HARMONIC ANALYSIS AND WIENER INTEGRALS

GEN-ICHIRÔ SUNOUCHI

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Introduction. Recently Cameron and Martin developed the theory of Wiener integrals. Their method of study based upon the measure of quasi-intervals in Wiener's generalized harmonic analysis (cf. Wiener [7]). In this note we prove their theorem by another method depending on harmonic development of Brownian motion. The known results and preliminary remarks are given in § 1. In § 2 and § 3, we prove some theorems concerning to the change of variables in Wiener integrals. These results are almost evident in view of our method, and somewhat general than the results of Cameron and Martin. The proof also is simpler than that of Cameron and Martin.

1. Harmonic analysis of Wiener integrals.*) The results of this article are well known, but we summarize them for the sake of completeness.

Let $\{x(t)\}$ be real valued functions of a real variable t, where $0 \leqq t \leqq 1$, and Brownian motion, then almost all functions $\{x(t)\}$ is continuous in this probability measure (cf. Wiener [7]). We call the Lebesgue integral of functionals $F[x] \equiv F[x(.)]$, where $x(.) \equiv x(t) \in C$ (all continuous functions of t, where $0 \leqq t \leqq 1$) by Wiener integral and denote

$$\int_C^W F[x] \, d_w x. \tag{1.1}$$

Then let $\varphi(t)$ and $\psi(t)$ be bounded variation in $[0, 1]$, we have

$$\int_C^W \left[\int_0^1 \varphi(t) \, dx(t) \int_0^1 \psi(t) dx(t) \right] d_w x = \frac{1}{2} \int_0^1 \varphi(t) \psi(t) \, dt. \tag{1.2}$$

This is well known, see for example, Wiener [7], Paley, Wiener and Zygmund [6] or Doob [3]. Especially let $\{\varphi_n(t)\}$ $(n = 1, 2, \cdots)$ be N. O. S. in $[0, 1]$, then the functionals

$$\sqrt{2}\, \Phi_n[x] \equiv \sqrt{2} \int_0^1 \varphi_n(t) dx(t) \tag{1.3}$$

is N. O. S. in the space C. Let $v(t)$ be any function of bounded variation and put

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*) During the preparation of this paper, G. Maruyama applied this method to another problems independently. See G. Maruyama, Kodai Math. Seminar Rep. 3 (1950) 41-44.

(1.4) $$V[x] \equiv \int_0^1 v(t)\, dx(t).$$

If $V[x]$ is developed in series such as

(1.5) $$V[x] \sim \sum_{n=1}^{\infty} c_n \Phi_n[x],$$

then the coefficient c_n is calculated by

(1.6) $$c_n = 2\int_C^W V[x]\, \Phi_n[x]\, d_w x = \int_0^1 v(t)\varphi_n(t)dt, \quad \text{(by (1.1))}.$$

That is, c_n is ordinary Fourier coefficient of $v(t)$. Provided that $\{\varphi_n(t)\}$ is L^2-complete, we have

(1.7) $$\int_C^W \left| V[x] - \sum_{n=1}^{N} c_n \varphi_n[x] \right|^2 d_w x$$
$$= \frac{1}{2}\int_0^1 \left| v(t) - \sum_{n=1}^{N} c_n \varphi_n(t) \right|^2 dt \to 0, \quad \text{as } N \to \infty,$$

that is

(1.8) $$\underset{N \to \infty}{\text{L. I. M.}} \sum_{n=1}^{N} c_n \Phi_n[x] = V[x].$$

Put $v(t) = 1$, if $0 < \tau$ and $v(t) = 0$, if $\tau \leqq 1$, then

(1.9) $$x(\tau) - x(0) = \underset{N \to \infty}{\text{L. I. M.}} \sum_{n=1}^{N} \left(\int_0^\tau \varphi_n(t)\, dt \right) \Phi_n[x],$$

and we can assume $x(0) = 0$ without loss of generality.

