# SOME REMARKS ON AN OPEN RIEMANN SURFACE WITH NULL BOUNDARY

TADASHI KURODA

(Received May 1, 1951)

**1.** In this paper we shall consider an open abstract Riemann surface with null boundary in the sense of Nevanlinna [4]. Recently Sario [8] has introduced the notion of the removable boundary of a Riemann surface. Sario and Pfluger [7] have obtained many interesting results concerning the removable boundary. We shall state some remarks on the null boundary and the removable boundary of a Riemann surface.

**2.** Let $F$ be an open abstract Riemann surface, $\Gamma$ be its ideal boundary and $F_n$ $(n = 0, 1, \cdots)$ be the subdomains of $F$ satisfying the following four conditions:

i) $F_n (n = 0, 1, \cdots)$ is open and relatively compact with respect to $F$,

ii) $$\overline{F}_n \subset F_{n+1} \qquad (n = 0, 1, \cdots),$$

iii) $$\bigcup_{n=0}^{\infty} F_n = F,$$

iv) the relative boundary $\Gamma_n$ of $F_n$ consists of a finite number of closed analytic curves.

Further let $u$ be the harmonic function in $F_n - \overline{F}_0$ such that

$$u = \begin{cases} 0 & \text{on } \Gamma_0 \\ \log \mu_n & \text{on } \Gamma_n \end{cases}$$

and

$$\int_{\Gamma_n} dv = 2\pi,$$

where $v$ is the conjugate harmonic function of $u$ and the integral is taken in the positive sense of $\Gamma_n$ with respect to the domain $F_n - \overline{F}_0$. We call $\mu_n$ the modulus of the domain $F_n - \overline{F}_0$.

Similarly we can define the modulus $\sigma_n$ of the open set $F_{n+1} - \overline{F}_n$.

Now we shall state a theorem without proof (Kuroda [1]).

THEOREM 1. *A Riemann surface $F$ has a null boundary, if and only if the modulus $\mu_n$ of the subdomain $F_n - \overline{F}_0$ satisfies the condition*

$$\lim_{n \to \infty} \mu_n = \infty.$$

**3.** Applying Theorem 1, we shall give another necessary and sufficient condition in order that a Riemann surface has a null boundary.

The following theorem which completes a theorem of Sario [9] is due to Professor K. Noshiro [6]. Here we shall give an alternative proof.

THEOREM 2. *If there exists a sequence of domains $F_n (n = 0, 1, \cdots)$ satisfying the conditions* i), ii), iii) *and* iv), *such that the infinite product*

$$\prod_{n=0}^{\infty} \sigma_n$$

*of the moduli of the open sets $F_{n+1} - \bar{F}_n$ $(n = 0, 1, \cdots)$ is divergent, the Riemann surface $F$ has a null boundary and conversely.*

PROOF. As the sufficiency was proved by Sario [9], we shall give a proof for the necessity only.

Suppose that $F$ has a null boundary. First we fix the relatively compact subdomain $F_0$ of $F$ arbitrarily. Then, by Theorem 1, we can take the compact domain $F_1$ which satisfies the conditions i), ii), iv) and such that the modulus $\sigma_0$ of the domain $F_1 - F_0$ is greater than $e$. Next by taking $F_1$ instead of $F_0$, we can take a relatively compact domain $F_2$ such that $F_2$ satisfies the conditions i), ii), iv) and the modulus $\sigma_1$ of the domain $F_2 - F_1$ is greater than $e$. Repeating the same process as above, we can take a relatively compact domain $F_{n+1}$ such that $F_{n+1}$ satisfies the conditions i), ii), iv) and further the modulus $\sigma_n$ of the open set $F_{n+1} - F_n$ is greater than $e$. Thus we get

$$\sigma_n > e \qquad (n = 0, 1, \cdots).$$

Moreover, we can easily take the sequence of the domains $F_n (n=0, 1, \cdots)$ such that the condition iii) is satisfied.

Therefore, the infinite product $\prod_{n=0}^{\infty} \sigma_n$ is divergent. (q. e. d.)

Next we shall give a simple proof for a theorem due to Nevanlinna [3] and Sario [10].

