ON THE THEORY OF NON-HOLONOMIC SYSTEMS IN THE FINSLER SPACE

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Introduction. G. Vranceanu[1] introduced the concept of a non-holonomic space which is more general than a Riemannian space and generalized the parallelism of Levi-Civita and geodesic curves in that space. From another standpoint Z. Horak[2] considered a non-holonomic region as a space with a non-holonomic dynamical system. These spaces were afterwards discussed by several authors. The non-holonomic system in a space of line-elements with an affine connection was first discussed by T. Hosokawa[3].

We now suppose that at each point x of an n-dimensional space n independent differential forms

$$ds^a = A^a(x^\alpha, dx^\beta) \qquad \begin{pmatrix} \alpha, \beta = 1, 2, \cdots, n; \\ a = 1, 2, \cdots, n \end{pmatrix} \tag{0.1}$$

are given for a displacement dx, where $A^a(x, dx)$ are homogeneous of degree one in the dx. If we write (0.1) in the form

$$ds^a = \lambda_\alpha(x, dx)dx^\alpha \qquad \left(\lambda^a_\alpha(x, dx) = \frac{\partial A^a}{\partial(dx^\alpha)}\right), \tag{0.2}$$

$\lambda^a_\alpha(x, dx)$ depend on the direction of dx only and have a non-vanishing determinant. As easily seen, λ^a_α is covariant in α. Hence we can define in the space of line-elements (x, x') a special non-holonomic[4] system by

$$ds^a = \lambda^a_\alpha(x, x')dx^\alpha, \tag{0.3 a}$$

which determines the displacement of a point in this system. ds^a coincides with the original $A^a(x, dx)$ when and only when the displacement lies in the direction of the line-element:

$$s'^a = \lambda^a_\alpha(x, x')x'^\alpha = A^a(x, x'). \tag{0.3b}$$

In this paper we treat such non-holonomic systems and derive some fundamental quantities. We find that by use of our system the well known Cartan connection of a Finsler-space can be expressed far more neatly than by general non-holonomic systems.

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1) G. VRANCEANU, *Sur les espaces non holonomes*; *Sur le calcul différentiel absolu pour les variétés non holonomes*, C. R., 183 (1926), p. 852 and 1083.

2) Z. HORAK, *Sur une généralisation de la notion de variété*, Publ. Fac. Sc. Univ. Masaryk, Brno, 86 (1927), pp. 1-20.

3) T. HOSOKAWA, *Ueber nicht-holonome Uebertragung in allgemeiner Mannigfaltigkeit T_n*, Jour. Fac. Sci. Hokkaido Imp. Univ., I, 2 (1934), pp. 1-11.

4) We use Greek indices in holonomic systems and Latin indices in non-holonomic systems.

§ 1. Fundamental properties of non-holonomic systems. From the given field of the n independent covariant vectors $\lambda^a_\alpha(x, x')$ we first derive that of the reciprocal contravariant vactors $\lambda^\alpha_a(x, x')$ as follows. If we solve the equations (0.1) for dx^α, the solutions may be of the form

$$dx^\alpha = B^\alpha(x, ds) \tag{1.1}$$

with homogeneous function B^α of degree one; we then put

$$\frac{\partial B^\alpha}{\partial(ds^a)} = {}^*\lambda^\alpha_a(x, ds) \quad \text{or} \quad = \lambda^\alpha_a(x, dx). \tag{1.2}$$

(In general, when we substitute ds in ${}^*f(x, ds)$ by (0.1) we write $f(x, dx)$ and vice versa; further, in case x' replaces dx we write s' in place of ds.) If we differentiate (1.1) with respect to dx^β, we get

$$\lambda^\alpha_a(x, x')\lambda^a_\beta(x, x') = \delta^\alpha_\beta. \tag{1.3}$$

Similarly differentiating (0.1) we get

$$\lambda^a_\alpha(x, x')\lambda^\alpha_b(x, x') = \delta^a_b. \tag{1.4}$$

(1.3) and (1.4) show the reciprocity of the two fields λ^a_α and λ^α_a: we write also by the above convention

$${}^*\lambda^\alpha_a(x, s')\,{}^*\lambda^a_\beta(x, s') = \delta^\alpha_\beta, \qquad {}^*\lambda^a_\alpha(x, s')\,{}^*\lambda^\alpha_b(x, s') = \delta^a_b. \tag{1.5}$$

We now introduce the following fundamental operation formally called the partial differentiation

$$\frac{\partial {}^*f}{\partial s^a} = \frac{\partial {}^*f}{\partial x^\alpha}\,{}^*\lambda^\alpha_a = \left(\frac{\partial f}{\partial x^\alpha} + \frac{\partial f}{\partial x'^\beta}\frac{\partial {}^*\lambda^\beta_b}{\partial x^\alpha}s'^b\right){}^*\lambda^\alpha_a; \tag{1.6}$$

the differentiation symbol $\partial/\partial s^a$ has only a formal meaning. We must note the obvious fact that $\partial f/\partial x^\alpha$ and $\partial {}^*f/\partial x^\alpha$ are differerent:

$$\frac{\partial f}{\partial x^\alpha} = \frac{\partial {}^*f}{\partial x^\alpha} + \frac{\partial {}^*f}{\partial s'^a}\frac{\partial \lambda^a_\beta}{\partial x^\alpha}x'^\beta. \tag{1.7}$$