In the following line we take

(1.10) $\varphi_n(t) = \sin n\pi t, \qquad (n = 1, 2, \cdots),$

then the series (1.9) converges in the mean almost $x(.) \in C$, that is

(1.11) $$\lim_{N \to \infty} \int_0^1 \left| x(\tau) - \sum_{n=1}^{N} \left(\int_0^\tau \varphi_n(t)\, dt \right) \Phi_n[x] \right|^2 d\tau = 0, \quad \text{a. e.}$$

(cf. Paley-Wiener [5]).

If $F[x]$ is depending only on finite number of $\Phi_n[x]$, i.e.

(1.12) $F[x] \equiv f(\Phi_1[x],\ \Phi_2[x],\ \cdots,\ \Phi_n[x]),$

then we have

(1.13) $$\int_C^W F[x] d_w x = \int_C^W f(\Phi_1[x],\ \cdots,\ \Phi_n[x]) d_w x$$
$$= \frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(u_1, \cdots, u_n) \exp\left(-\sum_{\nu=1}^{n} u_\nu^2 \right) du_1 \cdots du_n,$$

provided that the right hand series is convergent.

Any integrable functional $F[x]$ can be approximated in the sense of L-norm by these functionals (Paley-Wiener [5]). Thus if we write

(1.14) $$F[x] = \underset{n\to\infty}{\text{L. I. M.}}\, f_n(\Phi_1[x], \cdots, \Phi_n[x]) \equiv f(\Phi_1[x], \cdots, \Phi_n[x], \cdots)$$

then we have

(1.15) $$\int_C^W F[x]d_w x \equiv \int_C^W f(\Phi_1[x], \cdots, \Phi_n[x], \cdots)\, d_w x = \lim_{n\to\infty}\frac{1}{\pi^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(u_1, \cdots, u_n \cdots,)\exp\left(-\sum_{\nu=1}^{n} u_\nu^2\right) du_1\cdots du_n,$$

from the integration theory of infinite many variables. The right hand integral is ordinary Lebesgue integral and $\exp\left(\sum_{\nu=1}^{n} u_\nu^2\right)$ is convergence factor where (u_ν) is ordinary Fourier-Stieltjes coefficient of $x(t)$ (See (1.3)).

2. Change of variables of Wiener integrals under translations. Let $x_0(t)$ is a fixed continuous function and put

(2.1) $$y(t) = x(t) + x_0(t),$$

then if write

$$\Phi_n[y] = v_n, \quad \Phi_n[x] = u_n, \quad \Phi_n[x_0] = u_n^{(0)},$$

we have

(2.2) $$v_n = u_n + u_n^{(0)}.$$

Whence we have

(2.3) $$\int_C^W F[y]d_w y$$

$$= \lim_{n\to\infty}\frac{1}{\pi^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(v_1, \cdots, v_n, \cdots)\exp\left(-\sum_{\nu=1}^{n} v_\nu^2\right) dv_1\cdots dv_n$$

$$= \lim_{n\to\infty}\frac{1}{\pi^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(u_1 + u_1^{(0)}, \cdots, u_n + u_n^{(0)}, \cdots)$$

$$\exp\left(-\sum_{\nu=1}^{n}(u_\nu + u_\nu^{(0)})^2\right) du_1 \cdots du_n$$

$$= \exp\left(-\sum_{\nu=1}^{\infty}(u_\nu^{(0)})^2\right)\lim_{n\to\infty}\frac{1}{\pi^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(u_1 + u_1^{(0)}, \cdots, u_n + u_n^{(0)}\cdots,)$$

$$\exp\left(-\sum_{\nu=1}^{n} u_\nu^2\right)\exp\left(-2\sum_{\nu=1}^{n} u_\nu u_\nu^{(0)}\right) du_1 \cdots du_n$$

$$= \exp\left(-\int_0^1 [x_0'(t)]^2 dt\right)\int_C^W F[x+x_0] \exp\left[-2\int_0^1 x_0'(t)dx(t)\right] d_w x,$$

provided that $x_0'(t)$ is of bounded variation. Thus we get the following theorem.

