ON SUBPROJECTIVE SPACES II

TYÛZI ADATI

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§0. Introduction.

In the previous paper [1], we proved that, if the Christoffel symbols of the second kind in a Riemannian space V_n take the form, for a suitable coordinate system,

$$(0.1)\qquad \left\{ {\lambda \atop \mu\nu} \right\} = \varphi_\mu \delta_\nu^\lambda + \varphi_\nu \delta_\mu^\lambda + \varphi_{\mu\nu}\xi^\lambda,$$

where

$$(0.2)\qquad \xi^\lambda_{;\mu} = \alpha\delta^\lambda_\mu + \beta_\mu \xi^\lambda,$$

V_n is a subprojective space, and that the subprojective space is a conformally flat space admitting a concircular transformation.

In this paper, we shall prove some properties of the subprojective Riemannian space and study problems related to Rachevsky's condition (B).

§1. Riemannian space admitting $\left\{ {\lambda \atop \mu\nu} \right\} = \varphi_{\mu\nu}\xi^\lambda$.

In this section, we shall treat of the case when (0.1) becomes

$$(1.1)\qquad \left\{ {\lambda \atop \mu\nu} \right\} = \varphi_{\mu\nu}\xi^\lambda,$$

where ξ^λ is a torse-forming vector.

If V_n is a subprojective space, the next three conditions are satisfied [3], that is,

$$(1.2)\qquad \begin{array}{ll} \text{(A)} & R^\lambda_{\cdot\mu\nu\omega} = T^\lambda_{\cdot\omega} g_{\mu\nu} - T^\lambda_{\cdot\nu} g_{\mu\omega} + \delta^\lambda_\omega T_{\mu\nu} - \delta^\lambda_\nu T_{\mu\omega}, \\ \text{(A')} & T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu} = 0, \\ \text{(B)} & T_{\lambda\mu} = \rho g_{\lambda\mu} + \rho_\lambda \sigma_\mu, \end{array}$$

where

$$T_{\lambda\mu} = \frac{1}{n-2}\left(R_{\lambda\mu} - \frac{R}{2(n-1)} g_{\lambda\mu}\right),$$

and

$$\rho_\mu = \frac{\partial \rho}{\partial x^\mu}, \quad \sigma_\mu = \frac{\partial \sigma}{\partial x^\mu}, \quad \sigma = \sigma(\rho).$$

Putting

$$\xi^\lambda = \alpha\sigma^\lambda,$$

we have (0.2) and

$$(1.3)\qquad T_{\lambda\mu} = \rho g_{\lambda\mu} + u\xi_\lambda \xi_\mu,$$

$$(1.4)\qquad \rho_\mu = \alpha u \xi_\mu, \quad u_\mu + 2u\beta_\mu = q\xi_\mu.$$

Moreover, we have

(1.5) $$\alpha\beta_\mu - \alpha_\mu = (2\rho + u\xi^\sigma\xi_\sigma)\xi_\mu\,,$$

because of Ricci identities

$$\xi_{\lambda;\mu\nu} - \xi_{\lambda;\nu\mu} = -\xi_\sigma R^{\sigma}_{\cdot\lambda\mu\nu}\,.$$

Let us consider now differential equations

(1.6) $$z_{\lambda;\mu} = -\varphi_{\lambda\mu} z_\sigma \xi^\sigma,$$

where $\varphi_{\lambda\mu}$ is a symmetric tensor. Substituting (1.6) in Ricci identities

$$z_{\lambda;\mu\nu} - z_{\lambda;\nu\mu} = -z_\sigma R^{\sigma}_{\cdot\lambda\mu\nu},$$

we obtain

(1.7) $$z_\sigma R^{\sigma}_{\cdot\lambda\mu\nu} = z_\sigma\{\alpha(\varphi_{\lambda\mu}\delta^\sigma_\nu - \varphi_{\lambda\nu}\delta^\sigma_\mu) + 2U_{\lambda\mu\nu}\xi^\sigma\},$$

where

(1.8) $$2U_{\lambda\mu\nu} = \varphi_{\lambda\mu;\nu} - \varphi_{\lambda\nu;\mu} - \xi^\omega(\varphi_{\lambda\mu}\varphi_{\omega\nu} - \varphi_{\lambda\nu}\varphi_{\omega\mu}) + \varphi_{\lambda\mu}\beta_\nu - \varphi_{\lambda\nu}\beta_\mu.$$

Let us assume that $\varphi_{\lambda\mu}$ satisfies equation

(1.9) $$\alpha\varphi_{\lambda\mu} = 2\rho g_{\lambda\mu} + u\xi_\lambda\xi_\mu = T_{\lambda\mu} + \rho g_{\lambda\mu},$$

from which we have by covariant differentiation

$$\alpha_\nu\varphi_{\lambda\mu} + \alpha\varphi_{\lambda\mu;\nu} = T_{\lambda\mu;\nu} + \rho_\nu g_{\lambda\mu}.$$

Interchanging μ and ν and subtracting the resulting equation from the above, we have

$$(\alpha_\nu\varphi_{\lambda\mu} - \alpha_\mu\varphi_{\lambda\nu}) + \alpha(\varphi_{\lambda\mu;\nu} - \varphi_{\lambda\nu;\mu}) = (T_{\lambda\mu;\nu} - T_{\lambda\nu;\mu}) + (\rho_\nu g_{\lambda\mu} - \rho_\mu g_{\lambda\nu}).$$

