-\alpha}) = O(x^{\alpha-1}).$$

In case $n > 1/x$,

$$(2.5) \qquad H_n^{(\alpha)}(x) = \sum_{k=0}^{[1/x]} \psi_k(x)/(k+1)^\alpha + \sum_{k=[1/x]+1}^{n-1} \psi_k(x)/(k+1)^\alpha \equiv S_1 + S_2,$$

say. From the fact above proved, we have $S_1 = O(x^{\alpha-1})$. By Abel's transformation,

$$(2.6) \qquad S_2 = \sum_{[1/x]+1}^{n-2} \left[\frac{1}{(k+1)^\alpha} - \frac{1}{(k+2)^\alpha}\right] T_{k+1}(x) + \frac{1}{(n-1)^\alpha} T_n(x),$$

where $T_k(x) = \sum_{j=[1/x]+1}^{k-1} \psi_j(x)$. Since $T_k(x) = O(1/x)$ by (1.19), we have

$$(2.7) \qquad |S_2| \leqq x^\alpha O(1/x) + \frac{1}{n^\alpha} O(1/x) = O(x^{\alpha-1}).$$

From (2.4), (2.5), (2.6), and (2.7) we have the lemma.

LEMMA 2. *For $0 < \alpha < 1$ and $0 < m \leqq n$,*

$$(2.8) \qquad \left|\sum_{k=n-m}^{n-1} A_{n-k-1}^{(-\alpha)} \psi_k(x)\right| < A_\alpha x^{\alpha-1} \qquad (0 < x < 1).$$

PROOF. For $x \leqq 1/m$,

$$(2.9) \qquad \left|\sum_{k=n-m}^{n-1} A_{n-k-1}^{(-\alpha)} \psi_k(x)\right| \leqq \sum_{k=n-m}^{n-1} A_{n-k-1}^{(-\alpha)} \leqq A_\alpha \sum_{k=0}^{m} 1/(k+1)^\alpha \leqq A_\alpha m^{1-\alpha} \leqq A_\alpha x^{\alpha-1}.$$

In case $1/m < x$,

$$(2.10) \qquad \sum_{k=n-m}^{n-1} A_{n-k-1}^{(-\alpha)} \psi_k(x) = \left[\sum_{k=n-m}^{n-[1/x]-1} + \sum_{k=n-[1/x]}^{n-1}\right] A_{n-k-1}^{(-\alpha)} \psi_k(x) \equiv U_1 + U_2,$$

say. By Abel's transformation and (1.19)

$$\begin{aligned} |U_1| &\leqq \sum_{k=n-m}^{n-[1/x]-2} |A_{n-k-1}^{(-\alpha-1)} D_{k+1}(x)| + A_{[1/x]}^{(-\alpha)} |D_{n-[1/x]}(x)| + A_m^{(-\alpha)} |D_{n-m}(x)| \\ (2.11) \qquad &\leqq \frac{2}{x} \sum_{k=n-m}^{n-[1/x]-2} |A_{n-k-1}^{(-\alpha-1)}| + \frac{2}{x} A_{[1/x]}^{(-\alpha)} + \frac{2}{x} A_m^{(-\alpha)} \\ &\leqq \frac{A_\alpha}{x} \sum_{k=[1/x]}^{m-1} \frac{1}{k^\alpha} + \frac{A_\alpha}{x}\left[\frac{1}{x}\right]^{-\alpha} + \frac{A_\alpha}{x} m^{-\alpha} \\ &\leqq A_\alpha x^{\alpha-1}. \end{aligned}$$

On the other hand

3) A_α, B_α, ... denote the constants depending only on α.

$$|U_2| \leqq \sum_{k=n-[1/x]}^{n-1} A_{n-k-1}^{(-\alpha)} \leqq A_\alpha \sum_{k=0}^{[1/x]} \frac{1}{(k+1)^\alpha} \leqq A_\alpha x^{\alpha-1}. \tag{2.12}$$

Combining these results we have the lemma.

We shall denote the $(C, -\alpha)$ means of the series (2.2) by $N_n^{(\alpha)}(x; f)$. Then

$$\begin{aligned} N_n^{(\alpha)}(x;f) &= \frac{1}{A_{n-1}^{(-\alpha)}} \sum_{k=0}^{n-1} A_{n-k-1}^{(-\alpha)}(k+1)^{-\alpha} c_k \psi_k(x) \\ &= \frac{1}{A_{n-1}^{(-\alpha)}} \sum_{k=0}^{n-1} A_{n-k-1}^{(-\alpha)}(k+1)^{-\alpha} \psi_k(x) \int_0^1 \psi_k(t) f(t)\, dt \\ &= \frac{1}{A_{n-1}^{(-\alpha)}} \sum_{k=0}^{n-1} A_{n-k-1}^{(-\alpha)}(k+1)^{-\alpha} \int_0^1 \psi_k(x \dotplus t) f(t)\, dt \\ &= \int_0^1 f(x \dotplus t) \frac{1}{A_{n-1}^{(-\alpha)}} \sum_{k=0}^{n-1} A_{n-k-1}^{(-\alpha)}(k+1)^{-\alpha} \psi_k(t) dt. \end{aligned}$$

Thus

$$N_n^{(\alpha)}(x;f) = \int_0^1 f(x \dotplus t) N_n^{(\alpha)}(t) dt, \tag{2.13}$$

where we set

$$N_n^{(\alpha)}(t) = \frac{1}{A_{n-1}^{(-\alpha)}} \sum_{k=0}^{n-1} A_{n-k-1}^{(-\alpha)}(k+1)^{-\alpha} \psi_k(t). \tag{2.14}$$

