FIELDS OF ISOCLINE TANGENT PLANES ALONG A CURVE IN A EUCLIDEAN 4-SPACE

YUNG-CHOW WONG

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1. Introduction.

An R-surface in a Euclidean 4-space, R_4, is characterized by the property that its tangent planes are all isocline to one another. As a consequence, any curve on an R-surface admits a field of isocline tangent planes. The Cauchy problem in the determination of R-surfaces requires an answer to the question whether an arbitrary curve in R_4 has this property (§3). In the present note this question is answered in the affirmative (§4) and some properties of the fields of isocline tangent planes along a curve are given (§5). It is further shown (§6) that the planes in a one-parameter family of isocline planes in R_4 either are tangent to a curve or have a common fixed point. The note ends with a complete classification of one-parameter families of planes in R_4 (§7).

2. Preliminary formulas.

Two planes ξ, ξ^* in R_4 are said to be *isocline* to each other if the angle between a vector in ξ and its projection in ξ^* is independent of the position of the vector in ξ. Two planes can be isocline in either one of two senses; and the property of pairs of planes being isocline in one and the same sense is transitive. (See, for example, Manning [7], pp. 114–125, 180–198.)

A *frame* A-I_i $(i, j, k, \ldots. = 1, 2, 3, 4)$ in R_4 consists of 4 orientated mutually orthogonal unit vectors I_i attached to a point A. A nearby or consecutive frame A^*-I_i^* of A-I_i is determined when the infinitesimals ω's in the equations (Cartan [2])

$$dA = A^* - A = \omega_i I_i, \quad dI_i = I_i^* - I_i = \omega_{ij} I_j \tag{2.1}$$

are given. Here repeated indices imply summation, and the ω_{ij} are skew-symmetric in the indices i, j. Neglecting infinitesimals of the second and higher orders, equations (2.1) are equivalent to

$$dA^* = A - A^* = -\omega_i I_i^*, \quad dI_i^* = I_i - I_i^* = -\omega_{ij} I_j^*. \tag{2.1'}$$

Let us now find the condition for the plane $\xi : A$-I_1I_2 of the frame A-I_i and the plane $\xi^* : A^*$-$I_1^*I_2^*$ of the consecutive frame A^*-I_i^* defined by (2.1) to be isocline to each other. Consider in ξ the unit vector

$$I \equiv I_1 \cos\theta + I_2 \sin\theta = (I_1^* + dI_1^*)\cos\theta + (I_2^* + dI_2^*)\sin\theta.$$

Its component orthogonal to ξ^* is, by (2.1'),

$$-\{(\omega_{13}I_3^* + \omega_{14}I_4^*)\cos\theta + (\omega_{23}I_3^* + \omega_{24}I_4^*)\sin\theta\}.$$

Therefore the angle $d\psi$ between I and its projection in ξ^* is given by

$$(\sin d\psi)^2 = (\omega_{13}\cos\theta + \omega_{23}\sin\theta)^2 + (\omega_{14}\cos\theta + \omega_{24}\sin\theta)^2$$
$$= \frac{1}{2}(\omega_{13}^2 + \omega_{14}^2 + \omega_{23}^2 + \omega_{24}^2) + \frac{1}{2}(\omega_{13}^2 + \omega_{14}^2 - \omega_{23}^2 - \omega_{24}^2)\cos 2\theta$$
$$+ (\omega_{13}\omega_{23} + \omega_{14}\omega_{24})\sin 2\theta.$$

Hence $d\psi$ is independent of θ if and only if

$$\omega_{13}^2 + \omega_{14}^2 = \omega_{23}^2 + \omega_{24}^2, \quad \omega_{13}\omega_{23} + \omega_{14}\omega_{24} = 0,$$

which are easily seen to be equivalent to

$$(2.2) \qquad \omega_{24} = e\omega_{13}, \quad \omega_{14} = -e\omega_{23} \qquad (e = \pm 1),$$

that is,

$$(2.2') \qquad I_4 \cdot dI_2 = eI_3 \cdot dI_1, \quad I_4 \cdot dI_1 = -eI_3 \cdot dI_2 \qquad (e = \pm 1).$$

These are the conditions for the consecutive planes A-I_1I_2 and A^*-$I_1^*I_2^*$ to be isocline to each other. The bivaluedness of e confirms our previous statement that two planes may be isocline to each other in one sense or the other.

3. *R*-surface. Cauchy probem (cf. Cartan [3]).