Substituting (1.2) (A′) and (1.4), we obtain

(1.10) $$(\alpha_\nu\varphi_{\lambda\nu} - \alpha_\mu\varphi_{\lambda\mu}) + \alpha(\varphi_{\lambda\mu;\nu} - \varphi_{\lambda\nu;\mu}) = \alpha u(\xi_\nu g_{\lambda\mu} - \xi_\mu g_{\lambda\nu}).$$

From (1.5) and (1.9), we have

$$\alpha\beta_\mu - \alpha_\mu = \alpha\xi^\sigma\varphi_{\sigma\mu},$$

from which follows $\alpha_\mu = \alpha(\beta_\mu - \xi^\sigma\varphi_{\sigma\mu})$. Substituting in (1.10), we obtain, because of (1.8),

(1.11) $$2U_{\lambda\mu\nu} = u(\xi_\nu g_{\lambda\mu} - \xi_\mu g_{\lambda\nu}).$$

Making use of (1.2) (A), (1.9) and (1.11), we can find that (1.7) is satisfied identically and consequently (1.6) is completely integrable. Accordingly, if we represent n linearly independent solutions by

$$z^\alpha_\lambda \quad (\alpha = 1, 2, \ldots, n),$$

there exist n independent functions

(1.12) $$x^\alpha = x^\alpha(x^\lambda),$$

such that $z^\alpha_\lambda = \dfrac{\partial x^\alpha}{\partial x^\lambda}$. Considering (1.12) as a transformation of coordinates, we can easily conclude that the Christoffel symbols of the second kind may be transformed to the form

$$\left\{ {\alpha \atop \beta\gamma} \right\} = \bar{\varphi}_{\beta\gamma}\xi^\alpha.$$

Now we have from (0.2) and (1.6)

$$\left(\frac{1}{\alpha} z_\sigma \xi^\sigma\right)_{;\mu} = \frac{1}{\alpha^2} z_\sigma \xi^\sigma(\alpha\beta_\mu - \alpha_\mu - \alpha\varphi_{\omega\mu}\xi^\omega) + z_\mu = z_\mu,$$

from which follows

$$\overset{\alpha}{x} = \frac{1}{\alpha} z^\alpha_\sigma \xi^\sigma + d^\alpha,$$

where d^α are constants. Therefore we have

$$\xi^\alpha = z^\alpha_\sigma \xi^\sigma = \alpha(\bar{x}^\alpha - d^\alpha).$$

Hence we find the

THEOREM. *The Christoffel symbols of the second kind of a subprojective space can be reducible to the form, by a suitable transformation of coordinates,*

$$\left\{ {\lambda \atop \mu\nu} \right\} = \varphi_{\mu\nu}\xi^\lambda,$$

where ξ^λ is a concircular vector and $\varphi_{\mu\nu}$ a symmetric tensor. In this coordinate system, ξ^λ takes the form

$$\xi^\lambda = \alpha(x^\lambda - d^\lambda),$$

where α is a function of the x's and d^λ are constants.

Now if we put $u_{\lambda\mu} = \alpha\varphi_{\lambda\mu}$, (1.6) becomes

$$z_{\lambda;\mu} = - u_{\lambda\mu} z_\omega \sigma^\omega,$$

whose n independent solutions $\overset{*}{z}{}^\alpha_\lambda$ are equal to $\dfrac{\partial \overset{*}{x}{}^\alpha}{\partial x^\lambda}$, $\overset{*}{x}{}^\alpha$ being the canonical coordinate system. Since $\left\{ {\overset{*}{\lambda} \atop \mu\nu} \right\} = \overset{*}{u}_{\mu\nu} \overset{*}{x}{}^\lambda$ in the canonical system, we can easily obtain the above theorem.

§ 2. Fundamental quadratic differential form of subprojective space.

In the first place, we consider the fundamental quadratic differential form of a space which has constant Riemannian curvature. This fundamental form may be written in the form [2]

$$ds^2 = \sum_{i=1}^{n} \frac{(dx^i)^2}{U^2}, \tag{2.1}$$

where

$$U = \sum_{i=1}^{n} X_i, \qquad X_i = a(x^i)^2 + 2b_i x^i + c_i$$

and a, b_i and c_i are arbitrary constants satisfying the following condition

$$K = 4\left(a\sum c_i - \sum b_i^2\right), \quad K = \frac{R}{n(n-1)}.$$

Putting $b_i = 0$, we have

$$X_i = a(x^i)^2 + c_i, \qquad K = 4a\sum c_i,$$

from which follows

$$U = a\sum (x^i)^2 + \sum c_i = 4a\left\{\frac{1}{4}\sum (x^i)^2 + \frac{K}{16a^2}\right\}.$$

If we put $K = \pm 16a^2$, we have

$$U = \sqrt{\pm K}\left\{\frac{1}{4}\sum (x^i)^2 \pm 1\right\},$$

from which follows

$$ds^2 = \frac{(dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2}{\pm K\left\{\frac{1}{4}\sum_{i=1}^{n}(x^i)^2 \pm 1\right\}^2} \qquad (K \neq 0), \tag{2.2}$$

where the symbol $\pm$ takes $+$ or $-$ according as the scalar curvature is positive or negative.