THEOREM 3. *A Riemann surface $F$ has a null boundary, if and only if there does not exist the Green function on $F$.*

PROOF[1]. Denote by $u$ the harmonic function in the domain $F_n - F_0$ which defines the modulus $\mu_n$ of this domain, by $v$ the conjugate harmonic function of $u$ and by $g_n$ the Green function of $F_n$ which has its logarithmic pole in $F_0$ and vanishes on the relative boundary $\Gamma_n$ of $F_n$. By Green's formula, we have

$$(1) \qquad \int_{\Gamma_0} g_n \frac{\partial u}{\partial \nu} ds = \int_{\Gamma_n} u \frac{\partial g_n}{\partial \nu} ds = 2\pi \log \mu_n,$$

where the integrals are taken in the positive sense on $\Gamma_0$ and $\Gamma_n$ with respect to the domain $F_0$ and $F_n$, $\frac{\partial}{\partial \nu}$ represents the outer normal derivative on $\Gamma_0$ and $\Gamma_n$ with respect to $F_0$ and $F_n$ respectively and $ds$ is the line-

1) Recently the author learned that Virtanen [11] gave a similar proof as the author's.

element. It is clear that

$$\frac{\partial u}{\partial \nu} \geqq 0$$

on $\Gamma_0$ and

$$\int_{\Gamma_0} \frac{\partial u}{\partial \nu} ds = \int_{\Gamma_0} dv = 2\pi.$$

Hence, from (1), there exists at least one point $P_n$ on $\Gamma_0$ such that

$$g_n(P_n) = \log \mu_n .$$

Since these points $P_n$ $(n = 0, 1, \cdots)$ have at least one limiting point $P$ on $\Gamma_0$, we obtain

$$\lim_{n\to\infty} g_n(P) = \infty,$$

if and only if

$$\lim_{n\to\infty} \mu_n = \infty.$$

It is well-known that the Green function $g_n$ of $F_n$ is uniformly convergent in the wide sense on $F$. Therefore, by Theorem 1, we get our theorem.

4. Now we shall consider the removability of the ideal boundary $\Gamma$ of a Riemann surface $F$.

If every uniform bounded harmonic function on $F$ is a constant, we say that $\Gamma$ is $(u, M)$-removable. And if every uniform harmonic function on $F$ with a finite Dirichlet integral is a constant, we say that $\Gamma$ is $(u, D)$-removable.

We shall prove

THEOREM 4. *If $F$ has a null boundary, then $\Gamma$ is $(u, M)$-removable.*

PROOF. We construct the sequence of subdomains $F_n$ $(n = 0, 1, \cdots)$ of $F$ satisfying the condition i), ii), iii) and iv). Denote by $u$ the harmonic function which defines the modulus $\mu_n$ of $F_n - F_0$ and by $v$ conjugate harmonic function of $u$. We describe the niveau curve $\Gamma_\lambda : u = \lambda$ $(0 < \lambda \leqq \log \mu_n)$.

Suppose that there exists a uniform bounded harmonic function $U$ on $F(|U| \leqq M)$ and consider the Dirichlet integral

$$D(\lambda) = \int_{\Gamma_\lambda} U\, dV = \int_{\Gamma_\lambda} U \frac{\partial U}{\partial u} dv$$

of the function $U$ in the domain bounded by $\Gamma_\lambda$ and containing $F_0$, where $V$ is the conjugate harmonic function of $U$. By Schwarz's inequality we get

$$D^2(\lambda) \leqq M^2 \int_{\Gamma_\lambda} dv \int_{\Gamma_\lambda} \left(\frac{\partial U}{\partial u}\right)^2 dv$$

$$\leqq 2\pi M^2 \frac{dD(\lambda)}{d\lambda},$$

whence follows

$$\log \mu_n = \int_0^{\log \mu_n} d\lambda \leqq 2\pi M^2 \left( \frac{1}{D_0} - \frac{1}{D_n} \right),$$

where $D_n$ is the Dirichlet integral of $U$ in $F_n$.

Since, by assumption, $\log \mu_n$ is divergent, we obtain $D_0 = 0$ and hence the function $U$ must be a constant. (q. e. d.)

REMARK. This result was stated without proof by Nevanlinna [5]. The proof can be given by using Myrberg's theorem [2]. A. Sagawa has also given the same proof independently.

Now we shall state another proof of

THEOREM 5 (Nevanlinna [4]). *If $F$ has a null boundary, $\Gamma$ is $(u, D)$-removable.*

PROOF. Construct a sequence $F_n$ (n $= 0, 1, \cdots$) satisfying the conditions i), ii), iii), iv) and denote by $u$ the harmonic function which defines the modulus $\mu_n$ of the domain $F_n - \bar{F}_0$, by $v$ its conjugate function and $\Gamma_\lambda$ the niveau curve $u = \lambda$ $(0 < \lambda \leqq \log \mu_n)$.

