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KINSAKU TAKANO

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1. Introduction and summary

Recently E. Lukacs and O. Szász [3] gave a necessary condition which the reciprocal of a polynomial without multiple roots must satisfy in order to be a charcteristic function. The assumption that the polynomial has no multiple roots is, however, unnecessary, and moreover, if the degree of the polynomial is less than 4, the condition is not only necessary but also sufficient. The condition is not sufficient in case when the degree of the polynomial is equal to 5. The proof and example will be given in § 3. The principle of the proof is the same as one given by E. Lukacs and O. Szász.

Let L be the probability law defined by the density function

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & \text{if } x \geqq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

where α is positive, then the characteristic function of L is given by

$$\varphi(t) = \left(1 - \frac{it}{\alpha}\right)^{-1}.$$

The probability law L has a curious property; starting from the law L, one obtains the same symmetric law by $X = X_1 - X_2$, where X_1 and X_2 are independent random variables with the law L, or by $X = \varepsilon X_1$, where ε and X_1 are independent random variables, ε taking on ± 1 with equal probabilities, the law of X_1 being L. The identity of the two laws, which are in general different, is expressed by the formula

$$\frac{1}{2}[\varphi(t) + \varphi(-t)] = \varphi(t)\varphi(-t). \tag{1.1}$$

Under the condition that $\varphi(t)$ is a non-vanishing characteristic function, the solution of (1.1) is given by

$$\varphi(t) = \frac{1}{1 + i\omega(t)}, \tag{1.2}$$

where $\omega(t)$ is a real valued function of t such that

$$\omega(-t) = -\omega(t). \tag{1.3}$$

A question arises, if there is another characteristic function of this form than those obtained by putting $\omega(t) = ct$. P. Lévy [2] expressed himself that this question did not seem to have been solved. We will give an answer to this problem in the affirmative in § 4.

Professor T. Kawata has informed me the result of E. Lukacs and O. Szász [3], which has not been published as yet, and I have greatly profited from his valuable remarks. I wish to express my sincere thanks to him.

2. Auxiliary formulas and lemmas

FORMULAS. (See [3] § 3).

If $\Re(\alpha) > 0$, that is, α is a real positive number or a complex number with positive real part, and λ is a positive integer,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx}}{\left(1-\frac{it}{\alpha}\right)^{\lambda}}\,dt = \begin{cases} \frac{\alpha^{\lambda}}{\Gamma(\lambda)} x^{\lambda-1}e^{-\alpha x}, & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{2.1}$$

and

$$\int_{0}^{\infty} \frac{\alpha^{\lambda}}{\Gamma(\lambda)} x^{\lambda-1}e^{-\alpha x}e^{itx}dx = \left(1-\frac{it}{\alpha}\right)^{-\lambda}. \tag{2.2}$$

If $\Re(\beta) > 0$ and λ is a positive integer,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itx}}{\left(1+\frac{it}{\beta}\right)^{\lambda}}\,dt = \begin{cases} \frac{\beta^{\lambda}}{\Gamma(\lambda)} (-x)^{\lambda-1}e^{\beta x}, & \text{if } x < 0, \\ 0, & \text{if } x > 0, \end{cases} \tag{2.3}$$

and

$$\int_{-\infty}^{0} \frac{\beta^{\lambda}}{\Gamma(\lambda)} (-x)^{\lambda-1}e^{\beta x}e^{itx}dx = \left(1+\frac{it}{\beta}\right)^{-\lambda}. \tag{2.4}$$

In (2.1) and (2.3), if $\lambda = 1$, $\int_{-\infty}^{\infty}$ means $\lim_{T\to\infty}\int_{-T}^{T}$.

LEMMA 1. *(well-known). If a non-negative real valued almost periodic function $g(x)$ satisfies*

$$M(g(x)) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x)dx = 0,$$

then $g(x) \equiv 0$.

LEMMA 2. *Let*

$$g(x) = \sum_{j=1}^{m} (B_j e^{-ib_j x} + \bar{B}_j e^{ib_j x}) \tag{2.5}$$

where $0 < b_1 < b_2 < \cdots < b_m$, $B_j \neq 0\ (j = 1, 2, \cdots m)$, *and* $\bar{B}$ *expresses the conjugate complex number of a complex number* B, *then*

$$\liminf_{x\to\infty} g(x) < 0, \quad \liminf_{x\to-\infty} g(x) < 0.$$

PROOF. It is evident that

(2.6) $g(x)$ is a real valued almost periodic function,

(2.7) $M(g(x)) = 0$, and

$$M(|g(x)|^2) = 2\sum_{j=1}^{m} |B_j|^2 > 0. \tag{2.8}$$

If $g(x)$ is non-negative, from Lemma 1 and (2.7), it follows that $g(x) \equiv 0$ which contradicts (2.8). Therefore there exists at least one value x_0 such that $g(x_0) < 0$. Using (2.6), we have

$$\liminf_{x\to\infty} g(x) < 0, \qquad \liminf_{x\to-\infty} g(x) < 0.$$

3. THEOREM 3.1. (Eugene Lukacs and Otto Szász). *In order that the complex valued function of a real variable* t

(3.1) $\varphi(t) = \{c_0 + c_1(it) + c_2(it)^2 + \cdots + c_N(it)^N\}^{-1}$, $(i = \sqrt{-1}\,,\ c_N \neq 0)$,

be the characteristic function of a probability distribution, it is necessary that

