SOME REMARKS ON THE RIEMANN SUMS

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(Received November 29, 1950)

1. Let $f(x)$ be a function of period 1 and integrable in the Lebesgue sense in the interval $(0,1)$. Denote the n-th Riemann sum of $f(x)$ by

$$F_n(f,x) = F_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right). \tag{1}$$

If the sequence $F_n(x)$ converges to $\int_0^1 f(u)du$ as $n\to\infty$ for almost all x, we shall say that $f(x)$ has the property (R); and if the convergence is in the Cesàro sense of order α, instead of ordinary convergence, the function $f(x)$ is called to have the property $(R; C, \alpha)$. The following results are known:

THEOREM A. (J. MARCINKIEWICZ-A. ZYGMUND [3], p. 157; H. URSELL [6]). *For any p, $1\leqq p<2$, there exists a function $\in L^p\ (0,1)$, which has not the property (R).*

THEOREM B. (H. URSELL [6]). *If a function $\in L^2(0,1)$ is monotone in $(0,1)$, then it has the property (R).*

THEOREM C. (J. MARCINKIEWICZ-R. SALEM [2]). *If the Fourier coefficients a_n, b_n of a function $f(x)\in L^2(0,1)$ satisfy the condition*

$$\frac{1}{4}a_0^2 + \sum_{k=1}^{\infty}(a_k^2+b_k^2)k^{\varepsilon} < \infty \tag{2}$$

for an $\varepsilon>0$, then $f(x)$ has the property (R); and if

$$\sum_{k=3}^{\infty}(a_k^2+b_k^2)\log\log k < \infty, \tag{3}$$

then $f(x)$ has the property $(R; C, \alpha)$ for $\alpha>0$.

THEOREM D. (A. RAJCHMAN [4]). *There exists a bounded measurable function $f(x)$ such that the set of points, for which $F_n(f,x)$ does not tend to $\int_0^1 f(u)du$ as $n\to\infty$, forms an everywhere dense set in $(0,1)$*[1].

But it seems to be unknown whether we may weaken the additional conditions of monotonity of the function or (2) or (3), for a function $\in L^2\ (0,1)$ to have the property (R) or even $(R; C, \alpha)$. In this note we shall discuss some related problems using the Fourier expansion of functions.

2. If the function $f_t(x)$ with the Fourier series

1) For Rajchman's example, as we see immediately, the required set contains all the rational numbers.

$$(4)\qquad \frac{1}{2}a_0+\sum_{n=1}^{\infty}\varphi_n(t)(a_n\cos 2\pi nx+b_n\sin 2\pi nx),$$

where $\varphi_n(t)$ are the Rademacher functions, has a property P for almost all t, we shall say, following Paley and Zygmund (See [7] p. 125), that almost all the functions with the Fourier series

$$(5)\qquad \frac{1}{2}a_0+\sum_{n=1}^{\infty}\pm(a_n\cos 2\pi nx+b_n\sin 2\pi nx)$$

have the property P.

After this definition we shall aim to prove the following

THEOREM 1. *Suppose that one of the following conditions is satisfied:*

$$(1.1)\qquad \frac{1}{4}a_0^2+\sum_{k=1}^{\infty}(a_k^2+b_k^2)\log k<\infty;$$

$$(1.2)\qquad \sum_{k=n}^{\infty}(a_k^2+b_k^2)=o(1/\log n)\quad as\ n\to\infty;$$

(1.3) $\sum_{n=1}^{\infty}(a_n^2+b_n^2)<\infty$ *and the sequences* $|a_n|$ *and* $|b_n|$ *are non-increasing;*

(1.4) $\sum_{n=1}^{\infty}(a_n^2+b_n^2)<\infty$ *and* $a_k^2>(1-M/\log k)a_{k+1}^2,\ b_k^2>(1-M/\log k)b_{k+1}^2,$

for $k\geqq k_0$, *where* M *is a non-negative constant independent of* k, *and* k_0 *is an integer.*

Then almost all the functions with the Fourier series (5) *have the property* (R).