Let $U$ be a uniform, harmonic and non-constant function on $F$ and $V$ be its conjugate harmonic function. Without loss of generality we may suppose that $U$ is not identically equal to zero on $\Gamma_0$.

If we put

$$D(\lambda) = \int_{\Gamma_\lambda} U dV = \int_{\Gamma_\lambda} U \frac{\partial U}{\partial u} dv,$$

then, using Schwarz's inequality,

$$(2) \qquad D^2(\lambda) \leqq \int_{\Gamma_\lambda} U^2 dv \int_{\Gamma_\lambda} \left(\frac{\partial U}{\partial u}\right)^2 dv.$$

On the other hand if we put

$$(3) \qquad m(\lambda) = \int_{\Gamma_\lambda} U^2 \, dv,$$

then

$$\frac{dm(\lambda)}{d\lambda} = m'(\lambda) = 2\int_{\Gamma_\lambda} U \frac{\partial U}{\partial u} \, dv = 2D(\lambda) \ (>0)$$

and

$$\frac{d^2 m(\lambda)}{d\lambda} = m''(\lambda) = 2\frac{dD(\lambda)}{d\lambda} .$$

Hence, from (2),

$$\frac{m'(\lambda)}{m(\lambda)} \leqq 2 \frac{m''(\lambda)}{m'(\lambda)} .$$

By integrating from $\lambda = 0$ to $\lambda$, we have

$$(4) \qquad m(\lambda) \leqq k(m'(\lambda))^2 \leqq 4kD^2(\lambda), \quad k = \frac{m(0)}{4D_0^2} ,$$

where $D_0$ is the Dirichlet integral of $U$ with respect to $F_0$. Since $U$ is not identically equal to zero on $\Gamma_0$,

$$m(0) = \int_{\Gamma_0} U^2 dv > 0$$

and so $k$ is positive. From (2), (3) and (4) it follows that

$$D^2(\lambda) \leqq 4kD^2(\lambda)\frac{dD(\lambda)}{d\lambda}, \quad \text{or } d\lambda \leqq 4k\, dD(\lambda).$$

Integrating from $\lambda = 0$ to $\lambda = \log \mu_n$, we obtain

$$\log \mu_n \leqq 4k(D_n - D_0),$$

where $D_n$ is the Dirichlet integral of $U$ with respect to $F_n$. Since the modulus $\mu_n$ of $F_n - \bar{F}_0$ is divergent by the assumption and Theorem 1, we get

$$\lim_{n\to\infty} D_n = \infty.$$

Thus our assertion is proved. (q. e. d.)

## REFERENCES

[1] T. KURODA, On the type of an open Riemann surface, Proc. Jap. Acad. 27(1950).
[2] P. J. MYRBERG, Über die Existenz der Greenschen Funktionen auf einer gegebenen Riemannschen Fläche, Acta Math., 61(1933).
[3] R. NEVANLINNA, Über die Lösbarkeit des Dirichletschen Problems für eine Riemannsche Fläche, Nachr. Gesellschaft der Wissenschaften Gött. Fachgruppe I. Bd. 1 (1939).
[4] R. NEVANLINNA, Quadratische integrierbare Differentiale auf einer Riemannschen Mannigfaltigkeit, Ann. Acad. Sci. Fenn. A. I. 1(1941).
[5] R. NEVANLINNA, Sur l'existence de certaines classes de différentielles analytiques, C. R. 228 (1949).
[6] K. NOSHIRO, An open Riemann surface with null boundary, Nagoya Math. Journal 3(1951).
[7] A. PFLUGER, Über das Anwachsen eindeutiger analytischen Funktionen auf offene Riemannschen Fläche, Ann. Acad. Sci. Fenn. A. I. 64 (1949).
[8] L. SARIO, Über Riemannschen Fläche mit hebbarem Rand, Ann. Acad. Sci. Fenn. A. I. 50 (1948).
[9] L. SARIO, Question d'existence au voisinage de la frontière d'une surface de Riemann, C. R. 230 (1950).
[10] L. SARIO, Quelques propriétés à la frontière se rattachant à la classification des surfaces de Riemann, C. R. 230 (1950).
[11] K. I. VIRTANEN, Über die Existenz von beschränkten harmonischen Funktionen auf offene Riemannschen Fläche, Ann. Acad. Sci. Fenn. A. I. 75 (1950).

MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.