ON CONCIRCULAR GEOMETRY AND RIEMANN SPACES WITH CONSTANT SCALAR CURVATURES

SYUN-ICHI TACHIBANA

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S. Sasaki studied the spaces with normal conformal connexions whose groups of holonomy fix a point or a hypersphere and derived the following fundamental theorem: If the group of holonomy of a space C_n^* with a normal conformal connexion is a subgroup of the Möbius group which fixes a point (or a hypersphere), the C_n^* is a space corresponding to the conformal class of Riemann spaces including an Einstein space with a vanishing (or non-vanishing) scalar curvature [1, I]. He also generalized the Poincaré's representation for non-Euclidean geometry to any Einstein space with non-vanishing scalar curvature [1, III] and studied the spaces with normal conformal connexions whose groups of holonomy fix two points or hyperspheres [1, II]. Concerning these results K. Yano showed that these spaces are closely related to Einstein spaces which admit a concircular transformation and studied the relations between conformal and concircular geometries in these spaces [3]. In this paper we shall define in § 3 a space with a certain conformal connexion which corresponds to a class of concircularly related Riemann spaces. And making use of such a space we shall generalize in § 6 the Poincaré's representation to any Riemann space with non vanishing constant scalar curvature. In § 8 by considering spaces whose groups of holonomy fix two points or hyperspheres we shall obtain some results which are natural generalizations of those obtained by K. Yano [2, V]. Most of the results obtained in this paper will be found their analogues in the papers by S. Sasaki [1, I. II. III] and K. Yano [2, V], [3]. Therefore we shall not state in detail the proofs.

§1. Concircular geometry[1].

Let V_n be a Riemann space with a positive definite metric tensor g_{ij}[2]. In V_n consider a curve $x^i(s)$, where s is the arc length and x^i's represent local coordinates. We denote by $\delta/\delta s$ the covariant derivation with respect to $\left\{ {i \atop jk} \right\}$ along the curve. Then a curve $x^i(s)$ is called a Riemann circle if its first curvature is const. and its second one is 0. The differential equations are

$$(1.1)\qquad \frac{\delta n^i}{\delta s} + g_{jk} n^j n^k \frac{dx^i}{ds} = 0,$$

where

1) See K. Yano [2].

2) $i, j, k, \cdots\cdots$ run from 1 to n.

$$n^i \equiv \frac{\delta}{\delta s}\frac{dx^i}{ds}.$$

Now we take a conformal transformation

(1.2) $$\bar{g}_{ij} = \rho^2 g_{ij},$$

then the Christoffel's symbols are transformed by

(1.3) $$\overline{\left\{ {i \atop jk} \right\}} = \left\{ {i \atop jk} \right\} + \rho_j \delta^i_k + \rho_k \delta^i_k - \rho^i g_{jk},$$

where

$$\rho_j \equiv \frac{\partial}{\partial x^j} \log \rho, \qquad \rho^i = g^{ij}\rho_j.$$

Let us put also

(1.4) $$\rho_{jk} \equiv \rho_{j;k} - \rho_j \rho_k + \frac{1}{2}\rho^i \rho_i g_{jk},$$

where semi-colon denotes the covariant derivative with respect to $\left\{ {i \atop jk} \right\}$. In order that any Riemann circle is transformed into a Riemann circle by (1.2), it is necessary and sufficient that ρ satisfies the following differential equations:

(1.5) $$\rho_{jk} = \phi g_{jk},$$

where ϕ is a scalar function. A conformal transformation satisfying (1.5) and the geometry which deals with properties invariant under such transformations are called the concircular transformation and the concircular geometry respectively.