(3.2) $c_0 = 1$ *and all* c_j *are real,*

(3.3) *the polynomial with real coefficients*

$$Q(z) = 1 + c_1 z + c_2 z^2 + \cdots + c_N z^N$$

has no pure imaginary roots, and that

(3.4) *if* $a \pm ib\,(a \neq 0,\ b \neq 0)$ *is a pair of complex roots of the polynomial* $Q(z)$ *then it has at least one real root* c *such that* $\operatorname{sign} c = \operatorname{sign} a$ *and* $|c| \leqq |a|$.

PROOF. If the function $\varphi(t)$ is the characteristic function of a destribution then:

$$\varphi(0) = 1, \tag{3.5}$$

$$|\varphi(t)| \leqq 1, \tag{3.6}$$

$$\varphi(-t) = \overline{\varphi(t)}. \tag{3.7}$$

From (3.5), we have $c_0 = 1$. From (3.7), we have

$$1 + c_1(-it) + c_2(-it)^2 + \cdots + c_N(-it)^N$$
$$= 1 + \bar{c}_1(-it) + \bar{c}_2(-it)^2 + \cdots + \bar{c}_N(-it)^N$$

for all real t. Therefore all c_j must be real. (3.3) is derived from (3.6). In the sequel (3.2) and (3.3) are considered to be satisfied.

Let the zeros of the polynomial $Q(z)$ be

$$z = \alpha_j\ (j = 1, 2, \cdots, m) \quad \text{and} \quad z = -\beta_k \quad (k = 1, 2, \cdots, n)$$

where $\Re(\alpha_j) > 0$, $\Re(\beta_k) > 0$, and let their multiplicities be p_j and q_k. If α_j is not real, there exists $\alpha_{j'}$, which is conjugate complex with α_j and $p_j = p_{j'}$. The same is true for β_k's and q_k's.

The function (3.1) can then be written

$$\varphi(t) = \left[\prod_{j=1}^{m}\left(1 - \frac{it}{\alpha_j}\right)^{p_j} \prod_{k=1}^{n}\left(1 + \frac{it}{\beta_k}\right)^{q_k}\right]^{-1}. \tag{3.8}$$

If $\varphi(t)$ is decomposed into partial fractions, it is seen that

$$\varphi(t) = \sum_{j=1}^{m}\sum_{p=1}^{p_j} \frac{A_{jp}}{\left(1 - \dfrac{it}{\alpha_j}\right)^p} + \sum_{k=1}^{n}\sum_{q=1}^{q_k} \frac{B_{kq}}{\left(1 + \dfrac{it}{\beta_k}\right)^q} \tag{3.9}$$

where

(3.10) if α_j is real then A_{jp} is real, and if $\alpha_j = \overline{\alpha_{j'}}$, then $A_{jp} = \overline{A_{j'p}}$ and the same is true for β_k's and B'_{kq}s.

If $m = 0$, i.e., if there are no roots with positive real part the first term of (3.9) is omitted, and if $n = 0$, the second term.

Let

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\varphi(t)dt, \tag{3.11}$$

then, we have from (2.1)–(2.4),

$$f(x) = \begin{cases} \sum_{j=1}^{m}\sum_{p=1}^{p_j} \frac{A_{jp}\alpha_j^p}{\Gamma(p)} x^{p-1}e^{-\alpha_j x}, & (x > 0), \\ \sum_{k=1}^{n}\sum_{q=1}^{q_k} \frac{B_{kq}\beta_k^q}{\Gamma(q)} (-x)^{q-1}e^{\beta_k x}, & (x < 0), \end{cases} \tag{3.12}$$

and

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx}f(x)\,dx\,, \quad \int_{-\infty}^{\infty} f(x)dx = \varphi(0) = 1. \tag{3.13}$$

Using (3.10), $f(x)$ is seen to be real.