Applying $\partial/\partial s^b$ on (1.6) and permuting the indices a and b, we easily obtain

$$\frac{\partial^2 {}^*f}{\partial s^a \partial s^b} - \frac{\partial^2 {}^*f}{\partial s^b \partial s^a} = -\omega^c_{ab}\frac{\partial {}^*f}{\partial s^c} \tag{1.8}$$

where

$$\omega^c_{ab} = \left(\frac{\partial {}^*\lambda^c_\alpha}{\partial \lambda^\beta} - \frac{\partial {}^*\lambda^c_\beta}{\partial x^\alpha}\right){}^*\lambda^\alpha_a\,{}^*\lambda^\beta_b. \tag{1.9}$$

On the other hand, since

$$\frac{\partial {}^*f(x, s')}{\partial s'^a} = \frac{\partial f(x, x')}{\partial x'^\alpha}\lambda^\alpha_a,$$

$$\frac{\partial^2 {}^*f(x, s')}{\partial s'^a \partial s'^b} = \frac{\partial^2 f(x, x')}{\partial x'^\alpha \partial x'^\beta}\,{}^*\lambda^\alpha_a\,{}^*\lambda^\beta_b + \frac{\partial f}{\partial x'^\alpha}\frac{\partial {}^*\lambda^\alpha_a}{\partial s'^b},$$

we get

$$\frac{\partial^2 {}^*f(x, s')}{\partial s'^a \partial s'^b} - \frac{\partial^2 f(x, x')}{\partial x'^\alpha \partial x'^\beta}\,{}^*\lambda^\alpha_a\,{}^*\lambda^\beta_b = \Omega^c_{ab}\frac{\partial {}^*f}{\partial s'^c} \tag{1.10}$$

where

$$\Omega^c_{ab} = \frac{\partial {}^*\lambda^\alpha_a}{\partial s'^b}\,{}^*\lambda^c_\alpha. \tag{1.11}$$

The quantities ω^c_{ab} and Ω^c_{ab} will play an important rôle in the non-holonomic system (0.3).

We shall proceed to find the relations between these quantities ω^a_{bc}'s and Ω^a_{bc}'s under transformations of non-holonomic system. If two systems are determined by the functions A^a's and $'A^i$'s, then we have

(1.12) $$'s'^i = 'A^i(x, x'), \qquad d's^i = '\lambda^i_\alpha(x, x')dx^\alpha.$$

By (0.3b), (1.12) s'^i will directly be transformed into $'s'^a$ by the equations of the form

(1.13) $$s'^a = C^a(x, 's'^i), \qquad A^a = C^a(x, 'A^i),$$

where C^a are mutually independent in $'s'^i$ and homogeneous of degree one. From (0.1), (1.12), (1.13) we obtain by differentiation

(1.14) $$\lambda^a_\alpha = C^a_i{}'\lambda^i_\alpha, \qquad '\lambda^\alpha_i = C^a_i\lambda^\alpha_a,$$

where

$$C^a_i = \frac{\partial C^a}{\partial 's'^i}.$$

Hence ds^a are transformed as

(1.15) $$ds^a = C^a_i(x, 's')d's^i;$$

this is nothing but the non-holonomic transformation of our systems. The inverse equations of (1.13), (1.14), (1.15) run as

(1.13′) $$'A^i = 'C^i(x, A^a),$$

(1.14′) $$'\lambda^i_\alpha = 'C^i_a\lambda^a_\alpha, \qquad \lambda^\alpha_a = 'C^j_a{}'\lambda^\alpha_j \qquad \left('C^j_a = \frac{\partial 'C^j(x, s')}{\partial s'^a}\right),$$

(1.15′) $$d's^i = 'C^i_a ds^a.$$

Obviously we have

(1.16) $$'C^i_a C^b_i = \delta^b_a, \qquad 'C^i_a C^a_j = \delta^i_j.$$

Now, differentiating the second equation of (1.14) with respect to $'s'^j$ and noticing the homogeneity of C^b_k we get the transformation formula of the quantity Ω^c_{ab} in the form

(1.17) $$'\Omega^k_{ij} = \frac{\partial C^a_i}{\partial 's'^j}\, 'C^k_a + C^a_i C^b_j\, 'C^k_c\Omega^c_{ab}.$$

On the other hand, in virtue of (1.14′) we see the quantity ω^c_{ab} is transformed as follows

(1.18) $$'\omega^k_{ij} = C^a_i C^b_j\left\{\left(\frac{\partial^{*}{}'C^k_a}{\partial s^b} - \frac{\partial^{*}{}'C^k_b}{\partial s^a}\right) + 'C^k_c\omega^c_{ab}\right\} + \left(\frac{\partial C^d_l}{\partial 's^j} C^a_i - \frac{\partial C^d_l}{\partial 's^i} C^a_j\right)'s'^l\left(\frac{\partial 'C^k_a}{\partial s'^d} - 'C^k_c\Omega^c_{ad}\right).$$

