A CLASS OF SINGULAR INTEGRAL EQUATIONS

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Some years ago, Professor K. Kondô proposed to solve the following integral equation

$$(1) \qquad f(y) = \frac{1}{\pi}(P)\int_{-1}^{1}\frac{u(x)}{x-y}\,dx - \frac{1}{\pi}\int_{-1}^{1}k(y,x)u(x)dx, \quad (-1<y<1)$$

where (P) indicates Cauchy's principal value. This equation is related to a problem of aerodynamics. The case $k(x, y) \equiv 0$ was first solved by Fuchs-Hopf-Seewald (cf. G. Hamel [1] p. 145) employing Fourier series expansions. Subsequently K. Schröder [3] gave a beautiful solution with the aid of conjugate functions. The author [4] have ever given a solution of (1) following to Schlöder's method. Recently we have been able to see some periodicals published during the war, and learned that Professor E. Reissner [2] has already solved an integral equation of the type (1). But his method seems to be analogous to Fuchs-Hopf-Seewald, and the solution is given by its Fourier series. In this paper we give a method to solve the equation (1), which is different from that of Reissner and gives an easier solution in some cases.

We shall begin with a reciprocal formula of conjugate functions. Let $H(\theta)$ be even and $G(\theta)$ be odd, then we have

$$(2) \qquad G(\theta) = -\frac{1}{\pi}(P)\int_{0}^{\pi}H(\varphi)\frac{\sin\theta}{\cos\varphi-\cos\theta}\,d\varphi,$$

and

$$(3) \qquad H(\theta) = -\frac{1}{\pi}(P)\int_{0}^{\pi}G(\varphi)\frac{\sin\varphi}{\cos\theta-\cos\varphi}\,d\varphi + c,$$

where

$$c = \frac{1}{\pi}\int_{0}^{\pi}H(\varphi)d\varphi$$

(see Schröder [3]). Especially, if $H(\varphi)$ and $G(\varphi)$ belong to $L^2(0, \pi)$, then reciprocal formula (2) and (3) is valid almost everywhere.

If we put, in (1),

$$(4) \qquad y = \cos\theta\,(0 \leqq \theta \leqq \pi) \text{ and } x = \cos\varphi\,(0 \leqq \varphi \leqq \pi),$$

then we have

$$f(\cos\theta) = -\frac{1}{\pi}(P)\int_{\pi}^{0}\frac{u(\cos\varphi)\sin\varphi}{\cos\varphi-\cos\theta}\,d\varphi - \frac{1}{\pi}\int_{\pi}^{0}u(\cos\varphi)\,k(\cos\theta,\ \cos\varphi)\sin\varphi\,d\varphi,$$

that is

$$(5)\qquad f(\cos\theta)\sin\theta = \frac{1}{\pi}(P)\int_0^{\pi} \frac{u(\cos\varphi)\sin\varphi\sin\theta}{\cos\varphi-\cos\theta}\,d\varphi + \frac{1}{\pi}\int_0^{\pi} u(\cos\varphi)\sin\varphi\, k(\cos\theta,\ \cos\varphi)\sin\theta\, d\varphi.$$

If we put

(6) $\quad -f(\cos\theta)\sin\theta = F(\theta)$ and $k(\cos\theta,\ \cos\varphi)\sin\theta = K(\theta,\ \varphi)$

and suppose that $F(\theta)$ is odd in $(-\pi,\ \pi)$, and $K(\theta,\varphi)$ is odd for θ and even for φ, then

(7) $\quad u(\cos\varphi)\sin\varphi \equiv U(\varphi) \qquad (0 \leqq \varphi \leqq \pi)$

is even from the above reciprocal formula.

Then (5) becomes

$$(8)\qquad F(\theta) = -\frac{1}{\pi}(P)\int_0^{\pi} \frac{U(\varphi)\sin\theta}{\cos\varphi-\cos\theta}\,d\varphi - \frac{1}{\pi}\int_0^{\pi} U(\varphi)K(\theta,\varphi)d\varphi.$$

On the other hand, if $f_1(x)$ and $f_2(x)$ belong to the class L^2, then we have by the Parseval formula (see Zygmund [5] p. 76)

$$(9)\qquad \frac{1}{\pi}\int_0^{2\pi} f_1(x)f_2(x)\,dx = \frac{a_0^{(1)}a_0^{(2)}}{2} + \frac{1}{\pi}\int_0^{2\pi} \tilde{f}_1(x)\tilde{f}_2(x)dx,$$

where $\tilde{f}$ is conjugate to f and $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx.$