Now the fundamental quadratic differential form of a subprojective space V_n is represented by the equation [4]

$$ds^2 = f^2(x^n) f_{ij}(x^k) dx^i dx^j + (dx^n)^2 \qquad (i, j, k = 1, 2, \dots, n-1), \tag{2.3}$$

for a suitable coordinate system. In this case, since the hypersurfaces $x^n =$ const. are of constant curvature, by virtue of (2.2), (2.3) must be reducible to the form, by a suitable transformation of coordinates,

$$ds^2 = \frac{(dx^1)^2 + (dx^2)^2 + \dots + (dx^{n-1})^2}{\pm K(x^n)\left\{\frac{1}{4}\sum_{i=1}^{n-1}(x^i)^2 \pm 1\right\}^2} + (dx^n)^2, \tag{2.4}$$

where $K(x^n) = \dfrac{R(x^n)}{(n-1)(n-2)} \neq 0$[1], $R(x^n)$ being scalar curvatures of the hypersurfaces.

In fact, from (2.4) the Christoffel symbols of the hypersurfaces are given by

$$\left\{ {i \atop ii} \right\} = -\frac{1}{2V}x^i, \qquad \left\{ {i \atop jj} \right\} = \frac{1}{2V}x^i, \qquad (i, j, k \neq),$$
$$\left\{ {j \atop ji} \right\} = -\frac{1}{2V}x^i, \qquad \left\{ {i \atop jk} \right\} = 0$$

where $V = \dfrac{1}{4}\sum_{i=1}^{n-1}(x^i)^2 \pm 1$. If we represent curvature tensors, Ricci tensors and scalar curvatures of the hypersurfaces by $R^i_{\cdot jkl}$, R_{jk} and R respectively, we have

$$R^i_{\cdot jji} = \pm\frac{1}{V^2} \ (i \neq j). \qquad R_{jj} = \pm\frac{n-2}{V^2},$$

1) The case when $K =$ const. will be treated in the later paper.

where i is not summed. Thus we have readily $R=(n-1)(n-2)K$. Furthermore, since $\bar{R}^{i}_{\cdot jji}=K\bar{g}_{jj}$ and all other components of the curvature tensors are equal to zero, we find

$$\bar{R}^{i}_{\cdot jkl}=K(\bar{g}_{jk}\delta^{i}_{l}-\bar{g}_{jl}\delta^{i}_{k}).$$

Especially, when the space admits a concurrent vector field [5], we have

$$K(x^{n})=\pm\frac{k}{(x^{n})^{2}},$$

where k is a positive constant.

§3. Totally umbilical hypersurface in a conformally flat space.

A subprojective space is conformally flat and admits a family of ∞^{1} totally umbilical hypersurfaces. In this section, we shall consider the case that there exists a totally umbilical hypersurface in a conformally flat space C_n.

Let us define the totally umbilical hypersurface V_{n-1} by the equations

$$x^{\lambda}=x^{\lambda}(x^{i})\qquad(\lambda,\mu,\ldots=1,2,\ldots n;\ i,j,\ldots=\dot{1},\dot{2},\ldots,\dot{n}-\dot{1}).$$

If $g_{\lambda\mu}$ and g_{ij} are the fundamental tensors of C_n and V_{n-1} respectively, the Euler-Schouten's curvature tensor of V_{n-1} with respect to C_n takes the form

$$H_{ij}^{\cdot\cdot\lambda}=g_{ij}H^{\lambda}.$$

Consequently, if we represent the curvature tensors of C_n and V_{n-1} by $R^{\lambda}_{\cdot\mu\nu\omega}$ and $R^{i}_{\cdot jkh}$ respectively, the Gauss equations become

$$R^{i}_{\cdot jkh}=B^{i\mu\nu\omega}_{\lambda jkh}R^{\lambda}_{\cdot\mu\nu\omega}+H^{\lambda}H_{\lambda}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k}),\tag{3.1}$$

where

$$B^{i\mu\nu\omega}_{\lambda jkh}=B^{i}_{\cdot\lambda}B_{j}^{\cdot\mu}B_{k}^{\cdot\nu}B_{h}^{\cdot\omega},\qquad B_{i}^{\cdot\lambda}=\frac{\partial x^{\lambda}}{\partial x^{i}},\qquad B^{i}_{\cdot\lambda}=g^{ij}g_{\lambda\mu}B^{\cdot\mu}.$$

Since

$$R^{\lambda}_{\cdot\mu\nu\omega}=T^{\lambda}_{\cdot\omega}g_{\mu\nu}-T^{\lambda}_{\cdot\nu}g_{\mu\omega}+T_{\mu\nu}\delta^{\lambda}_{\omega}-T_{\mu\omega}\delta^{\lambda}_{\nu},$$

where

$$T_{\mu\nu}=\frac{1}{n-2}\left(R_{\mu\nu}-\frac{R}{2(n-1)}g_{\mu\nu}\right).$$

R being the scalar curvature of C_n, (3.1) may be reducible to

$$\begin{aligned}R^{i}_{\cdot jkh}=&\frac{1}{n-2}(B^{i\omega}_{\lambda h}R^{\lambda}_{\cdot\omega}g_{jk}-B^{i\nu}_{\lambda k}R^{\lambda}_{\cdot\nu}g_{jh}+B^{\mu\nu}_{jk}R_{\mu\nu}\delta^{i}_{h}-B^{\mu\omega}_{jh}R_{\mu\omega}\delta^{i}_{k})\\&+(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})\left(H^{\lambda}H_{\lambda}-\frac{R}{(n-1)(n-2)}\right),\end{aligned}\tag{3.2}$$