If we suppose (instead of (1.1)) that

$$\frac{1}{4}a_0^2+\sum_{k=1}^{\infty}(a_k+b_k^2)\log^{1+\epsilon}k<\infty$$

for an $\mathcal{E}>0$, then almost all the functions with the Fourier series (5) are continuous (See, [7] p. 127), and the conclusion of Theorem 1 is evident.

For the proof of Theorem 1 we need the f llowing

THEOREM 2. *Let us suppose, for the series*

$$(6)\qquad \sum_{k=1}^{\infty}c_k\varphi_k(t),$$

that one of the following conditions is fulfilled:

$$(2.1)\qquad \sum_{k=1}^{\infty}c_k\log k<\infty;$$

$$(2.2)\qquad C_n=\sum_{k=n}^{\infty}c_k^2=o(1/\log n)\quad as\ n\to\infty$$

(2.3) $\sum_{k=1}^{\infty} c_k^2 < \infty$ *and* $|c_k|$ *is a non-increasing sequence*;

(2.4) $\sum_{k=1}^{\infty} c_k^2 < \infty$ *and* $c_k^2 < (1 - M/\log k)\, c_{k+1}^2 \quad (k \geqq k_0)$,

where M *is a non-negative constant independent of* k, *and* k_0 *an integer.*

Then, for almost all t, *the series*

$$\Phi_n(t) = \sum_{k=1}^{\infty} c_{nk}\varphi_{nk}(t)$$

converges for every n *and* $\Phi_n(t) \to 0$ *as* $n \to \infty$.

PROOF OF THEOREM 2. The conditions (2.2) and (2.4) follow from (2.1) and (2.3) respectively, and we prove the theorem under the condition (2.2) and under (2.4). Let us denote, for $\delta > 0$,

$$C'_n = \sum_{k=1}^{\infty} c_{nk}^2 \quad \text{and} \quad \underset{t}{E} = \underset{t}{\mathrm{E}}[\,|\Phi_n(t)| > \delta] \qquad (n = 1, 2, \cdots).$$

By Khintchine's inequality (See, e. g., [7] p. 124 or [5]) we have

$$\exp(\lambda_n\delta)|E_n| \leqq \int_0^1 \exp\,(\lambda_n|\Phi_n(t)|)dt \leqq 2\,\exp\left(\frac{1}{2}\,\lambda_n^2 C'_n\right)$$

where $\lambda_n > 0$; putting $\lambda_n = \delta/C'_n$ we get

(7) $$|E_n| \leqq 2\,\exp\,(-\delta^2/(2C'_n)) \qquad (n = 1, 2, \cdots).$$

If the condition (2.2) is satisfied, we see for sufficiently large n ($\geqq n_0$ say) that

(8) $$C'_n \leqq C_n \leqq \delta^2/(4\log n),$$

and then from (7) $|E_n| \leqq 2\exp(-2\log n) = 2/n^2$ ($n \geqq n_0$) which is a term of a convergent series. Therefore, the series $\Sigma\,|E_n|$ being convergent for any $\delta > 0$, we complete the proof by the Borel-Cantelli lemma.

If the condition (2.4) is satisfied, we get for $n \geqq k_0$,

(9) $$\begin{aligned} C_n &= \sum_{k=1}^{\infty} c_{nk}^2 + \sum_{k=1}^{\infty} c_{nk+1}^2 + \cdots + \sum_{k=1}^{\infty} c_{nk+(n-1)}^2 \\ &= \sum_{k=1}^{\infty}\left\{\sum_{j=0}^{n-1}\prod_{i=j}^{n-1}\left(1 - \frac{M}{\log(nk+i)}\right)\right\} c_{nk+n}^2 \\ &\geqq \sum_{k=1}^{\infty} c_{nk+n}^2 \sum_{j=0}^{n-1}\left(1 - \frac{M}{\log n}\right)^{n-j} \\ &\geqq \frac{\log n}{M}\left(1 - \frac{M}{\log n}\right)\left(1 - \left(1 - \frac{M}{\log n}\right)^n\right)\sum_{k=1}^{\infty} c_{n(k+1)}^2 \\ &\geqq \frac{\log n}{2M}(C'_n - c_n^2) \qquad \text{for large } n. \end{aligned}$$