Next we shall derive the covariant derivative of tensors with respect to s' in our non-holonomic systems. The components of a vector v^α or a tensor $T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q}$ in a non-holnomic system are defined by

(1.19) $$v^a = \lambda^a_\alpha v^\alpha, \qquad T^{a_1\cdots a_p}_{b_1\cdots b_q} = \lambda^{a_1}_{\alpha_1}\cdots\lambda^{a_p}_{\alpha_p}\lambda^{\beta_1}_{b_1}\cdots\lambda^{\beta_q}_{b_q}T^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_q},$$

where v^α and $T^{\alpha_1\cdots}_{\beta_1\cdots}$ are components in the holonomic coordinate system x. In holonomic systems the partial derivatives of a vector $v^\alpha(x, x')$ with respect to x' are components of a tensor of degree two, but in a non-holonomic system the partial derivatives are not generally those of a tensor. Nevertheless the modified derivatives

$$v^a{}_{:b} = \frac{\partial^* v^a}{\partial s'^b} + \Omega^a_{db} v^d = {}^*v^a{}_{;b} + \Omega^a_{db} v^d \tag{1.20}$$

are components of a tensor of degree two, where ";" denotes partial differentiation with respect to s'. This can be easily seen in help of (1.17) or by the following. Indeed (1.20) are written by the definition of Ω^a_{bc} as

$$v^a{}_{:b} = \lambda^a_\alpha \lambda^\beta_b \frac{\partial v^\alpha}{\partial x'^\beta}. \tag{1.21}$$

By this reason, we call the tensor $v^a{}_{:b}$ the covariant derivative of the vector field v^a with respect to s'.

Generally for components of a tensor field $T^{a_1\cdots a_p}{}_{b_1\cdots b_q}$, we consider likewise

$$T^{a_1\cdots a_p}{}_{b_1\cdots b_q : b_{q+1}} = T^{a_1\cdots a_p}{}_{b_1\cdots b_q ; b_{q+1}} + \sum_{i=1}^{p} \Omega^{a_i}_{ab_{q+1}} T^{a_1\cdots a_{i-1} a\, a_{i+1}\cdots a_p}{}_{b_1\cdots b_q}$$
$$- \sum_{j=1}^{q} \Omega^b_{b_j b_{q+1}} T^{a_1\cdots a_p}{}_{b_1\cdots b_{j-1} b\, b_{j+1}\cdots b_q}. \tag{1.22}$$

It can be shown that $T^{a_1\cdots a_p}{}_{b_1\cdots b_q : b_{q+1}}$'s are non-holonomic components of the ordinary covariant derivative with respect to x', that is

$$T^{a_1\cdots a_p}{}_{b_1\cdots b_q : b_{q+1}} = \lambda^{a_1}_{\alpha_1} \cdots \lambda^{a_p}_{\alpha_p} \lambda^{\beta_1}_{b_1} \cdots \lambda^{\beta_q}_{b_q} \lambda^{\beta_{q+1}}_{b_{q+1}} T^{\alpha_1\cdots\alpha_p}{}_{\beta_1\cdots\beta_q ; \beta_{q+1}}. \tag{1.23}$$

Accordingly we call them the covariant derivative of the tensor field T with respect to s'.

2. Fundamental quantities in the Finsler space. The arc length of a curve $x^\alpha = x^\alpha(t)$ in a Finsler space is given by an integral

$$s = \int \mathfrak{L}(x, x')\, dt, \tag{2.1}$$

where $\mathfrak{L}(x, x')$ is homogeneous of degree one in the x'. To introduce an invariant connection we usually consider the manifold of line-elements (x, x'), each of which is composed of a point x and a direction x' in this point. As is well known, E. Cartan established the euclidean connection by setting four postulates[5]. This is the space with which we shall concern here.

In the Finsler space we consider a non-holonomic system defined by (0.3) and represent $\mathfrak{L}(x, x')$ in x and s':

$$\mathfrak{L}(x, x') = {}^*\mathfrak{L}(x, s') \tag{2.2}$$

which is homogeneous of degree one in the s'. By differentiation we get

$$\frac{\partial^* \mathfrak{L}(x, s')}{\partial s'^a} = \frac{\partial \mathfrak{L}(x, x')}{\partial x'^\alpha} \lambda^\alpha_a. \tag{2.3}$$

Putting the vector

[5] E. Cartan, *Les espaces de Finsler*, Actualités sci. et ind., 79 (1934).

$$l_\alpha = \frac{\partial \mathfrak{L}}{\partial x'^\alpha},$$

we obtain

(2.4) $$l_i = \frac{\partial {}^*\mathfrak{L}}{\partial s'^i}.$$

If we put

$$\mathfrak{F} = \frac{1}{2}\mathfrak{L}^2 \quad \text{and} \quad g_{\alpha\beta} = \frac{\partial^2 \mathfrak{F}}{\partial x'^\alpha \partial x'^\beta},$$

the tensor $g_{\alpha\beta}$ serves as the fundamental metric tensor in the manifold of line-elements. By virtue of (2.3) we see that