The curvature tensor $\bar{R}^i{}_{jkl}$ of $\bar{V}_n$ with the metric tensor $\bar{g}_{ij}$ is given by

(1.6) $$\bar{R}^i_{jkl} = R^i_{jkl} - \rho_{jk}\delta^i_l + \rho_{jl}\delta^i_k - g_{jk}\rho^i_l + g_{jl}\rho^i_k,$$

where

(1.7) $$R^i{}_{jkl} = \frac{\partial \left\{ {i \atop jk} \right\}}{\partial x^l} - \frac{\partial \left\{ {i \atop jl} \right\}}{\partial x^k} + \left\{ {h \atop jk} \right\}\left\{ {i \atop hl} \right\} - \left\{ {h \atop jl} \right\}\left\{ {i \atop hk} \right\},$$

and $\rho^i_l \equiv g^{ij}\rho_{jl}$.

If the conformal transformation (1.2) is a concircular one, we obtain from (1.5) and (1.7)

(1.8) $$\begin{aligned} \bar{R}^i_{jkl} &= R^i_{jkl} - 2\phi(g_{jk}\delta^i_l - g_{jl}\delta^i_k), \\ \bar{R}_{jk} &= R_{jk} - 2(n-1)\phi g_{jk}, \end{aligned}$$

and

$$\rho^2 \bar{R} = R - 2n(n-1)\phi,$$

that is

(1.9) $$\phi = -\frac{1}{2n(n-1)}(\rho^2 \bar{R} - R).$$

Substituting the last equation in (1.8), we see that the tensor

(1.10) $$Z^i{}_{jkl} \equiv R^i{}_{jkl} - \frac{R}{n(n-1)}(g_{jk}\delta^i_l - g_{jl}\delta^i_k)$$

is invariant under concircular transformations. $Z^i{}_{jkl}$ is the so-called concircular curvature tensor.

In the next place we consider integrability conditions of (1.5). From

(1.5) we have

(1.11) $$\rho_{j;k} = \psi g_{jk} + \rho_j \rho_k,$$

where

(1.12) $$\psi = \phi - \frac{1}{2} \rho_i \rho^i.$$

Differentiating (1.11) covariantly, we get

$$\rho_{j;k;l} = \psi_{,l} g_{jk} + \rho_k(\psi g_{jl} + \rho_j \rho_l) + \rho_j(\psi g_{kl} + \rho_k \rho_l).$$

Exchange k and l, and subtracting the equation thus obtained from the original one, we have

(1.13) $$\rho_i R^i{}_{jkl} = \psi_{,k} g_{jl} - \psi_{,l} g_{jk} + \psi(\rho_l g_{jk} - \rho_k g_{jl}).$$

If we contract ρ^j to (1.13), then the relation

(1.14) $$\psi_{,k} = \frac{\psi_{,l}\rho^l}{\rho_i \rho^i} \rho_k$$

holds good. By virtue of (1.13), we have

(1.15) $$\rho_i R^i{}_l = -(n-1)\left(\frac{\psi_{,j}\rho^j}{\rho_i \rho^i} - \psi\right)\rho_l.$$

(1.15) shows that any curve (we shall call it ρ-curve) which belongs to the congruence of curves determined by the vector field ρ_i is a Ricci-curve. It is known that ρ curves are geodesics and any hypersurface determined by an equation $\rho = \text{const.}$ (we shall call it ρ-hypersurface) is totally umbilical, furthermore the orthogonal trajectories of ρ-hypersurfaces are ρ-curves [2, II]. The following theorem is obtained easily.

THEOREM 1.1 *Any conformal transformation which makes $Z^i{}_{jkl}$ invariant is a concircular one.*

§ 2. Spaces with conformal connexions.