It is not without interest to say a few words about R-surfaces in R_4, of which each of the following properties is characteristic:

PROPERTY 1. The tangent planes of the surface are all isocline to one another.

PROPERTY 2. The surface is given in rectanglular coordinates x, y, u, v by the equations $u = u(x,y)$, $v = v(x,y)$, where $u(x,y)$, $v(x,y)$ are the real and imaginary parts of an analytic function:

$$f(x + \sqrt{-1}\,y) = u(x,y) + \sqrt{-1}\,v(x,y).$$

PROPERTY 3. The curvature ellipse at every point A of the surface is a circle with center at A.

It was Kwietniewski [6] who first started the study of R-surface by proving that Property 1 implies Property 2. Kommerell [5], to whom the name R-surface is due, then showed that Property 2 implies Property 3. The converse that Property 3 implies Property 2 was later established by Eisenhart [4]. Finally, Borůvka [1], using Cartan's method of moving frames, proved that a surface with Property 3 depends on 2 arbitrary functions of 1 variable. For more properties of R-surface, the reader is referred to Wong [9].

By means of (2.2) it is very easy for us to prove directly that *a surface in R_4 with Property 1 depends on 2 arbitrary functions of 1 variable.* In fact, let us attach to each point A of a surface in R_4 a frame A-I_i so that the plane A-I_1I_2 is tangent to the surface at A, then a surface with Property 1 is characterized by the equations

$$(3.1) \qquad \omega_3 = \omega_4 = 0, \quad \omega_{24} = e\omega_{13}, \quad \omega_{14} = -\omega_{23} \quad (e = \pm 1).$$

The ω_i, ω_{ij} are linear differential forms in the two parameters on which the surface depends, and must satisfy the equations of structure for R_4

$$d\omega_i = [\omega_k\omega_{ki}], \quad d\omega_{ij} = [\omega_{ik}\omega_{kj}], \tag{3.2}$$

where a d before a differential form indicates exterior differentiation.

On account of (3.2) and the equations (3.1) themselves, exterior differentiation of the first two equations in (3.1) gives

$$[\omega_1\omega_{13}] + [\omega_2\omega_{23}] = 0, \qquad -[\omega_2\omega_{13}] + [\omega_1\omega_{23}] = 0; \tag{3.3}$$

while that of the last two equations in (3.1) gives two equations which are identically satisfied.

Now the determinant of the polar matrix of (3.3) whose columns correspond to the forms ω_{13}, ω_{23} is

$$\begin{vmatrix} \omega_1 & \omega_2 \\ -\omega_2 & \omega_1 \end{vmatrix} = (\omega_1)^2 + (\omega_2)^2,$$

and is therefore of rank 2. Hence the system of equations (3.1) and (3.3) is in involution, and the surface in question exists and depends on 2 arbitrary functions of 1 variable. The characteristics of the system are the minimal curves $(\omega_1)^2 + (\omega_2)^2 = 0$.

Any one-dimensional solution of (3.1), i. e. a one-parameter family of frames A-I_i satisfying (3.1), such that $(\omega_1)^2 + (\omega_2)^2 \neq 0$, will determine one and only one R-surface. Such a one-dimensional solution is obtained if we can define, along any given curve (A), a field of frames A-I_i so that the plane A-I_1I_2 is tangent to (A) at A and is isocline to its consecutive planes. Therefore, the Cauchy problem in this case is to find out: What curves admit a field of isocline tangent plane? And, for such a curve, how far can this field of planes be determined? Although we can more or less guess the answer to these questions from the fact that the R-surface depends on 2 arbitrary functions of 1 variable, an explicit answer will be given in the next section.

4. Existence theorem.

An isocline field of tangent planes along a curve in R_4 is a family of planes tangent to the curve, one plane at each point, such that any two planes in the family are isocline to each other in one and the same sense. The field is of one *type* or the other according as its planes are isocline to one another in one sense or the other.

We now proceed to prove the following existence theorem.