(2.5) $$\mathfrak{F} = {}^*\mathfrak{F}, \qquad {}^*\mathfrak{F}_{;a} = \lambda^\alpha_a \frac{\partial \mathfrak{F}}{\partial x'^\alpha},$$

$$ {}^*\mathfrak{F}_{;a;b} = \lambda^\alpha_a \lambda^\beta_b g_{\alpha\beta}, \qquad {}^*\mathfrak{F}_{;a;b;c} = 2\lambda^\alpha_a \lambda^\beta_b \lambda^\gamma_c C_{\alpha\beta\gamma},$$

where we put $C_{\alpha\beta\gamma} = \frac{1}{2}\partial g_{\alpha\beta}/\partial x'^\gamma$ following Cartan. Thus the metric tensor $g_{\alpha\beta}$ and $C_{\alpha\beta\gamma}$ are given in the non-holonomic system by

(2.6) $$\begin{cases} g_{ab} = {}^*\mathfrak{F}_{;a;b} - {}^*\mathfrak{F}_{;d}\Omega^d_{ab} \\ C_{abc} = \frac{1}{2} g_{ab;c} = \frac{1}{2}(g_{ab;c} - \Omega^d_{ac} g_{db} - \Omega^d_{bc} g_{ad}). \end{cases}$$

As can be seen by (2.5), these components are symmetric in their indices and it holds

(2.7) $$C_{abc}s'^c = 0.$$

3. Parameters of connection. In our non-holonomic system we can introduce the covariant differential of a contravariant vector field v in the form

(3.1) $$\delta v^a = \lambda^a_\alpha \delta v^\alpha = \lambda^a_\alpha (dv^\alpha + \Gamma^{*\alpha}_{\beta\gamma} v^\beta dx^\gamma + C^\alpha_{\beta\gamma} v^\beta \delta x'^\gamma)$$

and put

(3.2) $$\delta v^a = dv^a + \Gamma^{*a}_{bc} v^b dx^c + C^{*a}_{bc} v^b \delta s'^c.$$

If we put $\Gamma^{*\alpha}_{\beta\gamma} x'^\beta x'^\gamma = 2G^\alpha$, $\delta x'^\alpha = dx'^\alpha + (\partial G^\alpha/\partial x'^\gamma)dx^\gamma$ following Cartan, then we have at once

(3.3) $$\begin{cases} \Gamma^{*a}_{bc} = \left\{\lambda^a_\alpha \Gamma^{*\alpha}_{\beta\gamma} - \frac{\partial \lambda^a_\beta}{\partial x^\gamma} + \frac{\partial \lambda^a_\beta}{\partial x'^\delta}\frac{\partial G^\delta}{\partial x'^\gamma}\right\}\lambda^\beta_b \lambda^\gamma_c, \\ C^{*a}_{bc} = \left\{\lambda^a_\alpha C^\alpha_{\beta\gamma} - \frac{\partial \lambda^a_\beta}{\partial x'^\gamma}\right\}\lambda^\beta_b \lambda^\gamma_c = g^{ae}C_{bce} + \Omega^a_{bc} = C^a_{bc} + \Omega^a_{bc}, \end{cases}$$

where g^{ab} is the contravariant tensor defined by g_{ab} as usual: $g_{ab}g^{ac} = \delta^c_b$. Further we get by (3.3)

(3.4) $$G^a_b = \Gamma^{*a}_{cb} s'^c = \lambda^a_\alpha \lambda^\beta_b \frac{\partial G^\alpha}{\partial x'^\beta} - \frac{\partial \lambda^a_\beta}{\partial x^\gamma} x'^\beta \lambda^\gamma_b.$$

In holonomic systems the parameters of the euclidean connection of Cartan are given by the following formulas:

$$\gamma^\gamma_{\alpha\beta} = \frac{1}{2} g^{\delta\gamma}\left(\frac{\partial g_{\alpha\delta}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\delta}\right),$$

(3.5) $$G^{\gamma} = \frac{1}{2}\gamma^{\gamma}_{\alpha\beta}\, x'^{\alpha} x'^{\beta},$$

$$\Gamma^{*\gamma}_{\alpha\beta} = \gamma^{\gamma}_{\alpha\beta} + \left(C_{\alpha\beta\delta}\frac{\partial G^{\delta}}{\partial x'^{\epsilon}} - C_{\alpha\epsilon\delta}\frac{\partial G^{\delta}}{\partial x'^{\beta}} - C_{\beta\epsilon\delta}\frac{\partial G^{\delta}}{\partial x'^{\alpha}}\right) g^{\gamma\epsilon}.$$

The parameters C^{*a}_{bc} are determined already by (3.3). We shall write in the following the parameters Γ^{*c}_{ab} in terms of the fundamental quantities g_{ab}, g^{ab}, C_{abc}, ω^{a}_{bc} and Ω^{a}_{bc}.