where $B^{i\omega}_{\lambda h}=B^{i}_{\cdot\lambda}B_{h}^{\cdot\omega}$ and $B^{\mu\nu}_{jk}=B_{j}^{\cdot\mu}B_{k}^{\cdot\nu}$. Contracting for i and h, we have

$$R_{jk}=\frac{n-3}{n-2}B^{\mu\nu}_{jk}R_{\mu\nu}+\left\{\frac{R}{(n-1)(n-2)}-\frac{1}{n-2}B^{\lambda}B^{\omega}R_{\lambda\omega}+(n-2)H^{\lambda}H_{\lambda}\right\}g_{jk}\tag{3.3}$$

where B^{λ} is a normal vector of V_{n-1}.

Now let us assume that tangential directions of V_{n-1} are Ricci directions. Then we have equations of the form

$$R_{\lambda\mu}B_i^{\cdot\lambda} = ag_{\lambda\mu}B_i^{\cdot\lambda}, \tag{3.4}$$

from which we have

$$B_{\lambda h}^{i\omega}R_{\cdot\omega}^{\lambda} = a\delta_h^i, \qquad B_{jk}^{\mu\nu}R_{\mu\nu} = ag_{jk}.$$

Therefore (3.2) is reducible to

$$R_{\cdot jkh}^{i} = \left(\frac{2a}{n-2} - \frac{R}{(n-1)(n-2)} + H^\lambda H_\lambda\right)(g_{jk}\delta_h^i - g_{jh}\delta_k^i),$$

and consequently V_{n-1} has constant Riemannian curvature.

Moreover, since all tangential directions of V_{n-1} are Ricci directions, we have from (3.4)

$$R_{\lambda\mu} = ag_{\lambda\mu} + bB_\lambda B_\mu, \tag{3.5}$$

where b is a certain scalar. Thus we find that the normals of V_{n-1} are also Ricci directions and consequently V_{n-1} has constant mean curvature.

Conversely, if a totally umbilical hypersurface V_{n-1} in C_n has constant Riemannian curvature and mean curvature, from (3.3) we have

$$B_{jk}^{\mu\nu}R_{\mu\nu} = ag_{jk},$$

that is,

$$(R_{\mu\nu} - ag_{\mu\nu})B_{jk}^{\mu\nu} = 0.$$

Thus we have equations of the form

$$R_{\mu\nu} = ag_{\mu\nu} + v_\mu B_\nu + v_\nu B_\mu,$$

where v_μ is a certain vector. However, since the normals of V_{n-1} are Ricci directions, $R_{\mu\nu}$ takes the form (3.5). Thus we have the

THEOREM. *In a conformal flat space C_n $(n > 3)$, in order that tangential directions of a totally umbilical hypersurface are all Ricci directions, it is necessary and sufficient that the hypersurface is of constant Riemannian curvature and mean curvature.*

§ 4. $\Pi_{\lambda\mu} = \rho g_{\lambda\mu} + \kappa\eta_\lambda\eta_\mu$ and concircular geometry.

In this section and the next, we shall treat of the problems connected with Rachevsky's condition (B). Using $\Pi_{\lambda\mu}$ in place of $-T_{\lambda\mu}$, we put

$$\Pi_{\lambda\mu} = -\frac{1}{n-2}\left(R_{\lambda\mu} - \frac{R}{2(n-1)}g_{\lambda\mu}\right). \tag{4.1}$$

We consider now a family of hypersurfaces

$$\eta(x^\lambda) = \text{const.}$$

in a Riemannian space V_n and assume that $\Pi_{\lambda\mu}$ takes the form

$$\Pi_{\lambda\mu} = \rho g_{\lambda\mu} + \kappa\eta_\lambda\eta_\mu, \tag{4.2}$$

where ρ and κ are any scalar functions of the x's. From (4.1) and (4.2) we have

$$R_{\lambda\mu} = \left\{ \frac{R}{2(n-1)} - (n-2)\rho \right\} g_{\lambda\mu} - (n-2)\kappa\eta_\lambda\eta_\mu.$$

Therefore any vector v^λ, which is orthogonal to η^λ, is the Ricci direction.

Conversely, if $R_{\lambda\mu}v^\lambda = av_\mu$ for any vector v^λ satisfying $v^\lambda\eta_\lambda = 0$, we have

$$(R_{\lambda\mu} - ag_{\lambda\mu})v^\lambda = 0,$$

from which we obtain equations of the form

$$R_{\lambda\mu} = ag_{\lambda\mu} + b\eta_\lambda\eta_\mu.$$

Thus $\Pi_{\lambda\mu}$ takes the form (4.2) and consequently follows the

THEOREM 4. 1. *In order that tangential directions of the hypersurfaces $\eta = const.$ in a V_n are Ricci directions, it is necessary and sufficient that $\Pi_{\lambda\mu}$ defined by* (4.1) *takes the form* (4.2).

In the subprojective space, we notice that ρ and κ are functions of η.

