NOTES ON FOURIER ANALYSIS (XLVI): A CONVERGENCE CRITERION FOR FOURIER SERIES

GEN-ICHIRÔ SUNOUCHI

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1. Introduction. The object of this paper is to generalize Young's convergence criterion for Fourier series. To simplify the writing, we shall suppose that the Fourier series

$$\varphi(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

in question is that of an even periodic function which is integrable in the Lebesgue sense. Then Pollard [4] generalizes Young's test as follows.

THEOREM. *The Fourier series of $\varphi(t)$ converges at the point $t = 0$ to the value zero, provided that*

$$\int_0^t \varphi(u)du = o(t), \qquad \text{as } t \to 0 \tag{1}$$

and

$$\int_0^t |d\{u\varphi(u)\}| = O(t), \qquad 0 \leqq t \leqq \eta. \tag{2}$$

On the other hand Hardy and Littlewood [1] proposed the problem, whether we can replace (1) and (2) by

$$\int_0^t \varphi(u)du = o\left(t \Big/ \log\frac{1}{t}\right), \quad \text{as } t \to 0 \tag{3}$$

and

$$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t), \quad 0 \leqq t \leqq \eta, \tag{4}$$

for some $\Delta > 1$. Later Randels [5] proved that this is impossible. Concerning this problem we shall prove the following theorem.

THEOREM. *The Fourier series of $\varphi(t)$ converges at the point $t = 0$ to the value zero, provided that there is a $\Delta \geqq 1$ such that*

$$\int_0^t \varphi(u)\, du = o(t^\Delta), \quad \text{as } t \to 0, \tag{5}$$

and

$$\int_0^t |d\{u^\Delta \varphi(u)\}| = O(t), \qquad 0 \leqq t \leqq \eta. \tag{6}$$

2. Proof of Theorem. It is sufficient to prove that

$$\lim_{\omega\to\infty}\int_0^{\pi}\varphi(t)\,\frac{\sin\omega t}{t}\,dt = 0.$$

Since $\varphi(t)$ is Lebesgue integrable, we have

$$\lim_{\omega\to\infty}\int_{\eta}^{\pi}\varphi(t)\,\frac{\sin\omega t}{t}\,dt = 0,$$

for any fixed $\eta > 0$. Let us now put

$$\alpha = (k/\omega)^{1/\Delta}$$

where k is a constant taken sufficiently large and put

$$\Phi(t) = \int_0^t \varphi(u)du = o(t^{\Delta}), \qquad \text{as } t\to 0.$$

Then we have

$$\int_0^{\alpha}\varphi(t)\,\frac{\sin\omega t}{t}\,dt = \left[\Phi(t)\,\frac{\sin\omega t}{t}\right]_0^{\alpha} - \int_0^{\alpha}\Phi(t)\,\frac{\omega t\cos\omega t - \sin\omega t\, dt}{t^2}$$

$$= I_1 + I_2,$$

say, where

$$|I_1| = o(\alpha^{\Delta-1}) = o\{(k/\omega)^{(\Delta-1)/\Delta}\} = o(1), \text{ as } \omega\to\infty$$

and

$$|I_2| = o\left(\omega\int_0^{\alpha} t^{\Delta-1}\right) = o(\omega\alpha^{\Delta}) = o\{\omega(k/\omega)^{\Delta/\Delta}\}$$

$$= o(1), \text{ as } \omega\to\infty.$$

Hence it is sufficient to prove that

$$\lim_{k\to\infty}\overline{\lim_{\omega\to\infty}}\left|\int_{\alpha}^{\eta}\varphi(t)\,\frac{\sin\omega t}{t}\,dt\right| = 0,$$

where $\alpha = (k/\omega)^{1/\Delta}$.

Let us put $\theta(t) = t^{\Delta}\varphi(t)$ and $\Theta(t) = \int_0^t |d\theta(u)|$, then $\Theta(t) = O(t)$ and $\theta(t) = O(t)$, since $\theta(0) = 0$ is an easy consequence of (5) and (6).

Our concerning integral is therefore

$$J = \int_{\alpha}^{\eta}\varphi(t)\,\frac{\sin\omega t}{t}\,dt = \int_{\alpha}^{\eta}\theta(t)\,\frac{\sin\omega t}{t^{\Delta+1}}\,dt$$

$$= -\int_{\alpha}^{\eta}\theta(t)dA(t),$$

where

$$A(t) = \int_t^{\eta}\frac{\sin\omega t}{t^{\Delta+1}}\,dt.$$

From the second mean value theorem, we get

$$A(t) = \frac{1}{t^{\Delta+1}} \int_t^{\xi} \sin \omega t \, dt = O\{\omega^{-1} t^{-(\Delta+1)}\}.$$

Then

$$-J = \int_\alpha^\eta \theta(t) dA(t) = [\theta(t)A(t)]_\alpha^\eta + \int_\alpha^\eta A(t) d\theta(t)$$

$$= J_1 + J_2,$$

say. We have now

$$J_1 = O(\omega^{-1}\alpha^{-\Delta}) = O\{\omega^{-1}(k/\omega)^{-\Delta/\Delta}\}$$
$$= O(k^{-1}) = o(1), \qquad \text{as } k \to \infty,$$

and

$$J_2 = \int_\alpha^\eta |A(t)||d\theta(t)| = \omega^{-1} \int_\alpha^\eta O\{t^{-(\Delta+1)}\} |d\theta(t)$$

$$= O\left\{\omega^{-1}[t^{-(\Delta+1)}\Theta(t)]_\alpha^\eta\right\} + O\left\{\omega^{-1}\int_\alpha^\eta \Theta(t) t^{-(\Delta+2)} \, dt\right\},$$

$$= K_1 + K_2,$$

say, where

$$K_1 = O(\omega^{-1}) + O(\omega^{-1}\alpha^{-\Delta}) = o(1) + O\{\omega^{-1}(k/\omega)^{-1}\} = O(k^{-1}) = o(1), \quad \text{as } k \to \infty$$

and

$$K_2 = O\left\{\omega^{-1}\int_\alpha^\eta t^{-(\Delta+1)} \, dt\right\} = O(\omega^{-1}[t^{-\Delta}]_\alpha^\eta) = o(1).$$

Thus we get the theorem.

REMARK 1. The condition (5) does not imply the convergence of the Fourier series of $\varphi(t)$. See Hsiang [2] or Izumi and Sunouchi [3].

REMARK 2. If (5) and (6) is valid for $0 \leqq t \leqq \eta$, then the analogous estimation gives

$$a_n = \int_0^\pi \varphi(t) \cos nt \, dt = O(n^{-1/\Delta}),$$

provided that $\Delta > 1$. Hence our test is closely connected with the test of Wang[6].

LITERATURE

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.