Differentiating $g_{\alpha\beta} = g_{ab}\lambda^{a}_{\alpha}\lambda^{b}_{\beta}$ and using (1.5), (1.11), (2.6), we get

$$\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} = \frac{\partial^{*} g_{ab}}{\partial s^{c}}\lambda^{c}_{\gamma}\lambda^{a}_{\alpha}\lambda^{b}_{\beta} + {}^{*}g_{ba}\left(\frac{\partial^{*}\lambda^{b}_{\alpha}}{\partial x^{\gamma}}\lambda^{b}_{\beta} + \lambda^{a}_{\alpha}\frac{\partial^{*}\lambda^{b}_{\beta}}{\partial x^{\gamma}}\right) + 2C_{abc}\frac{\partial\lambda^{c}_{\delta}}{\partial x^{\gamma}} x'^{\delta}\lambda^{a}_{\alpha}\lambda^{b}_{\beta}$$

and consequently

(3.6)
$$\begin{aligned}\gamma^{\gamma}_{\alpha\beta} = \frac{1}{2}\Big\{ & {}^{*}g^{ef}\left(\frac{\partial^{*} g_{ae}}{\partial s^{b}} + \frac{\partial^{*} g_{be}}{\partial s^{a}} - \frac{\partial^{*} g_{ab}}{\partial s^{e}}\right)\lambda^{a}_{\alpha}\lambda^{b}_{\beta}\lambda^{\gamma}_{f} + {}^{*}g^{ef}\lambda^{\delta}_{e}\lambda^{\gamma}_{f}\,{}^{*}g_{ab}\lambda^{a}_{\alpha}\left(\frac{\partial^{*}\lambda^{b}_{\delta}}{\partial x^{\beta}} - \frac{\partial^{*}\lambda^{b}_{\beta}}{\partial x^{\delta}}\right) \\ & + {}^{*}g^{ef}\lambda^{\delta}_{e}\lambda^{\gamma}_{f}\,{}^{*}g_{ab}\lambda^{a}_{\beta}\left(\frac{\partial^{*}\lambda^{b}_{\delta}}{\partial x^{\alpha}} - \frac{\partial^{*}\lambda^{b}_{\alpha}}{\partial x^{\delta}}\right) - \lambda^{\gamma}_{b}\left(\frac{\partial^{*}\lambda^{b}_{\alpha}}{\partial x^{\beta}} - \frac{\partial^{*}\lambda^{b}_{\beta}}{\partial x^{\alpha}}\right)\Big\} + \lambda^{\gamma}_{b}\frac{\partial\lambda^{b}_{\alpha}}{\partial x^{\beta}} \\ & + {}^{*}g^{ef}\lambda^{\delta}_{e}\lambda^{\gamma}_{f}C_{abc}\left(\frac{\partial^{*}\lambda^{b}_{\epsilon}}{\partial x^{\beta}}\lambda^{a}_{\alpha}\lambda^{c}_{\delta} + \frac{\partial^{*}\lambda^{a}_{\epsilon}}{\partial x^{\alpha}}\,{}^{*}\lambda^{b}_{\beta}\lambda^{c}_{\delta} - \frac{\partial^{*}\lambda^{c}_{\epsilon}}{\partial x^{\delta}}\lambda^{a}_{\alpha}\lambda^{b}_{\beta}\right)x'^{\epsilon}.\end{aligned}$$

Noticing $\frac{\partial\lambda^{f}_{\alpha}}{\partial x^{\beta}} x'^{\alpha} = \frac{\partial^{*}\lambda^{f}_{\alpha}}{\partial x^{\beta}} x'^{\alpha}$ we get further

(3.7)
$$\begin{aligned}\gamma^{\gamma}_{\alpha\beta}x'^{\alpha}x'^{\beta} = \lambda^{\gamma}_{f}\Big\{ & \frac{1}{2}\,{}^{*}g^{ef}\left(\frac{\partial^{*} g_{ae}}{\partial s^{b}} + \frac{\partial^{*} g_{be}}{\partial s^{a}} - \frac{\partial^{*} g_{ab}}{\partial s^{e}}\right)s'^{a}s'^{b} \\ & + \frac{1}{2}\left({}^{*}g^{ef}\,{}^{*}g_{ac}\omega^{c}_{eb} + {}^{*}g^{ef}\,{}^{*}g_{bc}\omega^{c}_{ea} - \omega^{f}_{ab}\right)s'^{a}s'^{b} + \frac{\partial\lambda_{\alpha}}{\partial x^{\beta}}x'^{\alpha}x'^{\beta}\Big\}.\end{aligned}$$

On the other hand we have from the first equations of (3.3)

$$\Gamma^{a}_{bc}s'^{b}s'^{c} = \lambda^{a}_{\alpha}\gamma^{\alpha}_{\beta\gamma}x'^{\beta}x'^{\gamma} - \frac{\partial\lambda^{a}_{\alpha}}{\partial x^{\beta}}x'^{\beta}x'^{\alpha}.$$