THEOREM 1. *Let $F(y)$ be a Wiener-integrable functional over C and $x_0(t)$ be a given function of C with $x_0'(t)$ of bounded variation in $0 \leqq t \leqq 1$. Then under the translation*

(2.4) $$y(t) = x(t) + x_0(t)$$

we have

(2.5) $$\int_C^W F[y]\, d_w y$$

$$= \exp\left(-\int_0^1 [x_0'(t)]^2 dt\right)\int_C^W F[x+x_0] \exp\left(-2\int_0^1 x_0'(t)\, dx(t)\right) d_w x.$$

This is a result of Cameron-Martin [1]. More generally we have

THEOREM 2.**) *Let $x_0(t)$ be an absolutely continuous function with $x_0'(t) \in L^2(0,1)$, then under the translation* (2.4), *we have the formula* (2.5), *where the integral*

(2.6) $$\int_0^1 x_0'(t)dx(t)$$

is Paley-Wiener's sense [5] (*that is defined by Fourier Stieltjes coefficient*).

The proof is evident from the above argument.

3. Change of vaiables of Wiener integrals under general class of linear transformations.

If $x(t) \in C$ and put

$$\int_0^1 \sin n\pi t\ dx(t) = u_n, \qquad \int_0^1 \cos n\pi t\ x(t)\ dt = U_n,$$

then we have

(3.1) $$u_n = n\pi U_n$$

and

(3.2) $$\int_C^W f(\Phi_1[x], \cdots, \Phi_n[x])\, d_w x$$

$$= \frac{1}{\pi^{n/2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(u_1, \cdots, u_n) \exp\left(-\sum_{\nu=1}^n u_\nu^2\right) du_1\cdots du_n$$

$$= n\,!\,\pi^{n/2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(U_1, \cdots, U_n) \exp\left(-\sum_{\nu=1}^n (\nu\pi U_\nu)^2\right) dU_1\cdots dU_n,$$

where

**) This is also proved in Maruyama's note, see loc. cit.

$$g(U_1, \cdots, U_n) = f(\pi U_1,\ 2\pi U_2,\ \cdots,\ n\pi U_n).$$

If

(3.3) $$y(t) \sim \sum_{n=1}^{\infty} V_n \cos n\pi t$$

(3.4) $$x(t) \sim \sum_{n=1}^{\infty} U_n \cos n\pi t$$

and

(3.5) $$K(t,s) \sim \sum_{m,n=1}^{\infty} a_{m,n} \cos m\pi t \cos n\pi s,$$

then under apropriate conditions

(3.6) $$y(t) = x(t) + \lambda \int_0^1 K(t,s) x(s)\, ds$$

is equivalent

(3.7) $$V_m = U_m + \lambda \sum_{n=1}^{\infty} a_{m,n} U_n, \qquad (m = 1, 2, \cdots),$$

that is

(3.8) $$V = A \cdot U$$

where (V) and (U) are single row matrices respectively and

(3.9) $$A = \begin{pmatrix} 1+\lambda a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & 1+\lambda a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & 1+\lambda a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

If we denote

$$D = \text{determinant } (A),$$

then we have by (3.2) formally

(3.10) $$\int_C^W F[y]\, d_w y$$

$$= \lim_{n\to\infty} n!\, \pi^{n/2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(V_1, \cdots, V_n, \cdots) \exp\Big(-\sum_{\nu=1}^{n} (\nu\pi V_\nu)^2 \Big)\, dV_1 \cdots dV_n$$

$$= \lim_{n\to\infty} n!\, \pi^{n/2} |D| \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g\Big(\cdots,\ U_m + \lambda \sum_{n=1}^{\infty} a_{mn} U_n,\ \cdots \Big)$$

$$\exp\Big[-\sum_{\nu=1}^{n} \Big\{ \pi^2 \nu^2 \Big(U_\nu + \lambda \sum_{n=1}^{\infty} a_{\nu n} U_n \Big)^2 \Big\} \Big]\, dU_1 \cdots dU_n \cdots$$

where

$$F[y] = f(\Phi_1[x],\ \cdots,\ \Phi_n[x], \cdots) = g(V_1, \cdots,\ V_n, \cdots).$$