Let us assume now that η^λ is a concircular vector. Then the fundamental quadratic differential form of V_n may be written in the form

$$ds^2 = f^2(x^n)f_{jk}(x^i)dx^j dx^k + (dx^n)^2$$

for a suitable coordinate system. In this case the above-mentioned hypersurfaces are defined by

$$x^n = \text{const.},$$

which are totally umbilical. If we represent Ricci tensors and scalar curvatures of the hypersurfaces by R_{ij} and R respectively, we can derive the next relations [1]

$$(4.3)\qquad \begin{cases} R_{ij} = \bar{R}_{ij} - \dfrac{1}{f^2}\{(n-2)f'^2 + ff''\}g_{ij}, \\ R_{nn} = -(n-1)\dfrac{f''}{f}, \\ R_{in} = 0, \end{cases}$$

$$(4.4)\qquad R = R - (n-1)\{(n-2)f'^2 + 2ff''\}\frac{1}{f^2}$$

and

$$(4.5)\qquad \begin{cases} \Pi_{ij} = -\dfrac{1}{n-2}\left(R_{ij} - \dfrac{R}{2(n-1)}g_{ij}\right) + \dfrac{1}{2}\dfrac{f'^2}{f^2}g_{ij}, \\ \Pi_{nn} = \dfrac{R}{2(n-1)(n-2)} + \dfrac{1}{2}\dfrac{1}{f^2}(2ff'' - f'^2), \\ \Pi_{in} = 0. \end{cases}$$

Since

$$(4.6)\qquad \bar{R} = \frac{1}{f^2} f^{jk}\bar{R}_{jk},$$

where $f^{jk}f_{jh} = \delta_h^k$ and R_{jk} are functions of x^i alone, we have

$$\frac{\partial R}{\partial x^n} = -\frac{2f'}{f}R. \tag{4.7}$$

Now, because of

$$\eta^\lambda = \frac{\partial x^\lambda}{\partial x^n} = \delta^\lambda_n, \qquad \eta_\lambda = g_{\lambda\mu}\eta^\mu = \delta^n_\lambda,$$

(4.1) reduces to

$$\Pi_{\lambda\mu} = \rho g_{\lambda\mu} + \kappa\delta^n_\lambda\delta^n_\mu, \tag{4.8}$$

from which we have

$$\Pi_{ij} = \rho g_{ij}, \qquad \Pi_{nn} = \rho + \kappa, \qquad \Pi_{in} = 0. \tag{4.9}$$

Comparing (4.9) with (4.5), we find that R_{ij} are proportional to g_{ij} and, when $n > 3$, from (4.6) we have

$$f^{jk}\bar{R}_{jk} = c = \text{const.}, \qquad R = \frac{c}{f^2}.$$

Thus we have the

THEOREM 4.2 [1]. *In order that a tensor $\Pi_{\lambda\mu}$ of a space admitting a concircular vector field η_λ satisfies a equation of the form (4.2), it is necessary and sufficient that the hypersurfaces $\eta =$ const. are all Einstein spaces.*

If we put

$$R_{ij} = \frac{R}{n-1} g_{ij},$$

from (4.5) we have

$$\Pi_{ij} = \Big(-\frac{R}{2(n-1)(n-2)} + \frac{1}{2}\frac{f'^2}{f^2}\Big) g_{ij},$$

$$\Pi_{nn} = \Big(-\frac{R}{2(n-1)(n-2)} + \frac{1}{2}\frac{f'^2}{f^2}\Big) + \frac{R}{(n-1)(n-2)} - \frac{f'^2}{f^2} + \frac{f''}{f}.$$

Comparing with (4.9), we obtain

$$\begin{aligned} \rho &= -\frac{R}{2(n-1)(n-2)} + \frac{f'^2}{2f^2} = -\frac{R}{2(n-1)(n-2)} - \frac{1}{n-2}\frac{f''}{f}, \\ \kappa &= \frac{R}{(n-1)(n-2)} - \frac{f'^2}{f^2} + \frac{f''}{f} = \frac{R}{(n-1)(n-2)} + \frac{d}{dx^n}\frac{f'}{f} \\ &= \frac{R}{(n-1)(n-2)} + \frac{n}{n-2}\frac{f''}{f}. \end{aligned} \tag{4.10}$$

Thus we have the

THEOREM 4.3. *If a tensor $\Pi_{\lambda\mu}$ of a space admitting a concircular vector field η_λ, where $\eta_\lambda = \frac{\partial\eta}{\partial x^\lambda}$, satisfies a equation of the form (4.2), then ρ, κ and R are functions of η alone ($n > 3$).*

From (4.10) we have

$$\rho_n \equiv \frac{\partial \rho}{\partial x^n} = -\frac{1}{2(n-1)(n-2)}\frac{\partial R}{\partial x^n} + \frac{f'}{f}\frac{d}{dx^n}\frac{f'}{f}.$$

Substituting (4.7), we have by virtue of (4.10)

$$\rho_n = \frac{f'}{f}\kappa,$$

from which follows

$$\rho_\mu = \frac{f'}{f}\kappa\delta^n_\mu \quad (n>3). \tag{4.11}$$

However because of

$$\eta_{\lambda;\mu} = \delta^n_{\lambda;\mu} = -\left\{{n \atop \lambda\mu}\right\} = \frac{f'}{f}(g_{\lambda\mu} - \delta^n_\lambda\delta^n_\mu), \tag{4.12}$$

we have from (4.8)

$$\Pi_{\lambda\mu;\nu} - \Pi_{\lambda\nu;\mu} = \left(\rho_\nu - \frac{f'}{f}\kappa\delta^n_\nu\right)g_{\lambda\mu} - \left(\rho_\mu - \frac{f'}{f}\kappa\delta^n_\mu\right)g_{\lambda\nu} + \kappa_\nu\delta^n_\lambda\delta^n_\mu - \kappa_\mu\delta^n_\lambda\delta^n_\nu$$
$$= \kappa_\nu\delta^n_\lambda\delta^n_\mu - \kappa_\mu\delta^n_\lambda\delta^n_\nu.$$