On the other hand, for a given $\varepsilon > 0$, if n is large enough, we have

$$\varepsilon > \sum_{k=[n/2]}^{n} c_k^2 \geqq \sum_{k=[n/2]}^{n} c_n^2 \prod_{m=k}^{n-1}\left(1-\frac{M}{\log m}\right)$$
$$\geqq c_n^2 \sum_{k=[n/2]}^{n}\left(1-\frac{M}{\log [n/2]}\right)^{n-k} \geqq c_n^2 \frac{\log [n/2]}{2M},$$

that is,

(10) $$c_n^2 = o(1/\log n) \quad \text{as } n \to \infty.$$

From (9) and (10) we have easily $C_n' = o(1/\log n)$; hence by the same arguments in the preceding case we complete the proof.

PROOF OF THEOREM 1. It is sufficient to consider the case where one of (1.2) and (1.4) is satisfied.

Since $F_n(f_t, x) \sim \frac{1}{2} a_0 + \sum_{k=1}^{\infty} \varphi_{nk}(t)(a_{nk}\cos 2\pi\, nkx + b_{nk}\sin 2\pi\, nkx)$, if

(11) $$\sum_{k=1}^{\infty}(a_{nk}^2\cos^2 2\pi\, nkx + b_{nk}^2 \sin^2 2\pi\, nkx) = o(1/\log n) \qquad \text{as } n\to\infty$$

for almost all x, then by the result of Theorem 2 we may prove Theorem 1 using the argument of Paley and Zygmund ([7], p. 125). If (1.2) is satisfied, then (11) is evident.

We suppose that (1.4) is satisfied, and for the sake of simplicity we consider the cosine series only, the sine series may be treated similarly. We may suppose that $x \neq 0, \ \neq \frac{1}{2} \pmod 1$. For a given $\varepsilon > 0$, if $n \geqq k_0$ is large enough. we have

(12) $$\varepsilon > \sum_{k=n}^{\infty} a_k^2 \cos^2 2\pi\, kx = \sum_{k=1}^{\infty}\sum_{j=0}^{n-1} a_{nk+j}^2 \cos^2 2\pi(nk+j)x$$
$$\geqq \sum_{k=1}^{\infty} a_{nk+n}^2 \sum_{j=0}^{n-1}\left\{\prod_{i=j}^{n-1}\left(1-\frac{M}{\log(nk+i)}\right)\right\}\cos^2 2\pi(nk+j)x$$
$$\geqq \frac{1}{2}\sum_{k=1}^{\infty} a_{n(k+1)}^2 \sum_{j=0}^{n-1}\left(1-\frac{M}{\log n}\right)^{n-j}(1+\cos 4\pi(nk+j)x)$$
$$\geqq \frac{1}{2}\sum_{k=1}^{\infty} a_{n(k+1)}^2 \left\{\frac{\log n}{4M} - S_n(x)\right\},$$

where, putting $(1 - M/\log n) = \alpha_n = \alpha$,

$$S_n(x) = \sum_{j=0}^{n-1} \alpha^{n-j}\cos 4\pi(nk+j)x$$
$$= \frac{\alpha^n \sin 4\pi\, nkx - \alpha^{n-1}\sin 4\pi(nk-1)x - \sin 4\pi n(k+1)x + \alpha^{-1}\sin 4\pi(nk+(n-1))x}{1-2\alpha^{-1}\cos 4\pi x + \alpha^{-2}}.$$

Since $x \neq 0, \ \neq \frac{1}{2} \pmod 1$, we have $\cos 4\pi\, nx \neq 1$, and the denominator of $S_n(x)$ is greater than a positive constant for large n. Since $\alpha^n \to 0$, as $n\to\infty$, the numerator of $S_n(x)$ is, in absolute value, not greater than $\alpha^n +$

$\alpha^{n-1} + 1 + \alpha^{-1} \leqq 3$ for large n. Hence $S_n(x)$ tends to zero with $1/n$. Then from (12) we conclude easily that

$$\sum_{k=1}^{\infty} a_{nk}^2 = o(1/\log n),$$

($a_n^2 = o(1/\log n)$ being deduced by the same argument as (10)), thus (11) is obtained in the cosine case.