Then we have

LEMMA 3. *For* $0 < \alpha < 1$,

$$N_n^{(\alpha)}(t) \leqq A_\alpha t^{\alpha-1} \qquad (0 < t < 1). \tag{2.15}$$

PROOF. We write

$$\begin{aligned} N_n^{(\alpha)}(t) &= \frac{1}{A_{n-1}^{(-\alpha)}} \left[\sum_{k=0}^{[n/2]-1} + \sum_{k=[n/2]}^{n-1} \right] A_{n-k-1}^{(-\alpha)}(k+1)^{-\alpha} \psi_k(t) \\ &= P_n + Q_n, \end{aligned} \tag{2.16}$$

say. By Abel's transformation

$$P_n = \frac{1}{A_{n-1}^{(-\alpha)}} \left\{ \sum_{k=0}^{[n/2]-2} H_{k+1}^{(\alpha)}(t) | A_{n-k-1}^{(-\alpha-1)} | + H_{[n/2]}^{(\alpha)}(t) A_{n-[n/2]}^{(-\alpha)} \right\}.$$

Therefore we have by Lemma 1

$$\begin{aligned} |P_n| &\leqq \frac{1}{A_{n-1}^{(-\alpha)}} \left\{ A_\alpha t^{\alpha-1} \sum_{k=0}^{[n/2]-2} | A_{n-k-1}^{(-\alpha-1)} | + A_\alpha t^{\alpha-1} A_{n-[n/2]}^{(-\alpha)} \right\} \\ &\leqq A_\alpha n^\alpha t^{\alpha-1} \left\{ \sum_{k=[n/2]}^{n} \frac{1}{k^{\alpha+1}} + n^{-\alpha} \right\} \\ &\leqq A_\alpha t^{\alpha-1}. \end{aligned} \tag{2.17}$$

On the other hand using Abel's transformation again, we have

$$Q_n=\frac{1}{A^{(-\alpha)}_{n-1}}\sum_{k=[n/2]}^{n-2}\left\{(k+1)^{-\alpha}-(k+2)^{-\alpha}\right\}\sum_{j=[n/2]}^{k}A^{(-\alpha)}_{n-j-1}\psi_j(t)+\frac{n^{-\alpha}}{A^{(-\alpha)}_{n-1}}\sum_{k=[n/2]}^{n-1}A^{(-\alpha)}_{n-k-1}\psi_k(t),$$

therefore

$$|Q_n| \leqq \frac{A_\alpha}{A^{(-\alpha)}_{-1}}\left\{\sum_{k=[n/2]}^{n-1}k^{-\alpha-1}+n^{-\alpha}\right\}\max_{[n/2]\leqq k<n}\left|\sum_{j=[n/2]}^{k}A^{(-\alpha)}_{n-j-1}\psi_j(t)\right| \tag{2.18}$$
$$\leqq A_\alpha \max_{[n/2]\leqq k<n}\left|\sum_{j=[n/2]}^{k}A^{(-\alpha)}_{n-j-1}\psi_j(t)\right|.$$

Thus by Lemma 2 we have

$$|Q_n| \leqq A_\alpha t^{\alpha-1}. \tag{2.19}$$

From (2.16), (2.17) and (2.19) we get the lemma.

Let us now proceed to the proof of Theorem. By (2.13) and (2.14),

$$|N_n^{(\alpha)}(x;f)| \leqq \int_0^1 |f(x \dotplus t)|\,|N_n^{(\alpha)}(t)|\,dt,$$

and by virtue of Lemma 3, we have

$$\sup_n |(N_n^{(\alpha)}x;f)| \leqq A_\alpha \int_0^1 f(x \dotplus t)\, t^{\alpha-1}dt. \tag{2.20}$$

Integrating both sides of this inequality with respect to x, we get

$$\int_0^1 \sup |N_n^{(\alpha)}(x:f)|\,dx \leqq A_\alpha\int_0^1 dx\int_0^1 |f(x\dotplus t)|\,t^{\alpha-1}\,dt$$
$$= A_\alpha\int_0^1 t^{\alpha-1}\,dt\int_0^1 f(x\dotplus t)\,dx \tag{2.21}$$
$$\leqq A_\alpha\int_0^1 |f(x)|\,dx.$$

From this inequality (2. 21) we can easily deduce the conclusion of Theorem 1. For example, we may argue as follows. Let us write $f(x) = g(x) + h(x)$, where $g(x)$ is a polynomial formed by Walsh functions and $h(x)$ satisfies the inequality $\int_0^1 |h(x)|\,dx < \varepsilon/A_\alpha$, ε being a given positive number and A_α being the constant which appears in the right hand side of the inequality (2.21). We shall denote the sum of the series $\sum_{n=0}^{\infty} c_n(n+1)^{-\alpha}\psi_n(x)$ (which converges almost everywhere) by $f^*(x)$. Then

$$f^*(x) = g^*(x) + h^*(x), \tag{2.22}$$

where $g^*(x)$ and $h^*(x)$ are defined by g and h in the same way as f^* is defined by f. By (2.21)

(2.23) $$\int_0^1 \sup_n |N_n^{(\alpha)}(x;h)|\,dx \leqq A_\alpha \int_0^1 |h(x)|\,dx < \varepsilon,$$

and so

(2.24) $$\int_0^1 |h^*(x)|\,dx < \varepsilon.$$

Therefore the measure of the set $E(\varepsilon)$ of points x on which $\sup_n |N_n^{(\alpha)}(x;h)| > \sqrt{\varepsilon}$ or $|h^*(x)| > \sqrt{\varepsilon}$ is $< 2\sqrt{\varepsilon}$. Since

(2.25) $$N_n^{(\alpha)}(x;f) - f^*(x) = N_n^{(\alpha)}(x;g) - g^*(x) + N_n^{(\alpha)}(x;h) - h^*(x),$$

we have outside $E(\varepsilon)$

(2.26) $$\limsup_{n\to\infty} |N_n^{(\alpha)}(x;f) - f^*(x)| \leqq \sup_n |N_n^{(\alpha)}(x;h)| + |h^*(x)| \leqq 2\sqrt{\varepsilon}.$$

Since $|E(\varepsilon)| \to 0$ as $\varepsilon \to 0$, Theorem 1 is proved.

Theorem 1 holds in the case of multiple WFS.