Since κ is a function of x^n, we obtain

$$\Pi_{\lambda\mu;\nu} - \Pi_{\lambda\nu;\mu} = 0. \tag{4.13}$$

Conversely, in the previous paper [1] we proved that, when the above equation holds, the hypersurfaces $x^n = \text{const.}$ are Einstein Spaces. Thus we have the

THEOREM 4.4 [4]. *In order that a tensor $\Pi_{\lambda\mu}$ of a space admitting a concircular vector field η_λ satisfies a equation* (4.2), *it is necessary and sufficient that*

$$\Pi_{\lambda\mu;\nu} - \Pi_{\lambda\nu;\mu} = 0 \quad (n>3).$$

If $\Pi_{\lambda\mu}$ satisfies (4.2) and η_λ is a concircular vector satisfying

$$\eta_{\lambda;\mu} = \alpha g_{\lambda\mu} + \beta\eta_\lambda\eta_\mu,$$

we have a relation, by virtue of (4.11) and (4.12),

$$\rho_\mu = \alpha\kappa\eta_\mu \quad (n>3). \tag{4.14}$$

Especially when $n=3$, if ρ (or κ) is a function of η, then R and κ (or ρ) also are functions of η and consequently (4.14) and (4.13) hold. Therefore in a three dimensional space V_3, if a gradient vector η_λ is a concircular vector and a tensor $\Pi_{\lambda\mu}$ satisfies (4.2), where ρ or κ is a function of η, then V_3 is a subprojective space.

Finally, we assume that $\Pi_{\lambda\mu}$ satisfies (4.2) and (4.13), and that ρ and κ are functions of η alone. Then

$$\Pi_{\lambda\mu;\nu} - \Pi_{\lambda\nu;\mu} = (\rho_\nu g_{\lambda\mu} - \rho_\mu g_{\lambda\nu}) + \kappa(\eta_{\lambda;\nu}\eta_\mu - \eta_{\lambda;\mu}\eta_\nu) = 0.$$

Multiplying by η^μ and summing for μ, we have

$$\rho_\nu\eta_\lambda - \eta^\mu\rho_\mu g_{\lambda\nu} + \kappa(\eta^\mu\eta_\mu\eta_{\lambda;\nu} - \eta^\mu\eta_{\lambda;\mu}\eta_\nu) = 0,$$

from which we have relations of the form

$$\eta_{\lambda;\nu} = \alpha g_{\lambda\nu} + \beta \eta_\lambda \eta_\nu .$$

Since we have from it

$$(\eta^\lambda \eta_\lambda)_{;\nu} = 2\eta^\lambda \eta_{\lambda;\nu} = 2(\alpha + \beta\eta^\lambda\eta_\lambda)\eta_\nu ,$$

$\eta^\lambda\eta_\lambda$, $\alpha + \beta\eta^\lambda\eta_\lambda$ and $\eta^\mu\rho_\mu$ are functions of η alone. Therefore α and β are also functions of η alone and consequently η_λ is a concircular vector. Hence we have the

THEOREM 4. 5. *If* $\Pi_{\lambda\mu} = \rho(\eta)g_{\lambda\mu} + \kappa(\eta)\eta_\lambda\eta_\mu$ *and* $\Pi_{\lambda\mu;\nu} - \Pi_{\lambda\nu;\mu} = 0$, *then* η_λ *is a concircular vector field.*

§ 5. Conformal transformation of $\Pi_{\lambda\mu} = \rho g_{\lambda\mu} + \kappa\eta_\lambda\eta_\mu$.

We shall seek a conformal transformation such that the form of the equation (4. 2) remains invariant. In the first place, we treat of the case when ρ and κ are functions of η, that is to say,

(5. 1) $$\Pi_{\lambda\mu} = \rho(\eta)g_{\lambda\mu} + \kappa(\eta)\eta_\lambda\eta_\mu .$$

Multiplying (5. 1) by $g^{\lambda\mu}$ and contracting for λ and μ, we have

(5. 2) $$-\frac{R}{2(n-1)} = n\rho + \kappa g^{\lambda\mu}\eta_\lambda\eta_\mu .$$

Differentiating with respect to x^μ, we have

(5. 3) $$-\frac{R_\mu}{2(n-1)} = n\rho_\mu + \kappa_\mu\eta^\lambda\eta_\lambda + \kappa(\eta^\lambda\eta_\lambda)_{;\mu} ,$$

where $R_\mu = \dfrac{\partial R}{\partial x^\mu}$ and $(\eta^\lambda\eta_\lambda)_{;\mu} = \dfrac{\partial}{\partial x^\mu}(\eta^\lambda\eta_\lambda)$.

