ON THE STRUCTURE OF A SPHERE BUNDLE

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1. In this note, we shall give some properties of a kind of sphere bundles. As for the definition of the fibre bundle, see for example, N. E. Steenrod [3].

THEOREM 1. *Let S^m be an m-dimensional sphere, and M^n be an $(n-m)$-sphere bundle over S^m $(n > m \geqq 2)$. In addition, let the homotopy groups $\pi_i(M^n)$ vanish for $i = 1, \cdots, m$. Then n must be equal to $2m-1$.*

PROOF.[1] In considering the homotopy groups of M^n, let x_0 be their base point, and S_0^{n-m} be the fibre over x_0. Then we get the following exact homotopy sequence

$$(1.1) \qquad \cdots\to\pi_i(M^n)\to\pi_i(M^n, S_0^{n-m})\to\pi_{i-1}(S_0^{n-m})\to\pi_{i-1}(M^n)\to\cdots.$$

According to the hypothesis on $\pi_i(M^n)$ and to the exactness of (1. 1), we get

$$(1.2) \qquad \pi_i(M^n, S_0^{n-m})\approx\pi_{i-1}(S_0^{n-m}) \qquad (2\leqq i\leqq m).$$

On the other hand, as M^n is compact, the covering homotopy theorem holds in this case, and according to the theorem of W. Hurewicz and N. E. Steenrod [2], we get

$$(1.3) \qquad \pi_i(M^n, S_0^{n-m}) \approx \pi_i(S^m) \begin{cases} \approx \text{infinite cyclic group, for } i = m, \\ \approx 0 \text{ for } 2\leqq i\leqq m-1, \text{ if } m\geqq 3. \end{cases}$$

From (1.2) and (1.3), we can easily obtain the required conclusion.

Evidently, this theorem can be extended to the generalized spaces, where the structure of homotopy groups and the dimensionalities are the same as we have quoted above.

2. In this section, we shall consider particularly, the orientable manifold M^{2m-1}, which is an $(m-1)$-sphere bundle over S^m. In addition, we shall assume that the projection $\pi : M^{2m-1}\to S^m$ is algebraically inessential.

Let the oriented sphere S^m be situated in an $(m+1)$-Euclidean space $(x_1, \cdots x_{m+1})$ by the equation

$$x_1^2+\cdots+x_{m+1}^2=1.$$

And we shall separate S^m into two hemispheres E_1^m, E_2^m respectively,

$$E_1^m=\{(x_i)\in S^m | x_{m+1}\geqq 0\},\quad E_2^m=\{(x_i)\in S^m | x_{m+1}\leqq 0\}.$$

We shall define $S_0^{m-1}=E_1^m\frown E_2^m$, $O_1=(0,\cdots 0,1)$, $O_2=(0,\cdots, 0, -1)$, and orient S_0^{m-1} coherently with E_1^m and conversely with E_2^m.

It is well known [1], that the fibre bundle over an element must be a product bundle. Therefore, there must exist two homeomorphisms φ_1 and

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1) Prof. K. Aoki suggested me the proof of this theorem.

φ_2 such that:

$$(2.1)\quad \begin{cases} \pi^{-1}(E_1^m) = \varphi_1(E_1^m \times S^{m-1}),\ \pi^{-1}(E_2^m) = \varphi_2(E_2^m \times S^{m-1}); \\ \varphi_1(E_1^m \times S^{m-1}) \cup \varphi_2(E_2^m \times S^{m-1}) = M^{2m-1}; \\ \varphi_1(E_1^m \times S^{m-1}) \cap \varphi_2(E_2^m \times S^{m-1}) = \varphi_1(S_0^{m-1} \times S^{m-1}) \\ \qquad\qquad = \varphi_2(S_0^{m-1} \times S^{m-1}) = D. \end{cases}$$

Therefore, if we define $\psi = \varphi_2^{-1}\varphi_1$, ψ is a homeomorphism from $S_0^{m-1} \times S^{m-1}$, the subset of $E_1^m \times S^{m-1}$, onto $S_0^{m-1} \times S^{m-1}$, the subset of $E_2^m \times S^{m-1}$.

We shall subdivide all the spaces considered here into simplicial complexes. Then, without any loss of generality, we can assume that all the mappings defined above are simplicial ones.