From the two last equations we arrive at

(3.8) $$\Gamma^{*a}_{bc}s'^{b}s'^{c} = \gamma^{a}_{bc}s'^{b}s'^{c},$$

where we put

(3.9) $$\gamma^{a}_{bc} = \frac{1}{2}\left\{{}^{*}g^{ae}\left(\frac{\partial^{*} g_{be}}{\partial s^{c}} + \frac{\partial^{*} g_{ec}}{\partial s^{b}} - \frac{\partial^{*} g_{bc}}{\partial s^{e}}\right) + {}^{*}g^{ae}\,{}^{*}g_{bd}\omega^{d}_{ec} + {}^{*}g^{ae}\,{}^{*}g_{cd}\omega^{d}_{eb} - \omega^{a}_{bc}\right\}.$$

These correspond to the Christoffel's symbols $\left\{ {\alpha \atop \beta\gamma} \right\}$.

From (2.5), (2.6), (3.4) we obtain

$$g^{\gamma\delta}\left(C_{\alpha\beta\epsilon}\frac{\partial G^{\epsilon}}{\partial x'^{\delta}}\right) = \lambda^{\gamma}_{e}\lambda^{a}_{\alpha}\lambda^{b}_{\beta}\,g^{ef}C_{abg}\left(G^{g}_{f} - \frac{\partial^{*}\lambda^{g}_{\epsilon}}{\partial x^{\delta}}x'^{\epsilon}\lambda^{\delta}_{f}\right),$$

by virtue of which we can derive from (3.5), (3.6)

$$\Gamma^{*\gamma}_{\alpha\beta}\lambda^{a}_{\gamma}\lambda^{\alpha}_{b}\lambda^{\beta}_{c} = \gamma^{a}_{bc} + g^{ae}C_{bcd}G^{d}_{e} - g^{ae}C_{bed}G^{d}_{c} - g^{ae}C_{ced}G^{d}_{b} + \frac{\partial^{*}\lambda^{a}_{\alpha}}{\partial x^{\beta}}\lambda^{\alpha}_{b}\lambda^{\beta}_{c}.$$

Using (1.6), (3.4) we calculate

$$\left(\frac{\partial\lambda^{a}_{\epsilon}}{\partial x'^{\delta}}\frac{\partial G^{\delta}}{\partial x'^{\gamma}} - \frac{\partial\lambda^{a}_{\beta}}{\partial x^{\gamma}}\right)\lambda^{\beta}_{b}\lambda^{\gamma}_{c} = \frac{\partial^{*}\lambda^{a}_{\beta}}{\partial s'^{d}}G^{d}_{c}\lambda^{\beta}_{b} - \frac{\partial^{*}\lambda^{a}_{\beta}}{\partial x^{\delta}}\lambda^{\beta}_{b}\lambda^{\delta}_{c} = -\Omega^{a}_{bd}G^{d}_{c} - \frac{\partial^{*}\lambda^{a}_{\beta}}{\partial x^{\gamma}}\lambda^{\beta}_{b}\lambda^{\gamma}_{c}.$$

From these equations and (3.3) we get finally

(3.10) $$\Gamma^{*a}_{bc} = \gamma^a_{bc} + {}^*g^{ad}(C_{bce}G^e_d - C_{cde}G^e_b - C_{bde}G^e_c) - \Omega^a_{bd}G^d_c.$$

However $G^e_a = \Gamma^{*e}_{fa}\, s'^f$, hence we must find the formula of G^e_a written in terms of the fundamental quantities. This can be easily done.

From (3.10) we have successively

(3.11) $$G^a_c = \gamma^a_{bc}s'^b - {}^*g^{ad}C_{cde}G^e_b s'^b,$$
$$G^a_c s'^c = \gamma^a_{bc}s'^b s'^c = 2G^a.$$

and hence

(3.12) $$G^a_c = \gamma^a_{bc}s'^b - 2\,{}^*g^{ad}C_{cde}G^e.$$

Thus the formulas: the second of (3.3), (3.9)-(3.12) completely determine the parameters of connection of Cartan in our non-holonomic system. We remark, however, the fact that $G^a_c s'^b = G^a_b$ but G^a_b does not coincide with $\partial G^a/\partial s'^b$ in general, and seek their relation. From the first of (3.3) we have

(3.13) $$G^\alpha = \lambda^\alpha_a\left\{G^a + \frac{1}{2}\frac{\partial \lambda^a_\gamma}{\partial x^\beta}x'^\gamma x'^\beta\right\},$$

and, differentiating this,

(3.14) $$\frac{\partial G^\alpha}{\partial x'^\gamma} = \lambda^\alpha_a\lambda^c_\gamma\frac{\partial G^a}{\partial s'^c} + \frac{1}{2}\left(\frac{\partial\, {}^*\lambda^a_\gamma}{\partial x^\delta} - \frac{\partial\, {}^*\lambda^a_\delta}{\partial x^\gamma}\right)x'^\delta\lambda^\alpha_a + \frac{\partial \lambda^\alpha_i}{\partial x'^\gamma}G^i + \lambda^\alpha_i\frac{\partial \lambda^i_\delta}{\partial x^\gamma}x'^\delta.$$