On the other hand, from (5. 1) we have

(5. 4) $$\Pi^{\lambda}_{\cdot\mu} = \rho\delta^\lambda_\mu + \kappa\eta^\lambda\eta_\mu .$$

Because of $\Pi^{\lambda}_{\cdot\mu;\lambda} = -\dfrac{1}{2(n-1)}R_\mu$, from (5. 4) we have

(5. 5) $$-\frac{1}{2(n-1)}R_\mu = \rho_\mu + (\kappa_\lambda\eta^\lambda + \kappa\eta^\lambda_{;\lambda})\,\eta_\mu + \frac{\kappa}{2}(\eta^\lambda\eta_\lambda)_{;\mu} .$$

Comparing (5. 5) with (5. 3), we find that $\eta^\lambda\eta_\lambda$ and R are functions of η. Thus we have the

THEOREM 5. 1. *If a tensor* $\Pi_{\lambda\mu}$ *of a space satisfies*

$$\Pi_{\lambda\mu} = \rho(\eta)g_{\lambda\mu} + \kappa(\eta)\eta_\lambda\eta_\mu ,$$

where $\eta_\lambda = \dfrac{\partial\eta}{\partial x^\lambda}$, *then* $\eta^\lambda\eta_\lambda$ *and* R *are functions of* η *alone.*

Let us consider now a conformal transformation

(5. 6) $$\bar{g}_{\mu\nu} = \sigma^2 g_{\mu\nu} .$$

If $\Pi_{\lambda\mu}$ is transformed by (5. 6) to $\bar{\Pi}_{\lambda\mu}$, we have

$$\bar{\Pi}_{\lambda\mu} = \Pi_{\lambda\mu} + \sigma_{\lambda;\mu} - \sigma_\lambda\sigma_\mu + \frac{1}{2}g^{\alpha\beta}\sigma_\alpha\sigma_\beta g_{\lambda\mu} ,$$

where $\sigma_\lambda = \dfrac{\partial \log \sigma}{\partial x^\lambda}$. Consequently, when $\bar{\Pi}_{\lambda\mu}$ satisfies (5.1), we have

$$\bar{\Pi}_{\lambda\mu} = \rho(\eta)g_{\lambda\mu} + \kappa(\eta)\eta_\lambda\eta_\mu + \sigma_{\lambda\mu}\,,$$

where

$$\sigma_{\lambda\mu} = \sigma_{\lambda;\mu} - \sigma_\lambda\sigma_\mu + \frac{1}{2}\, g^{\alpha\beta}\sigma_\alpha\sigma_\beta g_{\lambda\mu}\,.$$

Let us assume that

$$\bar{\Pi}_{\lambda\mu} = \bar{\rho}(\eta)\bar{g}_{\lambda\mu} + \bar{\kappa}(\eta)\eta_\lambda\eta_\mu\,,$$

where $\bar{\rho}$ and $\bar{\kappa}$ are functions of η. Then we have

(5.7) $$\sigma_{\lambda\mu} = (\bar{\rho}\sigma^2 - \rho)g_{\lambda\mu} + (\bar{\kappa} - \kappa)\eta_\lambda\eta_\mu.$$

However, according to the Theorem 5.1, we know that $g^{\lambda\mu}\eta_\lambda\eta_\mu$ is a function of η alone and consequently σ also a function of η alone, because of $\bar{g}^{\lambda\mu}\eta_\lambda\eta_\mu = \sigma^{-2}g^{\lambda\mu}\eta_\lambda\eta_\mu$.

Therefore from (5.7) we have equations of the form

$$\eta_{\lambda;\mu} = \alpha g_{\lambda\mu} + \beta\eta_\lambda\eta_\mu,$$

where α and β are functions of η. Thus we have the

THEOREM 5.2. *In order that the form of equations*

$$\Pi_{\lambda\mu} = \rho(\eta)g_{\lambda\mu} + \kappa(\eta)\eta_\lambda\eta_\mu,$$

where $\eta_\lambda = \dfrac{\partial\eta}{\partial x^\lambda}$, *remains invariant by a conformal transformation* $\bar{g}_{\mu\nu} = \sigma^2 g_{\mu\nu}$, *it is necessary and sufficient that* η^λ *is a concircular vector field and* σ *is a function of* η *alone.*

THEOREM 5.3. *In order that a subprojective space admitting a concircular vector field* η_λ, *where* $\eta_\lambda = \dfrac{\partial\eta}{\partial x^\lambda}$, *may be transformed to a subprojective space by a conformal transformation* $\bar{g}_{\mu\nu} = \sigma^2 g_{\mu\nu}$, *it is necessary and sufficient that* σ *is a function of* η.

Finally the case when ρ alone is a function of η, that is, equation

(5.8) $$\Pi_{\lambda\mu} = \rho(\eta)g_{\lambda\mu} + \kappa\eta_\lambda\eta_\mu$$

holds, will be treated. If we put

$$g^{\lambda\mu}\eta_\lambda\eta_\mu = \theta\,,$$

(5.3) and (5.5) become respectively

(5.9) $$-\frac{1}{(2n-1)}R_\mu = n\rho_\mu + \theta\kappa_\mu + \kappa\theta_\mu\,,$$

(5.10) $$-\frac{1}{2(n-1)}R_\mu = \rho_\mu + (\kappa_\lambda\eta^\lambda + \kappa\eta^\lambda_{;\lambda})\eta_\mu + \frac{\kappa}{2}\,\theta_\mu\,,$$

from which we find that R and κ are functions of η and θ. Let us assume that (5.8) reduces to the same form

(5.11) $$\bar{\Pi}_{\lambda\mu} = \bar{\rho}(\eta)\bar{g}_{\lambda\mu} + \bar{\kappa}\eta_\lambda\eta_\mu$$

by the conformal transformation (5.6). Then we find that κ and $\bar{R}$, which is a scalar curvature with respect to $g_{\lambda\mu}$, are functions of η, θ and σ.