THEOREM 2. *Let $f(x_1, x_2, \cdots, x_k)$ be integrable on the unit cube of k-dimensional Euclidean space and its WFS be*

(2.27) $$f(x_1, x_2, \cdots, x_k) \sim \sum_{n_1, n_2, \cdots, n_k = 0}^{\infty} c_{n_1, n_2, \cdots, n_k} \psi_{n_1}(x)\psi_{n_2}(x)\cdots\psi_{n_k}(x_k).$$

Then the series

(2.28) $$\sum_{n_1, n_2, \cdots, n_k = 0}^{\infty} c_{n_1, n_2, \cdots, n_k}(n+1)^{-\alpha_1}(n+1)^{-\alpha_2}\cdots(n+1)^{-\alpha_k}\psi_{n_1}(x_1)\psi_{n_2}(x_2)\cdots\psi_{n_k}(x_k)$$

is summable $(C, -\alpha_1, -\alpha_2, \cdots, -\alpha_k)$ almost everywhere for $0 < \alpha_1, \alpha_2, \cdots, \alpha_k < 1$.

3. Cesaro summability of positive order. Walsh [6] proved the following theorem:

THEOREM 3. *If $f(x)$ is integrable and if $\lim_{x\to x_0} f(x) = s$ exists, then*

(3.1) $$\sigma_n(x_0;f) \to s.$$

In this section, we shall give an alternative proof of this theorem.

LEMMA 4. *For $n \geqq 2$, we may write*

(3.2) $$\begin{aligned} K_{2^n}(t) &= (2^n+1)/2 && (0 \leqq t < 2^{-n}),\\ &= 2^{n-2} && (2^{-n} \leqq t < 2^{-n+1})\\ &= 2^{n-i-2} && (2^{-n+i} \leqq t < 2^{-n+i} + 2^{-n}),\\ &= 0 && (2^{-n+i} + 2^{-n} \leqq t < 2^{-n+i+1},\ i = 1, \cdots, n-1). \end{aligned}$$

This is known (see Yano [8]).

LEMMA 5. *Under the condition of theorem 3, we have*

(3.3) $$\sigma_{2^n}(x_0;f) \to s.$$

PROOF. We may write

(3.4) $$\sigma_{2^n}(x_0;f) = \int_0^1 f(x_0 \dotplus t) K_{2^n}(t)\,dt.$$

Thus by Lemma 4,

$$\sigma_{2^n}(x_0;f)=\frac{2^n+1}{2}\int_0^{2^{-n}}f(x_0\dotplus t)dt+\sum_{i=0}^{n-1}2^{i-2}\int_{2^{-i}}^{2^{-i}+2^{-n}}f(x_0\dotplus t)dt. \tag{3.5}$$

For given $\varepsilon>0$, there exists an n_0 such that $|f(x_0\dotplus t)-s|<\varepsilon$ for $0<|t|<2^{-n_0}$. Therefore

$$|\sigma_{2^n}(x_0;f)-s|\leqq\frac{2^n+1}{2}\int_0^{2^{-n}}|f(x_0\dotplus t)-s|\,dt+\sum_{i=0}^{n-1}2^{i-2}\int_{2^{-i}}^{2^{-i}+2^{-n}}|f(x_0\dotplus t)-s|\,dt$$

$$<\varepsilon\left(\frac{2^n+1}{2}+\sum_{i=n_0+1}^{n}2^{n-i+2}\right)2^{-n}+\sum_{i=0}^{n_0}2^{i-2}\int_{2^{-i}}^{2^{-i}+2^{-n}}|f(x_0\dotplus t)-s|\,dt \tag{3.6}$$

$$\leqq\varepsilon+o(1).$$

Thus the lemma is proved.

PROOF OF THEOREM 3. By virtue of (1.23), we may write

$$\sigma_n(x_0;f)-s=\frac{1}{n}\sum_{i=1}^{r}2^{n_i}\int_0^1[f(x_0\dotplus t)-s]\psi_{n-n(i)}(t)K_{2^{n_i}}(t)\,dt$$

$$+\frac{1}{n}\sum_{i=1}^{r}n^{(i)}\int_0^1[f(x_0\dotplus t)-s]D_{2^{n_i}}(t)\,dt \tag{3.7}$$

$$\equiv I_n+J_n$$

say. It is obvious that

$$\int_0^1[f(x_0\dotplus t)-s]\,D_{2^k}(t)\,dt=2^k\int_0^{2^{-k}}[f(x_0\dotplus t)-s]\,dt=o(1)$$

as $k\to\infty$. Therefore J_n is a weighted mean of a null-sequence, and it is easy to see that the weights are distributed in such a manner as to make the averages converge to zero with increasing n, that is

$$J_n=o(1). \tag{3.8}$$

On the other hand it was proved in the proof of above lemma that

$$\int_0^1|f(x_0\dotplus t)-s|K_{2^k}(t)dt=o(1)\qquad\text{as } k\to\infty. \tag{3.9}$$

Therefore I_n is also a weighted average of a null-sequence and by the same reason as above it follows that

$$I_n=o(1). \tag{3.10}$$

(3.8) and (3.10) prove the theorem.

REMARK. 1°. In Theorem 3, if x_0 is a dyadic rational and $f(x_0+0)$ exists, then it holds that

$$\sigma_n(x_0;f)\to f(x_0+0); \tag{3.11}$$

in fact if x_0 is a dyadic rational then $x_0\dotplus t$ lies in the right hand side of x_0 for sufficiently small t, therefore the above proof shows (3.11). 2°. If

$f(x)$ is continuous in the interval (a, b), then the summability in Theorem 3 is uniform in the interval $(a+\varepsilon, b-\varepsilon)$ for any $\varepsilon>0$. 3°. In the above proof it was proved that

(3.12) $$\int_0^1 |(f(x_0 \dotplus t)-s)K_n(t)|\,dt = o(1).$$

More generally than Theorem 3 we shall prove following theorem.