In a space with conformal connexion C_n, take a Veblen's repère R_A[1], then the defining equations of the connexion are given by

(2.1) $$dR_0 = dx^i R_i, \quad dR_j = \Pi^0_{jk} dx^k R_0 + \left\{ {i \atop jk} \right\} dx^k R_i + g_{jk} dx^k R_\infty, \quad dR_\infty = \Pi^i_{\infty k} dx^k R_i,$$

where

(2.2) $$R_0 R_0 = R_\infty R_\infty = R_0 R_i = R_\infty R_i = 0, \qquad R_0 R_\infty = -1. \quad R_i R_j = g_{ij}, \qquad \Pi^i_{\infty k} = g^{ij}\Pi^0_{jk}.$$

By a transformation (1.2) of the metric tensor the parameters of connexion are transformed as follows [4]:

(2.3) $$\overline{\Pi}^0_{jk} = \Pi^0_{jk} + \rho_{jk}, \quad \overline{\left\{ {i \atop jk} \right\}} = \left\{ {i \atop jk} \right\} + \rho_j \delta^i_k + \rho_k \delta^i_j - \rho^i g_{jk}, \quad \rho^2 \overline{\Pi}^i_{\infty k} = \Pi^i_{\infty k} + \rho^i_k.$$

The conformal curvature tensor of C_n is given by

1) $A, B, C, \cdots\cdots$ run over $0, 1, \cdots\cdots, n, \infty$.

$$(2.4) \qquad F^A_{Bkl} = \frac{\partial \Pi^A_{Bk}}{\partial x^l} - \frac{\partial \Pi^A_{Bl}}{\partial x^k} + \Pi^A_{Cl}\Pi^C_{Bk} - \Pi^A_{Ck}\Pi^C_{Bl},$$

where

$$\Pi^A_{B0} = \delta^A_B, \qquad \Pi^i_{0k} = \delta^i_k, \qquad \Pi^0_{0k} = \Pi^\infty_{0k} = \Pi^0_{\infty k} = \Pi^\infty_{\infty k} = 0,$$

$$\Pi^i_{jk} = \left\{ {i \atop jk} \right\}, \qquad \Pi^\infty_{jk} = g_{jk}.$$

C_n is not determined uniquely by a given Riemann space V_n. But if we assume that

$$(2.5) \qquad F^i_{jkl} = W^i_{jkl},$$

where

$$W^i_{jkl} = R^i_{jkl} - \frac{1}{n-2}(R_{jk}\delta^i_l - R_j\delta^i_k + g_{jk}R^i_l - g_{jl}R^i_k) + \frac{R}{(n-1)(n-2)}(g_{jk}\delta^i_l - g_{jl}\delta^i_k)$$

is the conformal curvature tensor given by H. Weyl, then the space C_n in consideration becomes a space with normal conformal connexion C^*_n. Between the set of spaces with normal conformal connexions and the set of classes each of which consists of Riemann spaces conformal to each other, there exists one to one correspondence. From (2.4) and (2.5), we obtain

$$\Pi^0_{jk} = -\frac{1}{n-2}\left(R_{jk} - \frac{R}{2(n-1)} g_{jk}\right),$$

and by virtue of the property of W^i_{jkl}, we have $F^i_{jki} = 0$.[1]

§ 3. Spaces with conformal connexions which are in one to one correspondence with concircular classes of Riemann spaces.

We shall denote by Ω_n spaces with conformal connexions which satisfy the condition

$$(3.1) \qquad F^i_{jkl} = Z^i_{jkl}.$$

Now consider two Riemann spaces V_n, $\bar{V}_n$ which correspond concircularly to each other. Making use of Veblen's repère we construct Ω_n and $\bar{\Omega}_n$ from V_n and V_n respectively, then $F^i_{jkl} = Z^i_{jkl} = \bar{Z}^i_{jkl} = \bar{F}^i_{jkl}$. From the last equation we obtain (2.2), after some computation.

Therefore Ω_n coincides with $\bar{\Omega}_n$. Conversely if $\bar{\Omega}_n$ corresponding to $\bar{V}_n$ coincides with Ω_n corresponding to V_n which is conformal to $\bar{V}_n$, we have

$$Z^i_{jkl} = F^i_{jkl} = \bar{F}^i_{jkl} = \bar{Z}^i_{jkl}.$$

Hence, on account of Theorem 1.1, $\bar{V}_n$ is concircular to V_n. Therefore we obtain the following

THEOREM 2.1 *The spaces Ω_n with conformal connexions such that the assumption (3.1) is satisfied are in one to one correspondence with classes each of which consists of Riemann spaces which are concircular to each other.*

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1) In order to define C^*_n, it may be used $F^i_{jki}=0$ instead of (2.5). See [4].