Now, let the $(m-1)$-dimensional homology basis of the subcomplex $S_0^{m-1} \times S^{m-1}$ of $E_1^m \times S^{m-1}$ be

$$p_1 \times S^{m-1}, \qquad S_0^{m-1} \times q_1;$$

and similarly that of $E_2^m \times S^{m-1}$ be

$$p_2 \times S^{m-1}, \qquad S_0^{m-1} \times q_2.$$

Then we get the following homology relation with suitable integers a and b,

$$\psi(S_0^{m-1} \times q_1) \sim a \cdot p_2 \times S^{m-1} + b \cdot S_0^{m-1} \times q_2 \quad \text{in} \quad S_0^{m-1} \times S^{m-1}.$$

If we apply φ_2 here, we get

$$(2.2)\qquad \varphi_1(S_0^{m-1} \times q_1) \sim a \cdot \varphi_2(p_2 \times S^{m-1}) + b \cdot \varphi_2(S_0^{m-1} \times q_2) \quad \text{in} \quad D.$$

We shall take here the cycle $\varphi_1(O_1 \times S^{m-1})$, which is evidently disjoint with D. Therefore we get from (2.2)

$$(2.3)\qquad \begin{aligned} &V_{M^{2m-1}}(\varphi_1(S_0^{m-1} \times q_1),\ \varphi_1(O_1 \times S^{m-1})) \\ &\quad = a \cdot V_{M^{2m-1}}(\varphi_2(p_2 \times S^{m-1}), \varphi_1(O_1 \times S^{m-1})) \\ &\qquad + b \cdot V_{M^{2m-1}}(\varphi_2(S_0^{m-1} \times q_2),\ \varphi_1(O_1 \times S^{m-1})), \end{aligned}$$

denoting by $V_{M^{2m-1}}$ the linking coefficient in the manifold M^{2m-1}.

The first term of the right hand side of (2.3) defines evidently a times of the Hopf invariant of the projection, which we shall denote by $c(\pi)$. The second term must vanish from the fact that $\varphi_2(S_0^{m-1} \times q_2)$ bounds $\varphi_2(E_2^{m-1} \times q_2)$. On the other hand, since $\varphi_1(S_0^{m-1} \times q_1)$ bounds in $\varphi_1(E_1^m \times S^{m-1})$, we can easily verify that

$$\begin{aligned} &V_{M^{2m-1}}(\varphi_1(S_0^{m-1} \times q_1),\ \varphi_1(O_1 \times S^{m-1})) \\ &= V_{\varphi_1(E_1^m \times S^{m-1})}(\varphi_1(S_0^{m-1} \times q_1),\ \varphi_1(O_1 \times S^{m-1})) \\ &= V_{E_1^m \times S^{m-1}}(S_0^{m-1} \times q_1,\ O_1 \times S^{m-1}) = 1. \end{aligned}$$

From this fact and from (2.3), we get $c(\pi) = \pm 1$; and re-orienting if necessary, we can omit the negative sign. Therefore, we have proved the following theorem:

THEOREM 2[2] *Let the orientable manifold M^{2m-1} be an $(m-1)$-sphere bundle over S^m, and let the projection be algebraically inessential, then the*

2) Prof. T. Kudo showed me another proof for this theorem, where the base sphere is, instead of being separated into two hemispheres, regarded as an element with the boundary pinched to a point.

Hopf invariant of the projection must be unity, by choosing the orientation of M^{2m-1} suitably.

The following theorem makes clearer the structure of such a sphere bundle.

THEOREM 3. *Using the notations and assumptions as above, we get the following homologies*

$$\left.\begin{aligned}\varphi_1(S_0^{m-1}\times q_1)-\varphi_2(S_0^{m-1}\times q_2)&\sim\pm\varphi_1(p_1\times S^{m-1})\\&\sim\pm\varphi_2(p_2\times S^{m-1})\end{aligned}\right\}\text{ in } D, \tag{2.4}$$

by choosing the orientations of M^{2m-1} suitably, and by reducing the non-orientable bundles into orientable ones.

PROOF. Let a, b, a' and b' be integers such that

$$\left.\begin{aligned}\varphi_1(S_0^{m-1}\times q_1)&\sim a\cdot\varphi_2(p_2\times S^{m-1})+b\cdot\varphi_2(S_0^{m-1}\times q_2)\\\varphi_2(S_0^{m-1}\times q_2)&\sim a'\cdot\varphi_1(p_1\times S^{m-1})+b'\cdot\varphi_1(S_0^{m-1}\times q_1)\end{aligned}\right\}\text{ in } D.$$

Then, if we solve these homologies on a, b, a' and b', the conclusion of the theorem can be easily obtained.

3. An example. We shall consider here the simplest case $m=2$. Let (ρ_1, θ_1), $(0\leqq\rho_1\leqq 1,\ 0\leqq\theta_1\leqq 2\pi)$ be the polar coordinates in the north hemisphere E_1^2 of S^2, and ξ $(0\leqq\xi\leqq 2\pi)$ be the angle coordinate of a circle S^1. If we take a point $(\rho_1, \theta_1)\in E_1^2$ and $\xi\in S^1$, then the point

$$(\rho_1\cos(\theta_1\pm\xi),\ \rho_1\sin(\theta_1\pm\xi);\ \cos(\pm\xi),\ \sin(\pm\xi)) \tag{3.1}$$

lies in $E_1^2\times S^1$. Conversely, if we take the point $(x_1, x_2; u_1, u_2)$, $(x_1^2+x_2^2\leqq 1$, $u_1^2+u_2^2=1)$ in $E_1^2\times S^1$, then the value ρ_1, satisfying the condition (3.1), is uniquely determined and the angles θ_1 and ξ are congruently determined by mod 2π. Therefore, (3,1) is considered to be a kind of product representation of $E_1^2\times S^1$.