THEOREM 4. *Under the same conditions as in Theorem* 3, *we have for* $\alpha>0$

(3.13) $$\sigma_n^{(\alpha)}(x_0; f) \to s.$$

PROOF. We may assume that $0<\alpha<1$. The kernel $K_n^{(\alpha)}(t)$ of (C, α) summability may be written as (cf. Yano [9; proof of Theorem 2])

$$K_n^{(\alpha)}(t) = \frac{1}{A_n^{(\alpha)}} \sum_{i=1}^{r} \psi_{n-n^{(i)}-1}(t) \left\{ \sum_{k=0}^{2^{n_i}-3} k A^{(\alpha-2)}_{n^{(i)}+k+1} K_{k+1}(t) + (2^{n_i}-2)\cdot A^{(\alpha-1)}_{n^{(i-1)}-2} K_{2^{n_i}-1}(t) + A^{(\alpha)}_{n^{(i-1)}-1} D_{2^{n_i}}(t) \right\}.$$

Therefore

(3.14) $$\begin{aligned} |\sigma_n^{(\alpha)}(x_0; f) - s| \leqq \frac{1}{A_{n-1}^{(\alpha)}} \sum_{i=1}^{r} \Bigg\{ & \sum_{k=1}^{2^{n_i}-3} k |A^{(\alpha-2)}_{n^{(i)}+k+1}| \int_0^1 |f(x_0 \dotplus t) - s) K_{k+1}(t)|\,dt \\ & + 2^{n_i} A^{(\alpha-1)}_{n^{(i-1)}-2} \int_0^1 |f(x_0 \dotplus t) - s) K_{2^{n_i}-1}(t)|\,dt \\ & + A^{(\alpha)}_{n^{(i-1)}-1} \int_0^1 |f(x_0 \dotplus t) - s| D_{2^{n_i}}(t)\,dt \Bigg\}. \end{aligned}$$

By (3.12) and the relation $\int_0^1 |f(x_0 \dotplus t) - s| D_{2^k}(t)\,dt = o(1)$, the right hand side of (3.14) is a weighted average of a null-sequence and it is easy to see that the weights are distributed in such a manner as to wake the average converge to zero with increasing n (cf. Yano [9; proof of Theorem 3]). and the theorem is proved.

4. Convergence. In this section we shall prove two analogues of Marcinkiewicz' theorem in case of TFS. (see Marcinkiewicz [2] and [3].)

THEOREM 5. *Let $f(x)$ be integrable. If $f(x)$ satisfies*

(4.1) $$\frac{1}{t}\int_0^t |f(x+u) - f(x)|\,du = O(1/\log 1/|t|)$$

for every point x belonging to a set E of positive measure, then the WFS *of $f(x)$ converges for almost every point of E.*

LEMMA 6. *Under the hypothesis of Theorem* 5, *we can write*

$$f(x) = g(x) + h(x), \tag{4.2}$$

where $g(x)$ satisfies

$$g(x+t) - g(x) = O(1/\log 1/|t|) \tag{4.3}$$

uniformly in $0 \leqq x \leqq 1$, and $h(x)$ vanishes on a perfect subset P of E and satisfies

$$\int_0^1 \frac{|h(x+t)|}{t}\, dt \tag{4.4}$$

for almost every point of P.

This is due to Marcinkiewicz [2: Lemma 1 and 2].

LEMMA 7. *Let $g(x)$ be of L^2 and its WFS be $g(x) \sim \sum_{n=0}^{\infty} d_n\, \psi_n(x)$. If*

$$\sum_{n=2}^{\infty} d_n^2 \log n < \infty \tag{4.5}$$

then $\sum_{n=0}^{\infty} d_n \psi_n(x)$ converges almost everywhere.

This is due to Paley [4]. (The alternative proof will be given in §6, Theorem 9).

LEMMA 8. *Under the notation of Lemma* 7, (4.5) *is equivalent to the following inequality*:

$$\int_0^1 dx \int_0^1 \frac{[g(x \dotplus t) - g(x)]^2}{t}\, dt < \infty. \tag{4.6}$$

PROOF. By (1.12)

$$g(x \dotplus t) - g(x) \sim \sum_{n=0}^{\infty} [\psi_n(t) - 1] d_n \psi_n(x). \tag{4.7}$$

Therefore by Parseval's theorem

$$\int_0^1 [g(x \dotplus t) - g(x)]^2 dx = \sum_{n=0}^{\infty} [1 - \psi_n(t)]^2 d_n^2. \tag{4.8}$$

If we write $n = 2^k + k'$, $0 \leqq k' < 2^k$, since $\psi_n(t)$ is equals to 1 in the interval $0 \leqq t < 2^{-k-1}$, we have

$$\int_0^1 \frac{[1-\psi_n(t)]^2}{t}\, dt = \int_{2^{-k-1}}^1 \frac{[1-\psi_n(t)]^2}{t}\, dt. \tag{4.9}$$

It is easily verified that there exist two constants A, B $(0 < A < B)$ such that

$$A \log n < \int_0^1 \frac{[1-\psi_n(t)]^2}{t} dt < B \log n. \tag{4.10}$$

Therefore we have

$$\int_0^1\int_0^1 \frac{[g(x \dotplus t) - g(x)]^2}{t} dx\,dt = \sum_{n=0}^{\infty} d_n^2 \int_0^1 \frac{[1-\psi_n(t)]^2}{t} dt \tag{4.11}$$

provided that either side of (4.11) exists.

(4.11) and (4.10) prove the lemma.

Now the proof of Theorem 5 is immediate in fact, by Lemma 6, $f(x) = g(x) + h(x)$ and g, h satisfy (4.3) and (4.4). It is easy to see that $g(x)$ satisfies (4.6), and so the WFS of $g(x)$ converges almost everywhere by Lemma 7 and 8. On the other hand $h(x)$ satisfies the Dini's condition at almost every point of P, and so the WFS of $h(x)$ converges at almost everypoint of P (See Fine [1; Theorem XII]). Therefore the WFS of $f(x) = g(x) + h(x)$ converges at almost every point of P, and the proof of theorem is completed.

THEOREM 6. *If $f(x)$ satisfies*

$$\int_0^1\int_0^1 \frac{|f(x+t) - f(x-t)|^p}{t} dx\,dt < \infty \qquad (2 \geqq p \geqq 1), \tag{4.12}$$

then the WFS of $f(x)$ converges almost everywhere.

PROOF. The proof of this theorem is almost identical to that of Marcinkiewicz [3], but for the completeness we shall repeat it.