It may be worth to notice that some one of the classes in consideration may consist of all Riemann spaces which relates trivially[1] to each other.

If we put $A = i$, $B = j$ in (2.4), then

$$F^{i}_{jkl} = R^{i}_{jkl} + \delta^{i}_{l}\Pi^{0}_{jk} - \delta^{i}_{k}\Pi^{0}_{jl} + g^{ih}\Pi^{0}_{hl}g_{jk} - g^{ih}\Pi^{0}_{hk}g_{jl}.$$

Substituting the last equation and (1.10) in (3.1), we get

$$\delta^{i}_{l}\Pi^{0}_{jk} - \delta^{i}_{k}\Pi^{0}_{jl} + g^{ih}\Pi^{0}_{hl}g_{jk} - g^{ih}\Pi^{0}_{hk}g_{jl} = -\frac{R}{n(n-1)}(g_{jk}\delta^{i}_{l} - g_{jl}\delta^{i}_{k}).$$

Contracting i and l, we obtain

$$(3.2) \qquad (n-2)\Pi^{0}_{jk} + g^{ih}\Pi^{0}_{ih}g_{jk} = -\frac{R}{n}g_{jk}.$$

Contracting g^{jk} with the last equation and substituting it in (3.2), we get

$$(3.3) \qquad \Pi^{0}_{jk} = cg_{jk}, \qquad \Pi^{j}_{\infty k} = c\delta^{j}_{k},$$

where

$$(3.4) \qquad c = -\frac{R}{2n(n-1)},$$

On the other hand we have from (1.10) $g^{jk}Z^{i}_{jki} = 0$, hence by virtue of (3.1) we get

$$(3.5) \qquad g^{jk}F^{i}_{jki} = 0.$$

It is easy to verify that in order to define Ω_n (3.5) may be used instead of (3.1).

§ 4. Theories of curves and hypersurfaces in Ω_n.

As Ω_n is a special case of C_n, the properties of C_n hold good in Ω_n. Therefore for example, the Frenet's formulas of curves in Ω_n i. e. the concircular Frenet's formulas of curves in V_n [2, III] are obtained from those in C_n [4, p. 131] with (3.3). The Gauss, Codazzi and Ricci's equations of subspaces of Ω_n are also obtained from corresponding ones of C_n by substituting (3.3) in these equations [4, p. 144], but the computation is complicated. In analogous way we may obtain another results, but we shall restrict ourselves to state a few results which are easily deducible.

THEOREM 4.1 *In order that a hypersurface X_{n-1} in Ω_n is proper*[2] *umbilical one, it is necessary and sufficient that any hypersphere which is tangent to X_{n-1} is invariant by the connexion of Ω_n along X_{n-1}.*

THEOREM 4.2 *In order that the induced conformal connexion on a hypersurface X_{n-1}: $x^i = x^i(x^a)$ in C^*_n makes X_{n-1} a Ω_{n-1} it is necessary and sufficient that X_{n-1} is umbilical and the equation*

1) It means that ρ in (1.2) is constant.

2) The word "proper" means that the mean curvature of the hypersurface is constant.

(4.1) $$Z^a_{bcd} = -W^i_{jkl}X^{\cdot}_i X^j_b X^k_c X^l_d$$ [1]

holds good, where Z^a_{bcd} *is the concircular curvature tensor of* X_{n-1} *and* $X^j_b = \dfrac{\partial x^j}{\partial x'^b}$, $X^a_i = g^{ab}g_{ij}X^j_b$.

§ 5. Ω_n whose group of holonomy fixes a point or a hypersphere.