THEOREM 4. 1. *Given any curve $(A(s))$ in R_4, which is not a straight line, and any plane ξ_0 tangent to the curve at the point $A(s_0)$. Then there exist along the curve two and only two isocline fields $(\xi(s))$ of tangent planes, one of each type, such that $\xi(s_0) = \xi_0$.*

PROOF. The curve $(A(s))$ is described by the point $A(s)$, the parameter being the arc length s. The curvatures κ_1, κ_2, κ_3 and the Frenet frame A-J_i of (A) are connected by the Frenet formulas (Schouten-Struik [8]), which will be written here as

$$(4.1) \qquad d_sA = J_1, \qquad d_sJ_i = \kappa_{ij}d_sJ_j \qquad (d_s = d/ds),$$

where

$$(4.2) \qquad \begin{gathered} \kappa_{12} = -\kappa_{21} = \kappa_1, \qquad \kappa_{23} = -\kappa_{32} = \kappa_2, \qquad \kappa_{34} = -\kappa_{43} = \kappa_3, \\ \text{all other } \kappa_{ij} \text{ are zero.} \end{gathered}$$

Any field of planes tangent to the curve (A), one plane at each point, may be considered as generated by the plane $A\text{-}I_1I_2$ of a one-parameter family of frames $A\text{-}I_i$ defined by

$$(4.3) \qquad I_1 = J_1, \qquad I_p = \alpha_{pq}J_q \qquad (p, q, r = 2, 3, 4),$$

where the α_{pq}, which are functions of s, are the elements of a direct orthogonal matrix so that

$$(4.4) \qquad \alpha_{pq}\alpha_{pr} = \alpha_{qp}\alpha_{rp} = \delta_{qr} = \begin{cases} 1 \text{ if } q = r, \\ 0 \text{ if } q \neq r, \end{cases}$$

$$(4.5) \qquad \{\text{Cofactor of } \alpha_{pq} \text{ in the determinant } (\alpha_{pq})\} = \alpha_{pq},$$

$$(4.6) \qquad \alpha_{2p}d\alpha_{2p} = 0.$$

We shall now show that the α's can be so chosen that the conditions (2.2′) for $A\text{-}I_1I_2$ to generate an isocline field are satisfied. Substituting in (2.2′) the values of I_i from (4.3) and then using (4.1), the result is

$$\begin{aligned} \alpha_{4p}d_s\alpha_{2p} + \alpha_{2p}\alpha_{4q}\kappa_{pq} &= e\alpha_{3q}\kappa_{1q}, \\ \alpha_{3p}d_s\alpha_{2p} + \alpha_{2p}\alpha_{3q}\kappa_{pq} &= -e\alpha_{4q}\kappa_{1q}, \end{aligned}$$

which, on account of (4.2) and (4.5), can be written as

$$(4.7) \qquad \begin{aligned} \alpha_{4p}d_s\alpha_{2p} &= e\alpha_{32}\kappa_{12} + \alpha_{34}\kappa_{23} + \alpha_{32}\kappa_{34}, \\ \alpha_{3p}d_s\alpha_{2p} &= -e\alpha_{42}\kappa_{12} - \alpha_{44}\kappa_{23} - \alpha_{42}\kappa_{34}. \end{aligned}$$

Now if we multiply the three equations in (4.7) and (4.6) by α_{4q}, α_{3q}, α_{2q}, respectively, and add, we have

$$\begin{aligned} \delta_{qp}d_s\alpha_{2p} = e\kappa_{12}(\alpha_{32}\alpha_{4q} - \alpha_{42}\alpha_{3q}) + \kappa_{23}(\alpha_{34}\alpha_{4q} - \alpha_{44}\alpha_{3q}) \\ + \kappa_{34}(\alpha_{32}\alpha_{4q} - \alpha_{42}\alpha_{3q}). \end{aligned}$$

After simplification by use of (4.5) and (4.2), the above equation can be written out as

$$(4.8) \qquad \begin{aligned} d_s\alpha_{22} &= \kappa_2\alpha_{23}, \\ d_s\alpha_{23} &= -\kappa_2\alpha_{22} + (e\kappa_1 + \kappa_3)\alpha_{24}, \\ d_s\alpha_{24} &= -(e\kappa_1 + \kappa_3)\alpha_{23}. \end{aligned}$$

We observe that equations contain only the unknowns α_{2q} but not all the α_{pq}. This is what we hoped would happen; for, from the nature of our problem, we are interested only in the α_{2q} which determine the plane $A\text{-}I_1I_2$ completely. Equations (4.8) are our fundamental equations, which will, among other things, enable us to prove the existence theorem.

If (A) is not a plane curve, its Frenet frame $A\text{-}J_i$ is uniquely determined (except for the senses of J_i, which are not important here). In this case, let us take ξ_0 to be the $A\text{-}I_1I_2$ at $A(s_0)$, and determine the initial values

$(\alpha_{2p})_0$ ot α_{2p} from the second equation in (4.3). Equations (4.8) for each value of e will then give a unique solution for α_{2p} whose initial values at $A(s_0)$ are $(\alpha_{2p})_0$. With this solution α_{2p}, the first two equations in (4.3) will determine a unique isocline field of planes tangent to (A).