If $f(x)$ satisfies (4.12), we shall say that $f(x)$ satisfies the condition $I_p(1 \leqq p \leqq 2)$.

First, we shall prove that if $f(x)$ satisfies the condition $I_p(1 \leqq p \leqq 2)$ then

$$\int_0^1\int_0^1 \frac{|f(x \dotplus t) - f(x)|^p}{t} dx\,dt \leqq \int_0^1\int_0^1 \frac{|f(x+t) - f(x)|^p}{t} dt\,dx + \int_0^1\int_0^1 \frac{|f(x-t) - f(x)|^p}{t} dt\,dx, \tag{4.13}$$

holds true.

In fact, by (1.8)

$$0 \leqq x - t \leqq x \dotplus t \qquad \text{for } 0 \leqq t \leqq x < 1$$

and

$$0 < t - x \leqq x \dotplus t \qquad \text{for } 0 \leqq x < t < 1.$$

Therefore

$$\int_0^1\int_0^1 \frac{|f(x \dotplus t) - f(x)|^p}{t} dx\,dt = \int_0^1 dx \int_0^x \frac{|f(x) - f(t)|^p}{x \dotplus t} dt$$

$$+\int_0^1 dx\int_x^1 \frac{|f(t)-f(x)|^p}{x+t}\,dt$$

(4.14)
$$\leqq \int_0^1 dx\int_0^x \frac{|f(t)-f(x)|^p}{x-t}\,dt+\int_0^1 dx\int_x^1 \frac{|f(t)-f(x)|^p}{t-x}\,dt$$
$$=\int_0^1 dx\int_0^x \frac{|f(x-t)-f(x)|^p}{t}\,dt+\int_0^1 dx\int_0^{1-x} \frac{|f(x+t)-f(x)|^p}{t}\,dt$$
$$\leqq \int_0^1\int_0^1 \frac{|f(x-t)-f(x)|^p}{t}\,dxdt+\int_0^1\int_0^1 \frac{|f(x+t)-f(x)|^p}{t}\,dx\,dt,$$

and this proves (4.13).

If $f(x)$ satisfies the condition I_p, then, since it is obvious that

(4.15)
$$\int_0^1\int_0^1 \frac{|f(x\pm t)-f(x)|^p}{t}\,dx\,dt<\infty,$$

it follows that

(4.16)
$$\int_0^1\int_0^1 \frac{|f(x\pm t)-f(x)|^p}{t}\,dt\,dx<\infty.$$

We shall say that $f(x)$ satisfies the condition $\dot{I}_p$ if $f(x)$ satisfies (4.16).

Now, if $f(x)$ satisfies $\dot{I}_1$ or $\dot{I}_2$, then Dini's test or Lemma 7, 8 shows that the WFS of $f(x)$ converges almost everywhere.

For p, $1<p<2$, following to Marcinkiewicz, we shall argue as follows. We can assume without any loss of generality that $f(x)$ is non-negative. Let us set .

(4.17)
$$A_n=\{x;f(x)\leqq n\},\ B_n=\{x;n<f(x)\leqq n+1\},\ C_n=\{x;f(x)>n+1\},$$
$$C_x=\{t;x\dotplus t\in C_n\}.$$

We may suppose $|A_n|>0$.

Moreover let us set

(4.18)
$$f_1(x)=f(x)\quad(x\in A_n\cup B_n);f_1(x)=n+1\qquad(x\overline{\in} A_n\cup B_n),$$
$$f_2(x)=f(x)-f_1(x).$$

If $f(x)$ satisfies the condition $\dot{I}_p$, then for almost every x

(4.19)
$$\int_0 \frac{|f(x\dotplus t)-f(x)|^p}{t}\,dt<\infty,$$

and for almost every x of A_n

(4.20)
$$\int_{C_x} \frac{|f(x\dotplus t)-f(x)|^p}{t}\,dt<\infty.$$

Since $|f(x\dotplus t)-f(x)|\geqq 1$ for $x\in A_n$, $t\in C_x$, we have for almost every x of A_n

$$\int_{Cx} \frac{|f(x \dotplus t) - f(x)|}{t} dt < \infty. \tag{4.21}$$

On the other hand, since for every x', x

$$|f(x') - f(x)| \geqq |f_2(x') - f_2(x)|, \tag{4.22}$$

we have by (4.21)

$$\int_{Cx} \frac{|f_2(x \dotplus t) - f_2(x)|}{t} dt = \int_0^1 \frac{|f_2(x \dotplus t)|}{t} dt < \infty$$

for almost every $x \in A_n$. Therefore the WFS of $f_2(x)$ converges at almost every point of A_n by Dini's test.

Since (4.22) holds for f_1 instead of f_2, $f_1(x)$ satisfies the condition $\dot{I}_p$ and is bounded. Hence $f_1(x)$ satisfies the condition $\dot{I}_2$ and so the WFS of $f_1(x)$ converges almost everywhere by Lemma 7 and 8.

It is proved that the WFS of $f(x)$ converges at almost every point of A_n. Since $|A_n| \to 1$ as $n \to \infty$, the proof of the theorem is completed.

5. Special series. In this section, we shall study the property of a special series $\sum_{n=0}^{\infty} c_n \psi_n(x)$, where $\{c_n\}$ is a given sequence which satisfies suitable conditions. The results obtained here are quite analogous to these in the case of TFS.

THEOERM 7. *If $c_n \to 0$ and $\{c_n\}$ is quasi-convex, the series*

$$\sum_{n=0}^{\infty} c_n \psi_n(x) \tag{5.1}$$

converges, save for $x = 0$, to an integrable function $f(x)$, and is the WFS of $f(x)$.

For the proof, we need the following two lemmas.

LEMMA 9. *If $K_n(x)$ is the "Fejér kernel" defined by (1.23), then*

$$\int_0^1 |K_n(x)|\, dx \leqq 2. \tag{5.2}$$

PROOF. Using the notations in (1.23)-(1.26), we have

$$n\int_0^1 |K_n(x)|\, dx \leqq \sum_{i=1}^{r} 2^{n_i} \int_0^1 K_{2^{n_i}}(x)\, dx + \sum_{i=1}^{r} n^{(i)} \int_0^1 D_{2^{n_i}}(x)\, dx. \tag{5.3}$$

Therefore the equalities

$$\int K_{2^{n_i}}(x) dx = 1, \qquad \int_0^1 D_{2^{n_i}}(x) dx = 1$$

prove the lemma.

