ON THE CENTRAL LIMIT THEOREM

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(Received June 24, 1951)

§ 1. Let $\{X_k(t)\}$ $(k = 0, 1, 2, \cdots)$ be a sequence of random variables defined in a probability space $(T. F. P)$. The so-called central limit theorem (Cramér [1]) states that when a sequence $\{X_k(t)\}$ satisfies some appropriate conditions, then we have

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n} X_k(t) \leqq a\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2}\, du$$

where $\frac{1}{\sqrt{n}} \sum_{k=0}^{n} X_k(t)$ denotes a suitably normalized variable. About this theorem we will consider the following two generalizations:

1°. Replacing the constant upper limit a of summation by a measurable function $g(t)$ defined in T.

2°. Replacing the number n of random variables of summation by a random function $N_n(t)$ defined in T.

On these generalizations J. C. Smith [2] has proved some theorems in the case where $\{X_k(t)\}$ is the system of Rademacher's functions. On the other hand M. Kac [3] has discussed the limit distribution of the type of $\frac{1}{\sqrt{n}} \sum_{k=0}^{n} f(2^k t)$, where $f(t)$ is a measurable function with period 1. In this note we consider the above generalizations in the case $X_k(t) = f(2^k t)$. Throughout this note we assume that $\frac{1}{\sqrt{n}} \sum_{k=0}^{n} f(2^k t)$ converges in law to the normal distribution $G(u)$.

§ 2. In this paragraph we prove a lemma.

LEMMA 1. *If E is a measurable set in $[0, 1]$ then*

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n} f(2^k t) \leqq a,\ E\right) = P(E)\, G(a)$$

where P is the Lebesgue measure.

PROOF. Devide $[0, 1]$ into 2^i equal parts and denote by Δ_i^j the set $[t;\ j/2^i \leqq t \leqq (j+1)/2^i]$ $j = 0, 1, 2, \cdots 2^i - 1$, and $i = 1, 2, \cdots$.

Then for any i and j

$$G(a) = \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n} f(2^k t) \leqq a\right)$$

$$= \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, \bigcup_{j=0}^{2^i-1} \Delta_i^j\right)$$

$$= \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\left(\sum_{k=0}^{i-1} + \sum_{k=i}^{n}\right) f(2^k t) \leqq a, \bigcup_{j=0}^{2^i-1} \Delta_i^j\right)$$

$$= \lim_{n\to\infty} \sum_{j=0}^{2^i-1} P\left(\frac{1}{\sqrt{n}}\sum_{k=i}^{n} f(2^k t) \leqq a, \Delta_i^j\right)$$

$$= 2^i \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=i}^{n} f(2^k t) \leqq a, \Delta_i^j\right)$$

$$= 2^i \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, \Delta_i^j\right) .$$

Therefore

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, \Delta_i^j\right) = P(\Delta_i^j)G(a).$$

Now, let M be the family of sets E which satisfy the following relation

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, E\right) = P(E)G(a).$$

Then we can easily conclude the following properties:

1° M includes $[0, 1]$ and Δ_i^j for any i and j.

2° If $E \subset E'$, and $E', E \in M$, then $E' - E \in M$.

For

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, E' - E\right)$$

$$= \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, E'\right) - \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, E\right)$$

$$= P(E')G(a) - P(E)G(a) = P(E' - E)G(a).$$

3° If $E = \bigcup_{j=1}^{\infty} \Delta_j$, $\Delta_j \in M$ and Δ_j, $\Delta_{j'}$ $(j \neq j')$ are non-overlapping, then $E \in M$.

For

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, E\right)$$

$$= \lim_{n\to\infty} \sum_{j=1}^{\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, \Delta_j\right), \tag{1}$$

and for every n we have

$$0 < P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a, \Delta_j\right) \leqq P(\Delta_j)$$

and

$$\sum_{j=0}^{\infty} P(\Delta_j) = P(E).$$

So the convergence of $\sum_{j=0}^{\infty} P\left(\frac{1}{n}\sum_{k=0}^{n} f(2^k t) \leqq a,\ \Delta_j\right)$ is uniform with respect to n. Hence we can exchange the order of limit and summation of (1). Thus the right hand side of (1) is

$$\sum_{j=0}^{\infty} \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t) \leqq a,\ \Delta_j\right)$$

$$= \sum_{j=0}^{\infty} P(\Delta_j)G(a) = P(E)G(a),$$

which is the required result.