From (5.7) we have

$$\sigma_{\lambda;\mu} = \left(\rho\sigma^2 - \rho - \frac{1}{2}\sigma^\nu\sigma_\nu\right)g_{\lambda\mu} + (\bar{\kappa} - \kappa)\eta_\lambda\eta_\mu + \sigma_\lambda\sigma_\mu, \tag{5.12}$$

from which we have

$$(\sigma^\lambda\sigma_\lambda)_{;\mu} = 2\sigma^\lambda\sigma_{\lambda;\mu} = 2\left\{\left(\rho\sigma^2 - \rho + \frac{1}{2}\sigma^\nu\sigma_\nu\right)\sigma_\mu + (\bar{\kappa} - \kappa)\sigma^\nu\eta_\nu\eta_\mu\right\}. \tag{5.13}$$

Therefore $\sigma^\lambda\sigma_\lambda$ is a function of σ and η.

In the first place, let us assume that σ is a function of η. Then from (5.13) $\sigma^\lambda\sigma_\lambda$ is a function of η and consequently we find that η_λ is a concircular vector, because coefficient of $g_{\lambda\mu}$ in (5.12) is a function of η.

Moreover, by virture of $\sigma^\lambda\sigma_\lambda = \left(\frac{a\sigma}{d\eta}\right)^2\theta$, θ is a function of η. Thus we have the

THEOREM 5.4. *When the form of the equation (5.8) remains invariant by a conformal transformation* $\bar{g}_{\lambda\mu} = \sigma(\eta)^2 g_{\lambda\mu}$, η_λ *is a concircular vector field and* κ, R *and* $\eta^\lambda\eta_\lambda$ *are functions of* η *alone.*

In the next place, we consider the case that θ is a function of η. Equations (5.12) may be written in the form

$$\sigma_{\lambda;\mu} = p g_{\lambda\mu} + q\eta_\lambda\eta_\mu + \sigma_\lambda\sigma_\mu, \tag{5.14}$$

where $p = \rho\sigma^2 - \rho - \frac{1}{2}\sigma^\nu\sigma_\nu$, $q = \bar{\kappa} - \kappa$. Accordingly p is a function of σ and η, and q is a function of σ, η and θ.

From (5.14) we have

$$\begin{aligned}\sigma_{\lambda;\mu\nu} = {} & p_\nu g_{\lambda\mu} + q_\nu\eta_\lambda\eta_\mu + q(\eta_{\lambda;\nu}\eta_\mu + \eta_\lambda\eta_{\mu;\nu}) \\ & + \sigma_{\lambda;\nu}\sigma_\mu + \sigma_\lambda\sigma_{\mu;\nu}.\end{aligned}$$

Substituting (5.14) and subtracting from it the equation obtained by interchanging μ and ν, we obtain

$$\begin{aligned}\sigma_{\lambda;\mu\nu} - \sigma_{\lambda;\nu\mu} &= -\sigma_\omega R^{\omega}_{\cdot\lambda\mu\nu} \\ &= (p_\nu - p\sigma_\nu)g_{\lambda\mu} - (p_\mu - p\sigma_\mu)g_{\lambda\nu} \\ &\quad + q\eta_\lambda(\sigma_\mu\eta_\nu - \sigma_\nu\eta_\mu) + \eta_\lambda(q_\nu\eta_\mu - q_\mu\eta_\nu) + q(\eta_{\lambda;\nu}\eta_\mu - \eta_{\lambda;\mu}\eta_\nu).\end{aligned}$$

Multiplying by $g^{\lambda\mu}$ and contracting for λ and μ, we have

$$\begin{aligned}-\sigma_\omega R^{\omega}_{\cdot\nu} = {} & (n-1)(p_\nu - p\sigma_\nu) + q(\eta^\lambda\sigma_\lambda\eta_\nu - \eta^\lambda\eta_\lambda\sigma_\nu) \\ & + (\eta^\lambda\eta_\lambda q_\nu - \eta^\lambda q_\lambda\eta_\nu) + q(\eta^\lambda\eta_{\lambda;\nu} - \eta^{\lambda}_{;\lambda}\eta_\nu).\end{aligned} \tag{5.15}$$

However, according to (5.8), the left-hand member of the above equation is a linear combination of σ_ν and η_ν, and in the right-hand member $\eta^\lambda\eta_{\lambda;\nu}$ is equal to $\frac{1}{2}\theta_\nu$. Thus (5.15) reduces to a linear combination of σ_ν, η_ν and θ_ν, that is to say, σ is a function of η and θ.

Consequenty if θ is a function of η, then σ is also a function of η.

Thus we find the

THEOREM 5.5. *When $\eta^\lambda\eta_\lambda$ is a function of η, where $\eta_\lambda = \frac{\partial\eta}{\partial x^\lambda}$, if the form of the equation (5.8) remains invariant by a conformal transformation $\bar{g}_{\lambda\mu} = \sigma^2 g_{\lambda\mu}$, then η^λ is a concircular vector field and κ, R and σ are functions of η alone.*

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MATHEMATICAL INSTITUTE, TOKYO COLLEGE OF SCIENCE.