On the other hand

$$(3.11)\qquad m^2\left(U_m+\lambda\sum_{n=1}^{\infty}a_{mn}U_n\right)^2 = m^2\left\{U_m^2+2\lambda U_m\sum_{n=1}^{\infty}a_{mn}U_n+\lambda^2\left(\sum_{n=1}^{\infty}a_{mn}U_n\right)^2\right\}.$$

From (3.3), (3.4) and (3.5), we have formally

$$(3.12)\qquad \sum_{m=1}^{\infty} m\pi \sin m\pi t\sum_{n=1}^{\infty}a_{mn}U_n \sim \frac{d}{dt}\int_0^1 K(t,s)x(s)ds$$

$$(3.13)\qquad \sum_{m=1}^{\infty} m^2\pi^2\left(\sum_{n=1}^{\infty}a_{mn}U_n\right)^2 = \left(\frac{d}{dt}\int_0^1 K(t,s)x(s)ds\right)^2$$

and

$$(3.14)\qquad \sum_{m=1}^{\infty} m^2\pi^2U_m\left(\sum_{n=1}^{\infty}a_{mn}U_n\right) = \int_0^1\left(\frac{d}{dt}\int_0^1 K(t,s)x(s)ds\right)dx(t).$$

If $K(t,s)$ is L^2 for each variables and both variables and

$$(3.15)\qquad \int_0^1\left(\frac{d}{dt}\int_0^1 K(t,s)x(s)ds\right)\in L^2,$$

the formulas (3.12), (3.13) and (3.14) are justified. Moreover it is sufficient for the absolute convergence of determinant D that

$$(3.16)\qquad \sum_{m,n=1}^{\infty}|a_{mn}|^2<\infty,\ \text{and}\ \sum_{m=1}^{\infty}|a_{mm}|<\infty,$$

from von Koch's theorem. If (3.16) holds, then above formal calculation can be justified by step by step, and D coincides with Fredholm's determinant, that is

$$(3.17)\qquad D=1+\sum_{\mu=1}^{\infty}\frac{\lambda^\mu}{\mu!}\int_0^1\cdots\int_0^1\begin{vmatrix}K(s_1,s_1)\cdots K(s_1,s_\mu)\\ \cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\\ K(s_\mu,s_1)\cdots K(s_\mu,s_\mu)\end{vmatrix}ds_1\cdots ds_\mu.$$

This is due to Marty [4]. If

$$(3.18)\qquad \frac{\partial K(t,s)}{\partial t}$$

exists almost everywhere and $\in L^2$, then $\sum_{m-1}^{\infty}|a_{mm}|<\infty$, for

$$\sum_{m=1}^{\infty} m^2\pi^2a_{mm}^2\leq\sum_{m,n=1}^{\infty}m^2\pi^2a_{mn}^2=\int_0^1\int_0^1\left|\frac{\partial K(t,s)}{\partial t}\right|^2 dtds<\infty,$$

and we have

$$\sum_{m=1}^{\infty}|a_{mm}|=\sum_{m=1}^{\infty}|a_{mm}|m\cdot 1/m$$

$$\leqq \left(\sum_{m=1}^{\infty} m^2 a_{mm}^2\right)^{1/2} \left(\sum_{m=1}^{\infty} 1/m^2\right)^{1/2} < \infty.$$

Thus we get the following theorem.