LEMMA 10. *If* $c_n \geqq c_{n+1} \to 0$ *and the series* $\sum_{n=0}^{\infty} c_n \psi_n(x)$ *converges, save for* $x=0$, *to an integrable function* $f(x)$, *then the series* $\sum_{n=0}^{\infty} c_n \psi_n(x)$ *is the Fourier series of* $f(x)$.

PROOF. For any fix m, the series $\sum_{n=0}^{\infty} c_n \psi_n(x)(1-\psi_m(x))$ converges uniformly to $f(x)(1-\psi_m(x))$, in fact by an argument as above we have

$$\left|(1-\psi_m(x))\sum_{n=p}^{q} c_n\psi_n(x)\right| \leqq \left|1-\psi_m(x)\right|\left|\sum_{n=p}^{q}\Delta c_n D_{n+1}(x) + \lambda_q D_{q+1}(x) - \lambda_p \lambda_p(x)\right|$$
$$\leqq |1-\psi_m(x)|4c_p/x.$$

Therefore it follows that

$$\int_0^1 f(x)dx - \int_0^1 f(x)\psi_m(x)dx = c_0 - c_m.$$

Since $c_m \to 0$ as $m \to \infty$, we have

$$\int_0^1 f(x)dx = c_0,$$

and then

$$\int_0^1 f(x)\psi_m(x)dx = c_m.$$

This proves the lemma.

We shall prove the theorem. Applying Abel's transformation twice we obtain

$$(5.4)\qquad s_n(x) = \sum_{k=0}^{n-3}(k+1)\Delta^2 c_k K_{k+1}(x) + (n-1)\Delta c_{n-2} K_{n-1}(x) + c_{n-1}D_n(x).$$

Since $|D_n(x)| < 2/x$ and $|K_n(x)| < 2/x$ $(0 < x < 1)$, the last two terms on the right tend to 0 with $1/n$ for $x \neq 0$, and therefore $s_n(x) \to f(x) = \sum_{k=0}^{\infty}(k+1)\Delta^2 c_k K_k(x)$. Since

$$|f(x)| \leqq \sum_{k=0}^{\infty}(k+1)\Delta^2 c_k |K_{k+1}(x)|$$

and the last series integrated over $(0,1)$ gives the finite value $\sum_{k=0}^{\infty} 2(k+1)\Delta^2 c_k$, $f(x)$ is integrable.

The remaining part of the theorem is immediate by Lemma 10.

In particular, the series

$$\sum_{n=1}^{\infty} \frac{\psi_n(x)}{n^\alpha} \ (\alpha > 0), \qquad \sum_{n=2}^{\infty} \frac{\psi_n(x)}{\log n}, \text{ etc} \tag{5.5}$$

are the Fourier series of integrable functions.

THEOREM 8. *If* $c(x)$, $x \geqq 0$, *is positive and convex function*, $c(x) \to 0$ *as* $x \to +\infty$, *and if for* $c_k = c(k)$, $k(c_k - c_{k+1})$ *is non-increasing and* $\sum_{n=0}^{\infty} c_n = \infty$, *then*

$$f(x) \sim \int_1^{1/x} t[c(t) - c(t+1)]dt \sim \int_n^{1/x} t|c'(t)|dt \qquad (x \to +0), \tag{5.6}$$

where

$$f(x) = \sum_{n=0}^{\infty} c_n \psi_n(x).$$

PROOF. Let us put $x_p = 1/2^p$. We have obviously

$$\begin{aligned} f(x_p) &= \sum_{k=0}^{2^p-1} c_k\psi_k(x_p) + \sum_{k=2^p}^{2^{p+1}-1} c_k\psi_k(x_p) + \cdots \\ &= \sum_{k=0}^{2^{p-1}-1} (c_k - c_{2^p-k-1}) + \sum_{k=0}^{2^{p-1}-1} (c_{2^p+k} - c_{2^{p+1}-k-1}) + \cdots \\ &\geqq \sum_{k=0}^{2^{p-1}-1} (k+1)\Delta c_k + \sum_{k=0}^{2^{p-1}-1} (k+1)\Delta c_{2^p+k} + \cdots \end{aligned} \tag{5.7}$$

From the assumptions on the sequence $\{c_n\}$, it follows that

$$\sum_{r=0}^{\infty} \sum_{k=0}^{2^{p-1}-1} (k+1)\Delta c_{r\cdot 2^p+k} \leqq f(x_p) \leqq 2 \sum_{k=0}^{2^{p-1}-1} (k+1)\Delta c_k. \tag{5.8}$$

Let x_p' be $2^{-p} \leqq x_p' < 2^{-p+1}$. Then, since $\psi_k(x_p') = \psi_k(x_p)$ $(0 \leqq k < 2^p)$,

$$\begin{aligned} f(x_p') &= \sum_{k=0}^{2^p-1} c_k\psi_k(x_p') + \sum_{k=2^p}^{\infty} c_k\psi_k(x_p') \\ &= \sum_{k=0}^{2^p-1} c_k\psi_k(x_p) + \sum_{k=2^p}^{\infty} c_k\psi_k(x_p'). \end{aligned} \tag{5.9}$$

Remembering that $D_{2^p}(x_p') = 0$, we have

$$\begin{aligned} \sum_{k=\ p}^{\infty} c_k\psi_k(x_p') &= \sum_{k=2^p}^{\infty} D_{k+1}(x_p')\Delta c_k - D_{2^p}(x_p')c_{2^p} \\ &= \sum_{k=2^p}^{\infty} D_{k+1}(x_p')\Delta c_k . \end{aligned} \tag{5.10}$$