Now we take an arbitrary Veblen's repère R_A in order to represent Ω_n analytically. In order that the group of holonomy of Ω_n in consideration fixes a point or a hypersphere it is necessary and sufficient that there exists a function ρ^A satisfying the following equation:

$$d(\rho^A R_A) = \tau(\rho^A R_A),$$

where τ is a Pfaffian [1, I]. If we put $\tau = \tau_k dx^k$, the above equation can be written also as follows:

(5.1) $$\rho^A{}_{|k} \equiv \frac{\partial \rho^A}{\partial x^k} + \Pi^A_{Bk}\rho^B = \tau_k \rho^A.$$

If we put

$$(g_{AB}) \equiv \begin{pmatrix} 0 & 0 & -1 \\ 0 & g_{ij} & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and

(5.2) $$\rho_B \equiv g_{AB}\rho^A,$$

(5.1) can be written in the following covariant form:

(5.3) $$\rho_{B|k} = \frac{\partial \rho_B}{\partial x^k} - \Pi^A_{Bk}\rho_A = \tau_k \rho_B.$$

If we write the last equation explicitly we get

(5.3)$_1$ $$\rho_{0,k} - \rho_k = \tau_k \rho_0,$$

(5.3)$_2$ $$\rho_{j;k} - c g_{jk}\rho_0 - g_{jk}\rho_\infty = \tau_k \rho_j,$$

(5.3)$_3$ $$\rho_{\infty,k} - c\rho_k = \tau_k \rho_\infty,$$

where commas denote the partial derivatives with respect to the coordinate (x^k). In the same way as spaces with normal conformal connexion C^*_n we can prove that if there exist points where $\rho_0 = 0$, their locus is an umbilical hypersurface [1, II], [3]. In the following discussion we restrict ourselves to the domain in Ω_n where these points do not exist. Then we can put $\rho_0 = -1$. Substituting it into (5.3)$_1$, we get $\tau_k = \rho_k$ and (5.3)$_{2,3}$ become

(5.4)$_1$ $$\rho_{j;k} + c g_{jk} - g_{jk}\rho_\infty = \rho_j \rho_k,$$

(5.4)$_2$ $$\rho_{\infty,k} - c\rho_k = \rho_\infty \rho_k.$$

In (5.4)$_1$ $\rho_{j;k}$ are symmetric with respect to j and k, so ρ_j is a gradient vector. Therefore there exists a scalar function such that

(5.5) $$\rho_j = \frac{\partial \log \rho}{\partial x^j}.$$

Now if we define $\bar{c}$ by

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1) See [4. p 144 (5. 11)]. $a, b, c, \cdots\cdots$ run over $1, \cdots\cdots, n-1$.

(5.6) $$\rho_\infty = \bar{c}\rho^2 - \frac{1}{2} g^{ij}\rho_i\rho_j,$$

$\bar{c}$ is a constant on account of (5.4). Substituting (5.6) in $(5.4)_1$, we get

$$\rho_{j;k} + c g_{jk} - \left(\bar{c}\rho^2 - \frac{1}{2} g^{il}\rho_i\rho_l\right) g_{jk} = \rho_j\rho_k ,$$

i.e

(5.7) $$\rho_{jk} = (\bar{c}\rho^2 - c)\, g_{jk}.$$

(5.7) shows that the conformal transformation $\bar{g}_{ij} = \rho^2 g_{ij}$ in consideration is concircular. Hence from (1.9) we get

$$\bar{c}\rho^2 - c = -\frac{1}{2n(n-1)}(\rho^2\bar{R} - R).$$

By virtue of the definition of c, (3.4), it follows that

(5.8) $$\bar{c} = -\frac{\bar{R}}{2n(n-1)}.$$

As $\bar{c}$ is a constant, so is $\bar{R}$. If we choose the Veblen's repère with respect to $\bar{g}_{ij}$, the invariant point or hypersphere in consideration is represented by $A \equiv R_\infty - \bar{c}R_0$. As $A^2 = 2\bar{c}$, A is a point or a hypersphere according to $\bar{c} = 0$ or $\neq 0$. Therefore we have the following

THEOREM 5.1. *Any space Ω_n whose group of holonomy is a subgroup of the Möbius group which fixes a point or a hypersphere corresponds to a concircular class of Riemann spaces including one with non vanishing or vanishing constant scalar curvature respectively. The converse is also true.*

In the following we shall denote, for brevity, the spaces with non-vanishing or vanishing constant scalar curvature by Θ_n^1 or Θ_n^2 respectively. If we do not need to distinguish them, we denote them simply by Θ_n.