If (A) is a plane curve but not a straight line, then $\kappa_1 \neq 0$, $\kappa_2 = 0$, and κ_3 is indeterminate. In this case, the Frenet frame for (A) is not unique; in fact, formulas (4.1) and (4.2) show that J_1, J_2 and κ_1 are uniquely determined by $d_s A = J_1, d_s J_1 = \kappa_1 J_2, d_s J_2 = -\kappa_1 J_1$, but J_3, J_4 and κ_3 have merely to satisfy the following two equations

$$d_s J_3 = \kappa_3 J_4, \qquad d_s J_4 = -\kappa_3 J_3. \tag{4.9}$$

Suppose first that ξ_0 is not the plane of (A), i. e. the plane in which the curve (A) lies. Then it follows from (4.9) that there exists a unique Frenet frame for (A) satisfying

$$\kappa_3 = -e\kappa_1, \tag{4.10}$$

and the initial condition that at $A(s_0)$, J_4 is orthogonal to ξ_0 and J_2. When this Frenet frame is used, equations (4.8) reduce to $d_s\alpha_{2p} = 0$, and have the unique solution

$$\alpha_{22} = (\alpha_{22})_0, \quad \alpha_{23} = (\alpha_{23})_0, \quad \alpha_{24} = (\alpha_{24})_0 = 0.$$

If ξ_0 is the plane of (A), let us use in (4.8) *any* Frenet frame A-J_i satisfying (4.9) and (4.10). Then the soution of (4.8) is

$$\alpha_{22} = (\alpha_{22})_0 = \pm 1, \qquad \alpha_{23} = (\alpha_{23})_0 = 0, \qquad \alpha_{24} = (\alpha_{24})_0 = 0,$$

and the isocline field of tangent planes required is the (single) plane of (A). Hence our theorem is completely proved.

REMARK. It is obvious that if the planes A-I_1I_2 in a family of frames A-I_i form an isocline family, then the planes A-I_3I_4 also form an isocline family. Therefore, Theorem 4.1 with the words "tangent" replaced by "normal" still holds and constitutes the existence theorem for isocline fields of normal planes along a curve in R_4.

5. Some properties.

Suppose that A-J_1I_2 and A-$J_1I_2^*$ are two isocline fields of tangent planes of the same type along a curve (A). Then the corresponding functions α_{2p}, α_{2p}^* will satisfy equations (4.8) with the same value of e. Now the angle ψ between the planes A-J_1I_2 and A-$J_1I_2^*$, which intersect at the tangent A-J_1, is given by

$$\cos\psi = I_2 \cdot I_2^* = \alpha_{2p}\alpha_{2p}^*.$$

If we differentiate the last member with respect to s and make use of (4.8), the result is found to be zero. From this and the linearity of the equations (4.8), we have

THEOREM 5.1. *Let $(\xi_{(1)})$, $(\xi_{(2)})$ be two fields of isocline tangent planes of the same type along a curve $(A(s))$, then the corresponding planes $\xi_{(1)}(s)$, $\xi_{(2)}(s)$ make a constant angle at the tangent along the curve. Let $(\xi_{(a)})$*

$(a = 1, 2, 3)$ *be three independent fields of isocline tangent planes of the same type along a curve* $(A(s))$, *then a tangent plane* $\xi(s)$ *of* (A) *will generate a 4th isocline field along the curve if and only if* $\xi(s)$ *is rigidly attached to the three planes* $\xi_{(a)}(s)$.

Here the words "independent" and "rigidly attached" are used in an obvious sense.

We shall now prove the following theorem.

THEOREM 5.2. *Let* (ξ) *be an isocline field of tangent planes along a curve* (A), *then any vector in* ξ *making a constant angle with the tangent* $A\text{-}J_1$ *will rotate along* (A) *at the same rate as the tangent* $A\text{-}J_1$.

PROOF. Let $A\text{-}I_1I_2$ generate the isocline field of tangent planes and let $I = \beta I_1 + \gamma I_2$, where I_1, I_2 are defined by (4.3) and β, γ are constants such that $\beta^2 + \gamma^2 = 1$. Differentiating I with respect to s and making use of (4.3), (4.1) and (4.8), we have after simplification

$$d_sI = \kappa_1\{\beta J_2 + \gamma(-\alpha_{22}J_1 + e\alpha_{24}J_3 - e\alpha_{23}J_4)\}.$$

Therefore, $(d_sI)^2 = (\kappa_1)^2 = (d_sJ_1)^2$, which proves the theorem. It is to be pointed out that the property stated in Theorem 5.2 is not sufficient to characterize an isocline field of tangent planes along a curve.