Let us write by $\tilde{K}_\varphi(\theta,\varphi)$ the conjugate function of $K(\theta,\varphi)$ with regard to the argument φ, then (8) becomes, by (9),

$$(10)\qquad F(\theta) = \tilde{U}(\theta) - \frac{1}{\pi}\int_0^{\pi} \tilde{U}(\varphi)\tilde{K}_\varphi(\theta,\varphi)\,d\varphi - \frac{c}{\pi}\int_0^{\pi} K(\theta,\varphi)\,d\varphi,$$

where

$$(11)\qquad c = \frac{1}{\pi}\int_0^{\pi} U(\varphi)\,d\varphi.$$

If we write

$$(12)\qquad F^*(\theta) = F(\theta) + \frac{c}{\pi}\int_0^{\pi} K(\theta,\ \varphi)\,d\varphi,$$

then we have

$$(13)\qquad \tilde{U}(\theta) = F^*(\theta) + \frac{1}{\pi}\int_0^{\pi} \tilde{K}_\varphi(\theta,\ \varphi)\tilde{U}(\varphi)\,d\varphi,$$

and this is Frehdholm's equation of the second kind for unknown $\tilde{U}(\theta)$.

Let the resolvent kernel of $\widetilde{K}_{\varphi}(\theta, \varphi)/\pi$ be $\mathfrak{K}(\theta, \varphi)$, then

$$\widetilde{U}(\theta) = F^{*}(\theta) - \int_{0}^{\pi} \mathfrak{K}(\theta, \varphi) F^{*}(\varphi)\, d\varphi. \tag{14}$$

Again from the reciprocal formula (3), we get

$$\begin{aligned} U(\theta) &= -\frac{1}{\pi}(P)\int_{0}^{\pi} \widetilde{U}(\varphi) \frac{\sin\varphi}{\cos\theta - \cos\varphi}\, d\varphi + c \\ &= -\frac{1}{\pi}(P)\int_{0}^{\pi} \left\{F^{*}(\varphi) - \int_{0}^{\pi} \mathfrak{K}(\varphi, u) F^{*}(u)\, du\right\} \frac{\sin\varphi}{\cos\theta - \cos\varphi}\, d\varphi + c \\ &= -\frac{1}{\pi}(P)\int_{0}^{\pi} \frac{F(\varphi)\sin\varphi}{\cos\theta - \cos\varphi}\, d\varphi \\ &\quad - \frac{c}{\pi^{2}}(P)\int_{0}^{\pi} \frac{\sin\varphi}{\cos\theta - \cos\varphi}\, d\varphi \int_{0}^{\pi} K(\varphi, u)\, du \\ &\quad + \frac{1}{\pi}(P)\int_{0}^{\pi} \frac{\sin\varphi}{\cos\theta - \cos\varphi}\, d\varphi \int_{0}^{\pi} \mathfrak{K}(\varphi, u) F(u)\, du \\ &\quad + \frac{c}{\pi^{2}}(P)\int_{0}^{\pi} \frac{\sin\varphi}{\cos\theta - \cos\varphi}\, d\varphi \int_{0}^{\pi} \mathfrak{K}(\varphi, u) \int_{0}^{\pi} K(u, t)\, dt + c, \end{aligned} \tag{15}$$

where

$$c = \frac{1}{\pi}\int_{0}^{\pi} U(\theta)\, d\theta,$$

which may be taken arbitrarily. Since the right hand of (15) is given by $F(\theta)$ and $K(\theta, \varphi)$, this is a complete solution of (1), where

$$u(\cos\varphi)\sin\varphi = U(\varphi), \qquad -f(\cos\theta)\sin\theta = F(\theta),$$
$$k(\cos\theta, \cos\varphi) = K(\theta, \varphi), \qquad \cos\theta = y \text{ and } \cos\varphi = x.$$

REMARK. For the calculation of conjugate function of $f(x)$ we can use the formula

$$\tilde{f}(x) = -\frac{1}{\pi}\int_{0}^{\infty} \frac{f(x+t) - f(x-t)}{t}\, dt$$

(see Zygmund [5], p. 164).

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