§ 6. A generalization of the Poincaré's representation.

S. Sasaki proved that the Poincaré's representations of non-Euclidean geometries can be generalized to any Einstein spaces with $c \neq 0$ making use of C_n^* whose group of holonomy fix a hypersphere [1, III]. Quite analogously the representations can be generalized to any Θ_n^1, replacing Einstein spaces, C_n^* and confomal circles by Θ_n^1, Ω_n and Riemann circles respectively. Therefore we obtain the following theorems.

THEOREM 6.1. *If the group of holonomy of Ω_n fixes a hypersphere (or a point) A, any Riemann circle, having a circle orthogonal to (or a circle passing through) A as its image, is a geodesic of the Θ_n^1 (or Θ_n^2) corresponding to A. The converse is also true.*

THEOREM 6.2. *Suppose that the group of holonomy of Ω_n fixes a hypersphere (or a point) A. Then any totally umbilical hypersurface, having a hypersphere orthogonal to A as its image, is a totally geodesic* *hypersurface of* Θ_n *corresponding to* A.

THEOREM 6.3. *The distance* s *between two points* P_0 *and* P_s *on a geodesic* g *of an* Θ_n^1 *is equal to* $\sqrt{-\frac{n(n-1)}{R}}$ *times the natural logarithm of the double ratio determined by points* P_0^*, P_s^*, P *and* P', *where* P *and* P' *are the points at which the development* g^* *of* g *in a tangent Möbius space of the space* Ω_n *corresponding to the concircular class of Riemann spaces and containing the given* Θ_n^1 *cut the invariant hypersphere* A, *and* P_0^*, P_s^* *are the image of* P_0, P_s *respectively.*

If Θ_n^1 *is the complete space with* $c>0$ *the above representation holds good not only in a tangent space of* Ω_n, *but also in the given* Θ_n^1.

In the previous paper [5] the author obtained the differential equations of pseudo-parallelism in Einstein spaces and computed parallel angles. These results can be also generalized to Θ_n^1.

§ 7. Concircular transformations of a Θ_n to another $\bar{\Theta}_n$.

In this section we shall state theorems concerning concircular transformations of a Θ_n to another $\bar{\Theta}_n$. These results are obtained from Theorem 5. 1 in the same way as the theorems 3, 4, 5, 6 in [1, I] are obtained from the Fundamental Theorem. [1, I]

THEOREM 7.1 *If a Riemann space* V_n *is concircular to* Θ_n *in* $r>1$ *ways, then* V_n *in consideration is concircular to a* Θ_n^2.

THEOREM 7.2 *If a* Θ_n^1 *can be mapped concircularly and non trivially on a* $\bar{\Theta}_n^1$, *it can be mapped on a* Θ_n^2.

THEOREM 7.3. *If a* Θ_n^1 *(or* Θ_n^2*) can be concircularly mapped on a* $\bar{\Theta}_n^2$ *(or* $\bar{\Theta}_n^1$*), it can be mapped concircularly and non trivially on a* $\bar{\bar{\Theta}}_n^1$ *(or* $\bar{\bar{\Theta}}_n^2$*).*

THEOREM 7.4. *If a* Θ_n^2 *is non trivially concircular to another* $\bar{\Theta}_n^2$, *it is concircular to a* Θ_n^1.

Summarizing these theorems, we can state the following theorem [3].