The following theorem is an easy consequence of (4.8).

THEOREM 5.3. *There does not exist any non-plane curve along which the plane containing the tangent and the first or second principal normal generates an isocline field. The plane containing the tangent and the last principal normal of a non-plane curve generates an isocline field if and only if the first and third curvatures of the curve are numerically equal.*

Without going into details we mention that further consequences of (4.8) can be obtained by considering the cases where $\alpha_{22} =$ const. or $\alpha_{24} =$ const. and by observing that (4.8) contains the curvatures of (A) only in the combinations κ_2, $(e\kappa_1 + \kappa_3)$.

6. One-parameter family of isocline planes in R_4.

In R_4 a one-parameter family of planes in which any pair of consecutive planes are isocline, but not parallel, to each other is called a *one-parameter family of isocline planes*. Any such family may be considered as generated by the plane $A\text{-}I_1I_2$ of a one-parameter family of frames $A\text{-}I_i$ satisfying the conditions

$$\omega_{24} = e\omega_{13}, \qquad \omega_{14} = -e\omega_{23} \qquad (e = \pm 1), \tag{6.1}$$

where the ω's are of the form $f(t)dt$, t being the parameter on which the family of frames depends. Since no consecutive planes in the family are parallel, the, ω's in (6.1) cannot all vanish at the same time, and therefore by continuity the e in (6.1) is always $+1$ or always -1. Hence, *any pair of planes in a one-parameter family of isocline planes are isocline to each* *other in one and the same sense.*

We shall now prove the following theorem.

THEOREM 6.1. *The planes of a one-parameter isocline family in R_4 either have a common point or are tangent to a curve.*

PROOF. Consider the point $B = A + \rho_a I_a$ $(a, b = 1, 2)$ in the plane $A\text{-}I_1I_2$, where ρ_a are functions of t. We have by (2.1)

$$dB = (\omega_a + d\rho_a + \rho_b\omega_{ba}) I_a + (\omega_p + \rho_b\omega_{bp}) I_p \qquad (p, q = 3, 4).$$

For the plane $A\text{-}I_1I_2$ to be tangent to the curve (B), or as a special case, passing through a common fixed point B, it is necessary and sufficient that $\omega_p + \rho_b\omega_{bp} = 0$, i.e. by (6.1),

$$\rho_1\omega_{13} + \rho_2\omega_{23} + \omega_3 = 0,$$
$$-\rho_1\omega_{23} + \rho_2\omega_3 + e\omega_4 = 0.$$

Since $(\omega_{13})^2 + (\omega_{23})^2 \neq 0$, these equations determine ρ_1, ρ_2, and consequently also the point B, uniquely as functions of t. Now let this point B take the place of A. Then $\rho_a = 0$, $\omega_p = 0$ and $dB = \omega_a I_a$. The point B is fixed or describes a curve according as $\omega_a = 0$ or $\neq 0$. Thus our theorem is proved.

7. One parameter family of planes in R_4.

By means of Cartan's [2] method of moving frames, it is not difficult to arrive at a complete classification of the one-parameter families of planes in R_4. We shall state the results without proof in the following theorem.

THEOREM 7.1. *In R_4 there are* 6 *categories of one-parameter families of planes. A family of Category I depends on* 5 *arbitrary functions of* 1 *variable and is the most general of such families. A family of Category II depends on* 4 *arbitrary functions of* 1 *variable, and every pair of consecutive planes in the family are* $\frac{1}{2}$ *-parallel. A family of Category III depends on* 3 *arbitrary functions of* 1 *variable, and the planes of the family are all parallel to a common fixed straight line. A family of Category IV depends on 3 arbitrary functions of 1 variable, and consists of isocline planes tangent to a curve. A family of Category V depends on 1 arbitrary function of 1 variable, and consists of isocline planes passing through a common fixed point; the family admits ∞^1 groups of ∞^1 displacements in R_4, each displacement leaving the planes of the family individually invariant. A family of Category VI depends on 1 arbitrary function of 1 variable, and consists of planes all parallel to a common fixed plane.*

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UNIVERSITY OF HONG KONG.