THEOREM 3. *Let*

(1°) *$K(t,s)$ belongs to L^2 for each variables and both variables,*

(2°) *$\partial K(t,s)/\partial t$ exists almost everywhere and belongs to L^2,*

(3°) $\int_0^1 \left(\frac{d}{dt}\int_0 K(t,s)x(s)ds\right) dx(t)$ *exists almost all $x(.) \in C$,*

(4°) $D \neq 0$,

then under the transformation

$$y(t) = x(t) + \int_0^1 K(t,s)x(s)ds,$$

we have

$$\text{(3.19)} \qquad \int_C^W F[y]d_w y = |D| \int_C^W F\left[x + \int_0^1 K(t,s)x(s)ds\right] \exp(-\Phi(x)) d_w x,$$

where

$$\text{(3.20)} \qquad \Phi[x] = \int_0^1 \left[\frac{d}{dt}\int_0^1 K(t,s)x(s)ds\right]^2 dt + 2\int_0^1 \left[\frac{d}{dt}\int_0^1 K(t,s)x(s)ds\right] dx(t),$$

and D is Fredholm's determinant of kernel $K(t,s)$.

For example, let the transformation be

$$\text{(3.21)} \qquad y(t) = x(t) + \lambda\int_0^t \tan \lambda(s-1)x(s)ds, \qquad -\pi/2 < \lambda < \pi/2,$$

then we have

$$\int_0^1 K(t,s)x(s)ds = \int_0^1 \lambda \tan \lambda(s-1)x(s)ds$$

and (3°) is

$$\text{(3.22)} \qquad \int_0^1 \lambda \tan \lambda(t-1)x(t)dx(t) = \int_0^1 \lambda \tan \lambda(t-1)d\{[x(t)]^2\}.$$

Thus we can apply the theorem to this case.

Since the condition (3°) contains $x(t)$, in order to exclude $x(t)$, we can proceed as follows.

Coresponding to the results of Cameron-Martin [2], put

$$\text{(3.25)} \qquad K(t,s) = \begin{cases} K^1(t,s), & \text{where } 0 \leqq t < s, \quad 0 < s \leqq 1, \\ K^2(t,s), & \text{where } s < t \leqq 1, \quad 0 \leqq s < 1, \\ 2^{-1}\{K^1(s,s) + K^2(s,s)\}, & \text{where } t = s, \quad 0 \leqq s \leqq 1, \end{cases}$$

and

$$\text{(3.24)} \qquad J(s) = K^2(s,s) - K^1(s,s).$$

Let

$$K^0(t,s) = \begin{cases} K^1(t,t), & \text{where } 0 \leqq t < s, \quad 0 < s \leqq 1 \\ K^2(t,t), & \text{where } s < t \leqq 1, \quad 0 \leqq s < 1 \\ 2^{-1}(K^1(s,s) + K^2(s,s)), & \text{where } t = s, \quad 0 \leqq s \leqq 1, \end{cases} \tag{3.25}$$

and

$$\mathfrak{K}(t,s) = K(t,s) - K^0(t,s) \tag{3.26}$$

be absolutely continuous with respect to t. Then

$$\begin{aligned} &\frac{d}{dt}\int_0^1 K(t,s)x(s)ds \\ &= \frac{d}{dt}\int_0^1 \mathfrak{K}(t,s)x(s)ds + \frac{d}{dt}\int_0^1 K^0(t,s)x(s)ds \\ &= I_1(t) + I_2(t), \quad say \end{aligned}$$

$$\begin{aligned} I_1(t) &= \lim_{h\to 0}\frac{1}{h}\left[\int_0^1 \mathfrak{K}(t+h,s)x(s)ds - \int_0^1 \mathfrak{K}(t,s)x(s)ds\right] \\ &= \lim_{h\to 0}\int_0^1 \frac{1}{h}\left[\mathfrak{K}(t+h,s) - \mathfrak{K}(t,s)\right] x(s)ds. \end{aligned}$$

If

$$\frac{\partial}{\partial t}\mathfrak{K}(t,s) = H(t,s), \cdots \qquad \text{a. e.,} \tag{3.27}$$

and

$$\int_0^1 \sup_{0\leqq t\leqq 1} H(t,s)ds < \infty, \tag{3.28}$$

then

$$I(t) = \int_0^1 H(t,s)x(s)ds$$

from the dominated convergence theorem of Lebesgue. In order to be of bounded variation, it is sufficient that

$$\int_0^1 \text{var.}_t\, H(t,s)ds < \infty. \tag{3.29}$$

For

$$\text{var.}\int_0^1 H(t,s)\,x(s)ds = \overline{\lim}\int_0^1 |\Delta_t H(t,s)|\,|x(s)|\,ds$$

$$\leqq \int_0^1 \overline{\lim}\,|\Delta_t H(t,s)|\,|x(s)|\,ds \leqq \int_0^1 (\text{var.}_t\, H(t,s))x(s)ds < \infty,$$

by Lebesgue's theorem.