3. THEOREM 3. *Let $\{a_n, b_n\}$ be the sequence of Fourier coefficients of a function $f(x)$. (i) If $\{a_n^2 + b_n^2\}$ forms a non-increasing sequence and if the series*

$$\sum_{k=1}^{\infty} (a_k^2 + b_k^2) \log k \tag{13}$$

converges, then $f(x)$ has the property (R). (ii) If $\{a_n^2 + b_n^2\}$ is non-increasing and if $f(x) \in L^2(0,1)$, then $f(x)$ has the property $(R;\, C, \alpha)$ $(\alpha > 0)$.

PROOF. (i) We may suppose that $a_0 = 0$. Clearly we have

$$\begin{aligned}\sum_{k=n}^{\infty} (a_k^2 + b_k^2) &= \sum_{k=1}^{\infty} \sum_{j=0}^{n-1} (a_{nk+j}^2 + b_{nk+j}^2) \\ &\geqq n \sum_{k=1}^{\infty} (a_{nk+n}^2 + b_{nk+n}^2) \\ &= n \left\{\sum_{k=1}^{\infty} (a_{nk}^2 + b_{nk}^2) - (a_n^2 + b_n^2)\right\}\end{aligned}$$

and

$$\frac{1}{2} n (a_n^2 + b_n^2) \leqq \sum_{k=[n/2]}^{n} (a_k^2 + b_k^2).$$

Hence we have

$$\sum_{k=1}^{\infty} (a_{nk}^2 + b_{nk}^2) \leqq \frac{3}{n} \sum_{k=[n/2]}^{\infty} (a_k^2 + b_k^2). \tag{14}$$

Since $F_n(x) \sim \sum_{k=1}^{\infty} (a_{nk} \cos 2\pi\, nkx + b_{nk} \sin 2\pi\, nkx)$, we deduce from (14) that

$$\begin{aligned}\sum_{n=1}^{\infty} \int_0^1 F_n^2(x)\, dx &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (a_{nk}^2 + b_{nk}^2) \leqq \sum_{n=1}^{\infty} \frac{3}{n} \sum_{k=[n/2]}^{\infty} (a_k^2 + b_k^2) \\ &\leqq 3 \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \sum_{n=1}^{2k+1} \frac{1}{n} \leqq \text{const.} \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \log k < \infty.\end{aligned}$$

So that $F_n(x) \to 0$ as $n \to \infty$ almost everywhere.

(ii) We may put $a_0 = 0$. Similarly as in (i) we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 F_n^2(x) dx = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} (a_{nk}^2 + b_{nk}^2) \leqq \text{const.} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) < \infty,$$

from which we see that $\frac{1}{n}\sum_{k=1}^{n} F_k^{\alpha}(x) \to 0$ and then easily

$$\lim_{n\to\infty} \frac{1}{A_n^{(\alpha)}} \sum_{k=0}^{n} A_{n-k}^{(\alpha-1)} F_k(x) = 0$$

for almost all x. q. e. d.

REFERENCES

1. B. JESSEN, On the approximation of Lebesgue integrals by Riemann sums, Ann. Math., 35(1934) pp. 248–251.
2. J. MARCINKIEWICZ–R. SALEM, Sur les sommes riemanniennes, Comp. Math., 7(1940) pp. 376–389.
3. J. MARCINKIEWICZ–A. ZYGMUND, Mean values of trigonometrical polynomials, Fund. Math., 28 (1937) pp. 131–166.
4. A. RAJCHMAN, Ueber eine paradoxe Eigenschaft gewisser bedingt konvergenter unendlicher Reihen, Math. Zeits., 26(1927) pp. 777–778.
5. T. TSUCHIKURA, Remark on the Rademacher System, Proc. Jap. Acad., 27(1951) pp. 141–145.
6. H. URSELL, On the behaviour of a certain sequence of functions derived from a given one, Jour. London Math. Soc., 12(1937) pp. 229–232.
7. A. ZYGMUND, Trigonometrical series, Warszawa, 1935.

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