Since M includes any closed interval and any open interval, any open set and any closed set are also included in M by 1°–3°. Here, P is the Lebesgue measure, so M includes any L-measurable set.

Using the above lemma we shall consider the generalizations mentioned in § 1.

§ 3. THEOREM 1. *Let $g(t)$ be a non-negative measurable function defined in $[0,1]$, then*

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\left|\sum_{k=0}^{n} f(2^k t)\right| \leqq g(t)\right) = \int_0^1 dt \int_{-g(t)}^{g(t)} dG(u),$$

where $G(u)$ denotes the normal distribution.

PROOF. It is sufficient to prove this theorem for the case where $g(t)$ is a simple function. Let $g(t)$ be a simple function such that $\{a_i,\ E_i\}$ $(i = 1, 2, \cdots\cdot)$. Then we have

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\left|\sum_{k=0}^{n} f(2^k t)\right| \leqq g(t)\right)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{\infty} P\left(\frac{1}{\sqrt{n}}\left|\sum_{k=0}^{n} f(2^k t)\right| \leqq a_i,\ g(t) = a_i\right). \tag{2}$$

The exchange of limit and summation of (2) is shown by the same way as in 3° of the preceding lemma. Hence, using Lemma 1, we have

$$\sum_{i=1}^{\infty} \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\left|\sum_{k=0}^{n} f(2^k t)\right| \leqq a_i,\ g(t) = a_i\right)$$

$$= \sum_{i=1}^{\infty} [G(a_i) - G(-a_i)]\, P(g(t) = a_i)$$

$$= \sum_{i=1}^{\infty} P(g(t) = a_i) \int_{-a_i}^{a_i} dG(u)$$

$$= \int_0^1 dt \int_{-g(t)}^{g(t)} dG(u).$$

Thus we get the theorem.

If two measurable functions $g_1(t)$ and $g_2(t)$ have the distribution functions $G_1(u)$ and $G_2(u)$ respectively and if $G_1(u) = G_2(u)$ for the continuity points of $G_1(u)$ and $G_2(u)$, then it is said that $g_1(t)$ and $g_2(t)$ have the same distribution function $G_1(u)$ (or $G_2(u)$).

COROLLARY 1. *Let $g_1(t)$ and $g_2(t)$ be non-negative and measurable functions having the same distribution function $\bar{G}(u)$. Then*

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \left| \sum_{k=0}^{n} f(2^k t) \right| \leqq g_1(t)\right)$$

$$= \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \left| \sum_{k=0}^{n} f(2^k t) \right| \leqq g_2(t)\right)$$

$$= \int_0^{\infty} d\bar{G}(v) \int_{-v}^{v} dG(u).$$

PROOF. From Theorem 1, we have

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \left| \sum_{k=0}^{n} f(2^k t) \right| \leqq g_1(t)\right) = \int_0^1 dt \int_{-g_1(t)}^{g_1(t)} dG(u) = \int_0^{\infty} d\bar{G}(v) \int_{-v}^{v} dG(u)$$

$$= \int_0^1 dt \int_{-g_2(t)}^{g_2(t)} dG(u) = \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \left| \sum_{k=0}^{n} f(2^k t) \right| \leqq g_2(t)\right).$$

§ 4. Next, we consider the second generalization.

THEOREM 2. *Let $N_n(t) = nN(t) + Q_n(t)$, where $N_n(t)$ and $N(t)$ are measurable functions which take only non-negative integers, and $Q_n(t) = o(n^{1/2})$. Then*

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \left| \sum_{k=0}^{N_n(t)} f(2^k t) \right| \leqq a\right)$$

$$= \sum_{M=0}^{\infty} P(N(t) = M) \int_{-aM^{-1/2}}^{aM^{-1/2}} dG(u)$$

$$= \int_0^1 dt \int_{-aN(t)^{-1/2}}^{aN(t)^{-1/2}} dG(u) .$$

PROOF. Put

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{N_n(t)} f(2^k t) = \frac{1}{\sqrt{n}}\sum_{k=0}^{nN(t)} f(2^k t) + S_n(t),$$