Hence we obtain from (3.4), (3.14)

(3.15) $$G^a_b = \frac{\partial G^a}{\partial s'^b} + \frac{1}{2}\omega^a_{bc}s'^c + \Omega^a_{cb}G^c.$$

By use of this formula the connection parameters (3.10) may be written in the form

(3.16) $$\begin{aligned}\Gamma^{*a}_{bc} = \gamma^a_{bc} &+ g^{ad}\left(C_{bce}\frac{\partial G^e}{\partial s'^d} - C_{cde}\frac{\partial G^e}{\partial s'^b} - C_{bde}\frac{\partial G^e}{\partial s'^c}\right) - \Omega^a_{bd}\frac{\partial G^d}{\partial s'^c}\\ &+ \frac{1}{2}\{g^{ad}(C_{cbe}\omega^e_{df} - C_{cde}\omega^e_{bf} - C_{bde}\omega^e_{cf}) - \Omega^a_{bd}\omega^d_{cf}\}\, s'^f\\ &+ \{g^{ad}(C_{cbe}\Omega^e_{df} - C_{cde}\Omega^e_{bf} - C_{bde}\Omega^e_{cf}) - \Omega^a_{bd}\Omega^d_{cf}\}\, G^f.\end{aligned}$$

At last we notice that the extremal curves are given in the non-holonomic system by the equations

(3.17) $$\frac{d^2s^a}{ds^2} + 2G^a\left(x, \frac{ds^a}{ds}\right) = 0.$$

4. Curvature and torsion tensors. If we denote the basis vectors of the natural reference system defined at each line-element (x, x') by e_α $(\alpha = 1, \cdots, n)$, we have the displacement of the centre $M(x)$

(4.1) $$dM = dx^\alpha e_\alpha,$$

which can be written in a non-holonomic system as

(4.2) $$e_a = \lambda^\alpha_a e_\alpha,$$

(4.3) $$dM = ds^a\lambda^\alpha_a e_\alpha = ds^a e_a.$$

The n vectors e_a $(a = 1, \cdots, n)$ are the basis vectors of the non-holonomic system. Using the symbol $\omega^e_a = \Gamma^{*e}_{ab}ds^b + C^{*e}_{ab}\delta s'^b$, $de_a = \omega^e_a e_e$ and denoting two infinitesimal displacements with d_1 and d_2, we have

(4.4) $$d_2 d_1 M - d_1 d_2 M = (d_2 d_1 s^a - d_1 d_2 s^a) e_a + (d_1 s^a \omega_a^e - d_2 s^a \omega_a^e) e_e.$$

From the first equations of (0.3a) it follows that

(4.5) $$d_2 d_1 s^a - d_1 d_2 s^a = \omega_{bc}^a d_1 s^b d_2 s^c + 2\Omega^a_{(b|d|} G^d_{c)} d_1 s^b d_2 s^c - \Omega^a_{bc} (\delta_2 s'^c d_1 s^b - \delta_1 s'^c d_2 s^b).$$

Thus, if we put

(4.6) $$d_2 d_1 M - d_1 d_2 M = \Omega^a e_a$$

we have

(4.7) $$\Omega^a = (\omega^a_{bc} + 2\Omega^a_{(b|d|} G^d_{c)} + 2\Gamma^{*a}_{(bc)}) d_{(1} s^b d_{2)} s^c + (C^{*a}_{bc} - \Omega^a_{bc}) [d_1 s^b \delta_2 s'^c].$$

If our space is a general space of line-elements, not necessarily Finslerian but with an affine connection, we obtain the torsion tensors of two kinds as follows:

(4.8) $${}'T^a_{bc} = \omega^a_{bc} + 2\Omega^a_{(b|f|} G^f_{c)} + 2\Gamma^{*a}_{(bc)}, \qquad {}''T^a_{bc} = C^{*a}_{bc} - \Omega^a_{bc}$$

if the space is Finslerian, we get from (3.9), (3.10)

(4.9) $$2\Gamma^{*a}_{(bc)} = -\omega^a_{bc} - 2\Omega^a_{(b|d|} G^d_{c)}$$

and from the second equation of (3.3) $C^{*a}_{bc} - \Omega^a_{bc} = C^a_{bc}$, hence

(4.10) $$\Omega^a = C^a_{bc} [d_1 s^b \delta_2 s'^c]$$

C^a_{bc} being the only torsion tensor.

On the other hand, if we put the covariant differential δv^a in the form

(4.11) $$\delta v^a = v^a_{\cdot b} ds^b + v^a_{\ddagger b} \delta s'^b,$$

we get the covariant derivatives of two kinds by virtue of (1.20), (3.2):

(4.12) $$(\alpha) \quad v^a_{\cdot b} = \frac{\partial^* v^a}{\partial s^b} - {}^*v^a_{;c} G^c_b + \Gamma^{*a}_{db} v^d,$$

$$(\beta) \quad v^a_{:b} = {}^*v^a_{;b} + \Omega^a_{cb} v^c.$$

As $C^{*a}_{cb} = \Omega^a_{cb} + C^a_{cb}$ and C^a_{cb} is a tensor, we obtain from (β) another covariant derivative

(4.13) $$v^a_{\ddagger b} = {}^*v^a_{;b} + C^{*a}_{cb} v^c.$$

We shall have therefore many tensors by combination of these derivatives.