THEOREM 7.5. *If a* Θ_n *with a constant* c *is non trivially concircular to another* $\bar{\Theta}_n$ *with a constant* $\bar{c}$, *then the* Θ_n *is also non trivially concircular to a* $\bar{\bar{\Theta}}_n$ *with any preassigned constant* $\bar{\bar{c}}$.

PROOF. If Θ_n is non trivially concircular to $\bar{\Theta}_n$, the partial differential equations

$$\rho_{jk} = (\bar{c}\rho^2 - c)\, g_{jk}$$

must be completely integrable. The necessary and sufficient condition for this is that the space Θ_n admits a family of ∞^1 totally umbilical hypersurfaces whose orthogonal trajectories are geodesic Ricci curves. But this condition does not depend on the constant $\bar{c}$.

§8. On the line element of Θ_n which is non trivially concircular to another $\bar{\Theta}_n$.

Suppose that Θ_n is non trivially concircular to another $\bar{\Theta}_n$. Let c and $\bar{c}$ be constants corresponding to Θ_n and $\bar{\Theta}_n$ respectively. As Θ_n admits a concircular transformation, if we choose a suitable coordinate system, its line element takes the following form [2, V]:

(8.1) $$ds^2 = f^2(x^n)\overset{*}{g}_{ab}(x^c)dx^a dx^b + (dx^n)^2. \qquad (a, b, c = 1, \cdots, n-1).$$

Therefore after some calculations we obtain

(8.2) $$R_{ab} = \overset{*}{R}_{ab} - [(n-2)f'^2 + ff'']\overset{*}{g}_{ab},$$
$$R_{nb} = 0,$$
$$R_{nn} = -(n-1)\frac{f''}{f},$$

where $\overset{*}{R}_{ab}$ is the Ricci's tensor constructed from $\overset{*}{g}_{ab}$ and dashes denote derivatives with respect to x^n. From (1.9), (1.12) we get

$$\psi = \rho^2\bar{c} - c - \frac{1}{2}\rho^i\rho_i.$$

Differentiating the last equation with respect to x^j, we have

$$\psi,_j = 2\rho^2\bar{c}\rho_j - (\psi\rho_j + \rho^i\rho_i\rho_j),$$

whence we get

$$\rho^j\psi,_j = (2\rho^2\bar{c} - \psi - \rho^i\rho_i)\rho^j\rho_j.$$

Substituting the last equation in (1.15), we obtain

(8.3) $$\rho_i R^i_l = -2(n-1)c\rho_l.$$

If we remember that in our coordinate system ρ-curves are x^n-curves, we can put $\rho^i = \alpha\delta^i_n$. Hence from (8.3) we have

$$R_{nn} = -2c(n-1).$$

From $(8.2)_3$ and the above equation, we get

$$f'' = 2cf.$$

Integrating the last equation we obtain the following theorem [2, V].

THEOREM 8.1. *The line element of Θ_n which is non trivially concircular to another Θ_n can be reduced to the following canonical form:*

(I) $ds^2 = (A\cos\sqrt{-2c}\,x^n + B\sin\sqrt{-2c}\,x^n)^2\overset{*}{g}_{ab}dx^a dx^b + (dx^n)^2$, *if* $c < 0$,

(II) $ds^2 = (Ax^n + B)^2\overset{*}{g}_{ab}dx^a dx^b + (dx^n)^2$, *if* $c = 0$,

(III) $ds^2 = (Ae^{\sqrt{2c}\,x^n} + Be^{-\sqrt{2c}\,x^n})^2\overset{*}{g}_{ab}dx^a dx^b + (dx^n)^2$, *if* $c > 0$,

where $\overset{*}{g}_{ab}(x^c)\,dx^a dx^b$ *is a line element of* Θ_{n-1} *whose scalar curvature is*

(I) $$\overset{*}{R} = \frac{(n-2)}{n}(A^2 + B^2),$$

(II) $$\overset{*}{R} = (n-1)(n-2)A^2,$$

(III) $$\overset{*}{R} = \frac{4(n-2)}{n}AB.$$

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MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY, SENDAI.