As well as in E_1^2, we shall take in E_2^2 the polar coordinates (ρ_2, θ_2), and in S^1 the angle coordinate η, and represent $E_2^2\times S^1$ as follows:

$$(\rho_2\cos(\theta_2\pm\eta),\ \rho_2\sin(\theta_2\pm\eta);\ \cos(\pm\eta),\ \sin(\pm\eta)). \tag{3.2}$$

Next, we shall define two maps $\varphi_i: E_i^2\times S^1\to S^3$, $(i=1, 2)$ by defining the images of (3.1) and (3.2) by the coordinates

$$\begin{gathered}\left(\frac{\rho_1}{\sqrt{1+\rho_1^2}}\cos(\theta_1\pm\xi),\ \frac{\rho_1}{\sqrt{1+\rho_1^2}}\sin(\theta_1\pm\xi),\ \frac{\cos(\pm\xi)}{\sqrt{1+\rho_1^2}},\ \frac{\sin(\pm\xi)}{\sqrt{1+\rho_1^2}}\right),\\\left(\frac{\cos(\pm\eta)}{\sqrt{1+\rho_2^2}},\ \frac{\sin(\pm\eta)}{\sqrt{1+\rho_2^2}},\ \frac{\rho_2}{\sqrt{1+p_2^2}}\cos(\theta_2\pm\eta),\ \frac{\rho_2}{\sqrt{1+\rho_2^2}}\sin(\theta_2\pm\eta)\right)\end{gathered} \tag{3.3}$$

respectively. If we denote the coordinates of (3.3) as (y_1, y_2, y_3, y_4), they must satisfy the following relations respectively:

$$\begin{aligned}&y_1^2+y_2^2\leqq y_3^2+y_4^2,\quad \Sigma y_i^2=1 &&: H_1^3\\&y_1^2+y_2^2\geqq y_3^2+y_4^2,\quad \Sigma y_i^2=1 &&: H_2^3\end{aligned} \tag{3.4}$$

Therefore H_1^3 and H_2^3 are two subsets of S^3. Here, we can easily verify that φ_1 and φ_2 are homeorphisms.

Next, in choosing coordinates in E_i^2 as (ρ_i, θ_i), the coordinates of the point on the common region, namely on the boundary $p \in S_0^1$ can be chosen to be $(1, \theta)$ or $(1, -\theta)$ according to the cases that p belongs to E_1^2 or to E_2^2 respectively. Therefore, from (3.3) we get by the continuously depending rotations

$$\varphi_1(p \times S^1) = \varphi_2(p \times S^1), \quad \text{for every } p \in S_0^1 .$$

Therefore, S^3 is an orientable circle bundle over S^2.

If we take two points $p_1 = (1, 0) \in S_0^1$, $q_1 = (0) \in S^1$, the 1-dimensional homology basis of the space $\varphi_1(S_0^1 \times S^1)$ is obtained as follows:

$$\text{(3.5)} \qquad \begin{aligned} &\varphi_1(p_1 \times S^1): \left(\frac{\cos(\pm\xi)}{\sqrt{2}}, \frac{\sin(\pm\xi)}{\sqrt{2}}, \frac{\cos(\pm\xi)}{\sqrt{2}}, \frac{\sin(\pm\xi)}{\sqrt{2}}\right), \\ &\varphi_1(S_0^1 \times q_1): \left(\frac{\cos\theta_1}{\sqrt{2}}, \frac{\sin\theta_1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right). \end{aligned}$$

Similarly that of $\varphi_2(S_0^1 \times S^1)$ is obtained as follows:

$$\text{(3.6)} \qquad \begin{aligned} &\varphi_2(p_2 \times S^1): \left(\frac{\cos(\pm\eta)}{\sqrt{2}}, \frac{\sin(\pm\eta)}{\sqrt{2}}, \frac{\cos(\pm\eta)}{\sqrt{2}}, \frac{\sin(\pm\eta)}{\sqrt{2}}\right), \\ &\varphi_2(S_0^1 \times q_2): \left(\frac{1}{\sqrt{2}}, 0, \frac{\cos\theta_2}{\sqrt{2}}, \frac{\sin\theta_2}{\sqrt{2}}\right). \end{aligned}$$

And then we obtain easily

$$\begin{aligned} \varphi_1(S_0^1 \times q_1) - \varphi_2(S_0^1 \times q_2) &\sim \pm\varphi_1(p_1 \times S^1) \text{ in } D \\ &\sim \pm\varphi_2(p_2 \times S^1) \text{ in } D. \end{aligned}$$

These are the conclusions of Theorem 3.

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