After a complicated calculation, we get

(4.14) $$v^a_{[\cdot b \cdot c]} = K^a_{dbc} v^d + {}'K^f_{bc} v^a_{;f}$$

where

(4.15) $$\begin{aligned} K^a_{dbc} &= \Gamma^{*a}_{d[b\cdot c]} + \Gamma^{*e}_{d[c} \Gamma^{*a}_{|e|b]} - \Omega^a_{df} {}'K^f_{bc}, \\ {}'K^f_{bc} &= -G^f_{[b,c]} + G^f_{[b;|g|} G^g_{c]} + \Gamma^{*e}_{[bc]} G^f_e, \\ \Gamma^{*a}_{db\cdot c} &= \Gamma^{*a}_{db,c} - \Gamma^{*a}_{db;f} G^f_c + \Gamma^{*a}_{fc} \Gamma^{*f}_{db} - \Gamma^{*e}_{bc} \Gamma^{*a}_{de} - \Gamma^{*e}_{dc} \Gamma^{*a}_{eb} \quad (\Gamma^{*a}_{bc,d} = \partial \Gamma^{*a}_{bc} / \partial s^d). \end{aligned}$$

Further we have the following results

(4.16) $$v^a_{:b\cdot c} - v^a_{\cdot c:b} = L^a_{dbc} v^d + {}'L^e_{bc} v^a_{:e},$$

where

(4.17) $$\begin{aligned} L^a_{dbc} &= \Omega^a_{db\cdot c} - \Gamma^{*a}_{dc;b} + \Omega^e_{cb} \Gamma^{*a}_{de} - \Omega^a_{de} {}'L^e_{bc}, \\ {}'L^e_{bc} &= -\Gamma^{*e}_{bc} + G^e_{c;b} - \Omega^f_{cb} G^e_f, \\ \Omega^a_{db\cdot c} &= \Omega^a_{db,c} - \Omega^a_{db;f} G^f_c + \Gamma^{*a}_{fc} \Omega^f_{db} - \Gamma^{*e}_{bc} \Omega^a_{de} - \Gamma^{*e}_{dc} \Omega^a_{eb}, \end{aligned}$$

and

(4.18) $$v^a_{[:b:c]} = \{\Omega^a_{[b|d|;c]} + \Omega^a_{e[c}\Omega^e_{|d|b]}\}v^d = 0,$$

because

(4.19) $$\Omega^a_{[b|c|;e]} + \Omega^a_{e[c}\Omega^e_{|e|b]} = 0.$$

To the end we obtain

(4.20)
$$v^a_{*b\cdot c} - v^a_{\cdot c*b} = M^a_{dbc}v^d + 'M^e_{bc}v^a_{\cdot e} + ''M^e_{bc}v^a_{:e},$$
$$v^a_{:b*c} - v^a_{*c:b} = N^a_{dbc}v^d + 'N^e_{bc}\, v^a_{e:}, \qquad v^a_{[*b*c]} = O^a_{ebc}v^e,$$
$$\delta_{[2}\delta_{1]}v^a = \{S^a_{ebc}\delta_{[2}s'^c\delta_{1]}s'^b + P^a_{ebc}d_{[2}s^c\delta_{1]}s'^b + R^a_{ebc}d_{[1}s^bd_{2]}s^c\}\, v^e,$$

where

$$M^a_{dbc} = L^a_{dbc} - C^a_{de}\,'L^e_{bc} + C^a_{db\cdot c}, \qquad 'M^e_{bc} = C^e_{bc}, \qquad ''M^e_{bc} = 'L^e_{bc},$$
$$N^a_{dbc} = -C^a_{db:c}, \qquad 'N^e_{bc} = -C^e_{bc} = -'M^e_{bc}, \qquad O^a_{ebc} = C^a_{e[b:c]} + C^a_{d[c}C^d_{|e|b]},$$
$$S^a_{ebc} = O^a_{ebc}, \qquad P^a_{ebc} = M^a_{ebc}, \qquad R^a_{ebc} = K^a_{ebc} - C^a_{ef}\,'K^f_{bc}.$$

Such tensors can be also represented by the fundamental quantities g_{ab}, C_{abc}, ω^a_{bc}, Ω^a_{bc} and their derivatives with respect to the s and s'.

REMARK. This paper was read at the meeting of the Mathematical Society of Japan in Nov., 1948. Recently the present author could read a paper of V. Wagner[6] sent to Prof. A. Kawaguchi, which had many connections with mine and in some respects was more general. Especially V. Wager considered m-dimensional non-holonomic referring manifolds. (March, 1949).

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6) V. WAGNER, *The inner geometry of non-linear non-holonomic manifolds*, Rec. Math., N.S. 13 (1943), pp. 135–167.