If we write $k = r2^p + s$, $0 \leqq s \leqq 2^p$, $r = 1, 2, \cdots$; then by virtue of (1.22) and the fact $D_{2^p}(x_p') = 0$, we obtain

$$\sum_{k=2^p}^{\infty} D_{k+1}(x_p')\Delta c_k = \sum_{r=1}^{\infty}\sum_{s=0}^{2^p-1} \{D_{2^p}(x_p')D_r(2^p x_p') \tag{5.11}$$

$$+ \psi_r(2^p x_p')D_{s+1}(x)\}\Delta c_{r\cdot 2^p+s} = \sum_{r=1}^{\infty}\sum_{s=0}^{2^p-1} \psi_r(2^p x_p')D_{s+1}(x_p')\Delta c_{r\cdot 2^p+s}.$$

Thus we have

$$\left|\sum_{k=2^p}^{\infty} D_{k+1}(x_p')\Delta c_k\right| \leqq \sum_{r=1}^{\infty}\left|\sum_{s=0}^{2^p-1} \Delta c_{r\cdot 2^p+s} D_{s+1}(x_p')\right| \tag{5.12}$$

$$= \sum_{r=1}^{\infty}\left|\sum_{s=0}^{2^p-1} c_{r\cdot 2^p+s}\psi_s(x_p')\right|.$$

By a same consideration as in the calculation of (5.7), it follows that

$$\left|\sum_{s=0}^{2^p-1} c_{r\cdot 2^p+s}\psi_s(x_p')\right| \leqq \sum_{s=0}^{2^p-1}(s+1)\Delta c_{r\cdot 2^p+s}. \tag{5.13}$$

Therefore, by (5.8), (5.9) and (5.13) we have for $2^{-p} \leqq x < 2^{-p+1}$

$$\frac{1}{2}\sum_{k=0}^{2^p-1}(k+1)\Delta c_k \leqq f(x) \leqq 2\sum_{k=0}^{2^p-1}(k+1)\Delta c_k. \tag{5.14}$$

The relation (5.6) can be deduced from (5.14) and the assumption on $c(x)$ as in Zygmund [10; p. 155], and the proof of Theorem 8 is completed.

From this theorem we have, in particular, $\sum_{n=1}^{\infty} \psi_n(x)/n^{\alpha} \sim x^{\alpha-1}$ as $x \to +0$ for $0 < \alpha < 1$, and

$$\sum_{k=2}^{\infty} \psi_k(x)/\log k \sim 1/x\log^2 x \tag{5.15}$$

as $x \to +0$.

6. Convergence factor. It was proved by Paley [4] that $\{1/\log^{1/p}(n+2)\}$ $(1 \leqq p \leqq 2)$ is the convergence facter for the WFS of functions of L^p class.

In this section, we shall prove that $\{1/\log^{1/2}(k+2)\}$ and $\{1/\log(k+2)\}$ are the convergence factórs for the WFS of squarely integrable functions and integrable functions respectively. The proofs are quite analogous to those in the case of TES. (See Yano [7].)

Let $f(x)$ be an integrable function and its WFS be

$$f(x) \sim \sum_{n=0}^{\infty} c_n\psi_n(x). \tag{6.1}$$

Lut us put

$$s_n^*(x) = \sum_{k=0}^{n-1} \frac{c_k}{\log(k+2)} \psi_k(x), \tag{6.2}$$

$$s_n^{**}(x) = \sum_{k=0}^{n-1} \frac{c_k}{\log^{1/2}(k+2)} \psi_k(x). \tag{6.3}$$

Then we can prove following theorem.

THEOREM 9. 1°. *If $f(x) \in L^2$, then*

$$\int_0^1 \sup_n |s_n^{**}(x)|^2 dx \leqq A \int_0^1 |f(x)|^2 dx, \tag{6.4}$$

where A is an absolute constant.

2°. *If $f(x)\log(1+f^2(x)) \in L$, then*

$$\int_0^1 \sup_n |s_n^*(x)| dx \leqq A \int_0^1 |f(x)| \log(1+f^2(x)) dx + B, \tag{6.5}$$

where A and B are absolute constants.

3°. *If $f(x) \in L$ then*

$$\left\{\int_0^1 \sup_n |s_n^*(x)|^r dx\right\}^{1/r} \leqq A_r \int_0^1 |f(x)| dx \qquad (0 < r < 1), \tag{6.6}$$

where A is a constant depending only on r.

LEMMA 11. *Let $g(x)$ be an integrable function and $g^*(x)$ be defined by*

$$g^*(x) = \sup_{0<h<1} \frac{1}{2h} \int_{-h}^{h} |g(x+t)| dt, \tag{6.7}$$

then we have

$$\sup_{0<h<1} \frac{1}{h} \int_0^h |g(x \dotplus t)| dt \leqq 2g^*(x). \tag{6.8}$$

PROOF. Let us denote the characteristic function of the interval $0 \leqq t \leqq h$ by $\rho_h(t)$, then

$$\int_0^h |g(x \dotplus t)| dt = \int_0^1 \rho_h(t) |g(x \dotplus t)| dt = \int_0^1 \rho(x \dotplus t) |g(t)| dt. \tag{6.9}$$

The function $\rho_h(x \dotplus t)$ is equal to 1 only for the value of t for which $0 \leqq x \dotplus t \leqq h$ and vanishes elsewhere. Let $x = \sum_{n=1}^{\infty} x_n/2^n$ and $t = \sum_{n=1}^{\infty} t_n/2^n$ be the dyadic expansions of x and t respectively, then by virtue of (1.8)

$$x \dotplus t = \sum_{n=1}^{\infty} |x_n - t_n|/2^n.$$

Therefore, if t satisfies the inequality $0 \leqq x \dotplus t \leqq h$, then

$$\sum_{n=1}^{\infty} \frac{|x_n - t_n|}{2^n} \leqq h.$$

From this it follows

$$x - h \leqq t \leqq x + h.$$

From these considerations, we have

$$(6.10) \qquad \int_0^1 \rho_h(x \dotplus t)|g(t)|dt \leqq \int_{x-h}^{x+h} |g(t)|dt \leqq 2hg^*(x).$$

This proves the lemma.