On the other hand

$$I_2(t) = \frac{d}{dt}\left(\int_0^t K^2(s,s)x(s)ds + \int_t^1 K^1(s,s)x(s)ds\right)$$
$$= K^2(t,t)x(t) - K^1(t,t)x(t) = J(t)x(t)$$

by (3.24).

If $J(t)x(t) \in L^2$, then we have

$$2\int_0^1 J(t)x(t)dx(t) = \int_0^1 J(t)d[\{x(t)\}^2], \quad \text{a.e.}$$

and if $J(t) \in BV$, then the right hand integral is ordinary Riemann-Stieltjes integral. Summing up these results we have the following theorem which is due to Cameron and Martin [2].

THEOREM 4. *Let $K^1(t,s)$ be continuous on the closed triangle $[0 \leqq t \leqq s, 0 \leqq s \leqq 1]$ and let it vanish on the line segment $t = 0$ $[0 \leqq s \leqq 1]$; let $K^2(t,s)$ be continuous on the closed triangle $[0 \leqq s \leqq t, 0 \leqq t \leqq 1]$; let*

$$K(t,s) = \begin{cases} K^1(t,s), & \text{where } 0 \leqq t < s, \quad 0 < s \leqq 1 \\ K^2(t,s), & \text{where } s < t \leqq 1, \quad 0 \leqq s \leqq 1 \\ 2^{-1}(K^1(s,s) + K^2(s,s)), & \text{where } t = s, \ 0 \leqq s \leqq 1, \end{cases}$$

$J(s) = K^2(s,s) - K^1(s,s), \quad 0 \leqq s \leqq 1,$

$D =$ *Fredholm's determinant of* $K(t,s)$.

Assume further that $K(t,s)$ is such that the following conditions are satisfied:

(1°) *For almost all s, $K(t,s)$ is absolutely continuous in t on $0 \leqq t \leqq 1$ after the jump at $t = s$ is removed by the addition of a step function.*

(2°) *There exists a measurable function $H(t,s)$ which is of bounded variation in t for each s and which for almost all t, s in the square $[0 \leqq t \leqq 1, 0 \leqq s \leqq 1]$ is equal to $\partial K(t,s)/\partial t$.*

(3°) *The function $H(t,s)$ mentioned in (2°) can be so chosen that*

$$\int_0^1 \sup_{0 \leqq t \leqq 1} |H(t,s)| ds < \infty, \qquad \int_0^1 \operatorname*{var}_{0 \leqq t \leqq 1} [H(t,s)]\, ds < \infty.$$

and

$$\int_0^1 \int_0^1 |H(t,s)|^2 dt\, ds < \infty,$$

(4°) *$J(s)$ is of bounded variation on $0 \leqq s \leqq 1$,*

(5°) $D \neq 0$.

Then under the transformation

$$y(t) = x(t) + \int_0^1 K(t,s)x(s)ds,$$

we have

$$\int_C^W F[y]d_w y = |D| \int_C^W F\left[x + \int_0^1 K(.,s)x(s)ds\right] \exp\,(-\Phi[x])d_w x,$$

where

$$\Phi[x] = \int_0^1 \left[\frac{d}{dt} \int_0^1 K(t, s) x(s) ds \right]^2 dt$$
$$+ 2\int_0^1 \left[\int_0^1 \frac{\partial}{\partial t} K(t, s) x(s) ds \right] dx(t) + \int_0^1 J(t) d\{[x(t)]^2\}.$$

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.