then

$$\lim_{n\to\infty} P(|S_n(t)| > \varepsilon)$$

$$= \lim_{n\to\infty}\Big(P|S_n(t)| > \varepsilon, \bigcup_{M=0}^{\infty}(N(t) = M, \bigcup_{k=-\infty}^{\infty} Q_n(t) = k)\Big)$$

$$= \sum_{M=0}^{\infty} \lim_{n\to\infty} P\Big(|S_n(t)| > \varepsilon,\ N(t) = M, \bigcup_{k=-\infty}^{\infty} Q_n(t) = k\Big)$$

$$= \sum_{M=0}^{\infty}\lim_{n\to\infty} P\Big(\frac{1}{\sqrt{n}}\Big|\sum{}' f(2^k t)\Big| > \varepsilon\Big)$$

where Σ' denotes the summation from nM to $nM + |Q_n(t)|$ or from $nM - |Q_n(t)|$ to nM according to $Q_n(t) \geqq 0$ or $Q_n(t) < 0$. But from our assumption $\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t)$ converges in law to $G(u)$ and $Q_n(t) = o(n^{1/2})$. Hence

$$\lim_{n\to\infty} P\Big(\frac{1}{\sqrt{n}}\Big|\sum{}' f(2^k t)\Big| > \varepsilon\Big) = 0.$$

So we have

$$\lim_{n\to\infty} P\Big(\frac{1}{\sqrt{n}}\Big|\sum_{k=0}^{N_n(t)} f(2^k t)\Big| \leqq a\Big)$$

$$= \lim_{n\to\infty} P\Big(\frac{1}{\sqrt{n}}\Big|\sum_{k=0}^{nN(t)} f(2^k t)\Big| \leqq a\Big)$$

$$= \lim_{n\to\infty}\sum_{M=0}^{\infty} P\Big(\frac{1}{\sqrt{n}}\Big|\sum_{k=0}^{nM} f(2^k t)\Big| \leqq a,\ N(t) = M\Big) \tag{3}$$

$$= \lim_{n\to\infty}\sum_{M=0}^{\infty} P\Big(\frac{1}{\sqrt{nM}}\Big|\sum_{k=0}^{nM} f(2^k t)\Big| \leqq aM^{-1/2},\ N(t) = M\Big)$$

$$= \sum_{M=0}^{\infty}\lim_{n\to\infty} P\Big(\frac{1}{\sqrt{nM}}\Big|\sum_{k=0}^{nM} f(2^k t)\Big| \leqq aM^{-1/2},\ N(t) = M\Big).$$

By Lemma 1, the last hand side of (3) equals to

$$\sum_{M=0}^{\infty} P(N(t) = M)\int_{-aM^{-1/2}}^{aM^{-1/2}} dG(u) = \int_0^1 dt \int_{-aN(t)^{-1/2}}^{aN(t)^{-1/2}} dG(u).$$

In Theorem 2, when $M = N(t) = 0$ then we interprete that $aM^{-1/2} = aN(t)^{-1/2}$ denotes ∞.

COROLLARY 2. *Let*

$$N_n'(t) = nN'(t) + Q_n'(t)$$

and

$$N_n''(t) = nN''(t) + Q_n''(t)$$

satisfy the conditions of $N_n(t)$, $N(t)$ *and* $Q_n(t)$ *of Theorem* 2. *If* $N'(t)$ *and* $N''(t)$ *have the same distribution function* $\bar{G}(u)$, *then*

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\left|\sum_{k=0}^{N_n'(t)} f(2^k t)\right| \leqq a\right)$$

$$= \lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}}\left|\sum_{k=0}^{N_n''(t)} f(2^k t)\right| \leqq a\right)$$

$$= \int_0^\infty d\bar{G}(u) \int_{-av^{-1/2}}^{av^{-1/2}} dG(u).$$

PROOF. It is evident from Theorem 2.

The theorems of J. C. Smith [2] are obtained when we put

$$f(t) = \text{sign } (\sin 2\pi t)$$

in Theorem 1 and Theorem 2.

LITERATURE

[1] H. CRAMER, Random variable and its probability distribution, Cambridge (1937).

[2] J. C. SMITH, On the asymptotic distribution of the sum of Rademacher's functions, Bull. Amer. Math. Soc, 51 (1945).

[3] M. KAC, On the limiting distribution of the type of $\frac{1}{\sqrt{n}}\sum_{k=0}^{n} f(2^k t)$, Annals of Math. 47 (1946).

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