LEMMA 12. *If $\chi_n(t)$, $0 \leqq t \leqq 1$, $n = 0, 1, 2, \cdots$, are periodic functions with period 1 and absolutely continuous and satisfy the inequalities*

$$(6.11) \qquad \text{a)} \int_0^1 |\chi_n(t)|dt \leqq K_1, \qquad \text{b)} \int_0^1 t \left| \frac{d}{dt} \chi_n(t) \right| dt \leqq K_2,$$

K_1 and K_2 being independent of n. Then

$$(6.12) \qquad \sup_n \left| \int_0^1 g(x \dotplus t)\chi_n(t) \right| dt \leqq Ag^*(x),$$

where A is independent of g.

PROOF. Using Lemma 11, the proof is quite same as in Zygmund's book [10; pp. 246–247].

LEMMA 13. *Let*

$$(6.13) \qquad H_n(t) = \sum_{k=0}^{n-1} \frac{\psi_k(t)}{\log(k+2)},$$

then

$$(6.14) \qquad |H_n(t)| \leqq \chi_n(t) \qquad (0 < t < 1;\ n = 0, 1, 2, \cdots,)$$

where $\chi_n(t)$, $n = 0, 1, \cdots$, satisfy the conditions in the preceding lemma.

PROOF. Using the relation (5.15), the proof is quite same as that in Yano [7; Lemma 1].

LEMMA 14. *If $g(x) \in L$, then*

$$(6.15) \qquad \sup_n \left| \int_0^1 g(t) H_n(x \dotplus t)dt \right| \leqq Ag^*(x),$$

where $H_n(t)$ is defined by (6.13) and A is independent of $g(x)$.

This is obvious by Lemma 12 and 13.

PROOF OF THEOREM 9.1°. Let $n(x)$, $0 \leqq x < 1$, be a measurable function taking non-negative integral values less than n. We have to prove that for any $g(x) \in L^2$ it holds that

$$(6.16)\quad J=\int_0^1 g(x)\sum_{k=0}^{n(x)-1}\frac{c_k\psi_k(x)}{\sqrt{\log(k+2)}}\,dx\leqq A\Big(\int_0^1 g^2(x)dx\Big)^{1/2}\Big(\int_0^1 f^2(x)dx\Big)^{1/2}.$$

We may write

$$(6.17)\quad J=\int_0^1 g(x)\sum_{k=0}^{n-1} p_k(x)\frac{c_k\psi_k(x)}{\sqrt{\log(k+2)}}\,dx,$$

where $p_k(x)$ $(k=0,1,2,\cdots, n-1)$ are the characteristic functions of the suitable sets

$$(0,1)=E_0\supset E_1\supset\cdots\supset E_{n-1}.$$

Then

$$J=\int_0^1 f(t)\,dt\int_0^1 g(x)\sum_{k=0}^{n-1} p_k(x)\frac{\psi_k(x\dotplus t)}{\sqrt{\log(k+2)}}\,dx$$

$$(6.18)\quad \leqq\Big(\int_0^1 f^2(t)dt\Big)^{1/2}\left(\int_0^1 dt\left[\int_0^1 g(x)\sum_{k=0}^{n-1}p_k(x)\frac{\psi_k(x\dotplus t)}{\sqrt{\log(k+2)}}\,dx\right]^2\right)^{1/2}.$$

Since $\displaystyle\int_0^1 \psi_k(x\dotplus t)\psi_l(y\dotplus t)\,dt=\psi_k(x)\psi_l(y)\int_0^1\psi_k(t)\psi_l(t)\,dt,$

the square of the last factor of (6.18) is equal to

$$\int_0^1\int_0^1\sum_{k=0}^{n-1} g(x)g(y)p_k(x)p_k(y)\frac{\psi_k(x\dotplus y)}{\log(k+2)}\,dxdy$$

$$(6.19)\quad =\sum_{k=0}^{n-2}\int_0^1\int_0^1 g(x)g(y)\Delta(p_k(x)p_k(y))\,H_{k+1}(x\dotplus y)\,dxdy$$

$$+\int_0^1\int_0^1 g(x)g(y)p_{n-1}(x)p_{n-1}(y)H_n(x\dotplus y)\,dxdy$$

$$=I_1+I_2.$$

Now

$$I_1=\sum_{k=0}^{n-1}\int_0^1\int_0^1 g(x)g(y)p_k(y)\Delta p_k(x)H_{k+1}(x\dotplus y)\,dxdy$$

$$(6.20)\quad +\sum_{k=0}^{n-2}\int_0^1\int_0^1 g(x)g(y)p_{k+1}(x)\Delta p_k(y)H_{k+1}(x\dotplus y)dxdy$$

$$=I_1'+I_1''.$$

By Lemma 14, we get

$$(6.21)\quad |I_1'|=\left|\sum_{k=0}^{n-2}\int_0^1 g(x)\Delta p_k(x)dx\int_0^1 g(y)p_k(y)H_{k+1}(x\dotplus y)dy\right|$$

$$\leq A \sum_{k=0}^{n-2} \int_0^1 |g(x)| \Delta p_k(x) g^*(x) dx$$

$$\leq A \int_0^1 g(x) g^*(x) dx \leq A \int_0^1 g^2(x) dx.$$

Similarly we have

$$|I_1''| \leq A \int_0^1 g^2(x) dx, \quad |I_2| \leq A \int_0^1 g^2(x) dx. \tag{6.22}$$

From (6.18), (6.19), (6.20), (6.21) and (6.22), we have (6.16) and Theorem 9, 1° is proved.

The remaining parts of the theorem can be shown by Lemma 13 and the maximal theorem of Hardy and Littlewood.

By (6.4) and (6.6), we deduce that $\{1/\sqrt{\log(n+2)}\}$ and $\{1/\log(n+2)\}$ are the convergence factors for the WFS of squarely integrable functions and integrable functions respectively.

The failure of the inequality

$$\int_0^1 \sup |s_n^*(x)| dx \leq A \int_0^1 |f(x)| dx \tag{6.23}$$

can be seen by the example

$$f(x) = \sum_{n=3}^{\infty} \frac{\psi_n(x)}{\log \log n}.$$

The corresponding example for TFS was given by Zygmund [10].

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