Since

$$|\varphi(t)| \sim |c_N t^N|^{-1}, \quad (t \to \pm\infty),$$

it follows that $|\varphi(t)|$ is integrable over $(-\infty, \infty)$, assuming that $N \geqq 2$. Therefore, if $\varphi(t)$ is a characteristic function, then $f(x)$ must be the probability density corresponding to $\varphi(t)$ (Cramér [1], p. 94). Since $f(x)$ is continuous, it is necessary that $f(x) \geqq 0$ for every x in order that $f(x)$ is a probability density.

Conversely, if $f(x) \geqq 0$, from (3.13) it follows that $\varphi(t)$ is a characteristic function (This fact is used for the proof of Theorem 3.2).

To derive the condition (3.4), it is sufficient to prove that if none of α_j corresponding to the smallest $\Re(\alpha_j)$ are real, then there exists at least one value x such that $f(x) < 0$.

Let

$$\alpha_j = a_j + ib_j \quad (a_j, b_j \text{ real}) \quad (j = 1, 2, \cdots, m).$$

We may assume that

$$a_1 \leqq a_2 \leqq \cdots \leqq a_m.$$

We can find m_0 such that either

$$\begin{aligned} &a_1 = a_2 = \cdots = a_{m_0} < a_{m_0+1} \leqq \cdots \leqq a_m, \text{ or} \\ &m_0 = m \text{ and } a_1 = a_2 = \cdots = a_m. \end{aligned} \tag{3.14}$$

Let

$$\max(p_1, p_2, \cdots, p_{m_0}) = s,$$
$$\max(p_{m_0+1}, \cdots, p_m) = t,$$

then the function (3.12) can be written

$$f(x) = x^{s-1}e^{-ax}g(x) + O(x^{s-2}e^{-ax}) + O(x^{t-1}e^{-a_{m_0+1}x}) \tag{3.15}$$

as $x \to +\infty$, where

$$a = a_1 < a_{m_0+1}, \tag{3.16}$$

$$g(x) = \sum_{1 \leqq j \leqq m_0, p_j = s} \frac{A_{js}\alpha_j^s}{\Gamma(s)} e^{-ib_j x}. \tag{3.17}$$

If $s = 1$ the second term of (3.15) is omitted, and if $m_0 = m$, the third term.

Because of (3.15) and (3.16), we have

$$f(x) = x^{s-1}e^{-\alpha x}(g(x) + o(1)), \quad (x \to \infty). \tag{3.18}$$

We may assume that b_j's which appear in the right hand side of (3.17) and are positive are arranged as

$$0 < b_1 < b_2 < \cdots < b_{m_1}.$$

(3.17) can then be written

$$g(x) = \sum_{j=1}^{m_1} \left(B_j e^{-ib_j x} + \bar{B}_j e^{ib_j x} \right)$$

with $B_j \neq 0 \ (j = 1, \cdots m_1)$. From Lemma 2, it follows that

$$\liminf_{x \to +\infty} g(x) < 0. \tag{3.19}$$

From (3.18) and (3.19), it follows that there exists at least one value x such that $f(x) < 0$. Q. E. D.

COROLLARY. *If the polynomial with real coefficients*

$$Q(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_{n-2} z^{n-2} + a_n z^n$$

of degree n without term of degree $n-1$ has only one value of real roots, then $1/Q(it)$ cannot be a characteristic function.

PROOF. If $Q(z)$ is decomposed into real factors it is seen that

$$Q(z) = a_n (z - \alpha)^k \prod_{j=1}^{l} (z^2 - 2\beta_j z + \gamma_j), \quad (k + 2l = n).$$

As the coefficient of z^{n-1} in the right hand side is equal to zero,

$$-2\sum_{j=1}^{l} \beta_j - k\alpha = 0, \quad \text{or,} \quad 2\sum_{j=1}^{l} \alpha\beta_j = -k\alpha^2 < 0.$$

Therefore, there exists at least one j such that

$$\alpha\beta_j < 0.$$

However, β_j is the real part of a complex root of $Q(z)$ and α is the only one value of real roots of $Q(z)$. Therefore $Q(z)$ does not satisfy (3.4).

Q. E. D.

THEOREM 3.2. *If the polynomial of third degree with real coefficients*

$$Q(z) = 1 + a_1 z + a_2 z^2 + a_3 z^3$$

has a real root α and complex roots $\beta \pm i\gamma$, and if

$$\operatorname{sign} \alpha = \operatorname{sign} \beta \quad \textit{and} \quad |\alpha| \leqq |\beta|,$$

then

$$\varphi(t) = \frac{1}{Q(it)}$$

is a characteristic function.

PROOF. Since, if $\varphi(t)$ is a characteristic function, $\varphi(-t)$ is also one, we may assume that $0 < \alpha \leqq \beta$ and $\gamma > 0$. If $f(x)$ is defined by (3.11), then we have (3.13). Therefore, it is sufficient to prove that $f(x) \geqq 0$.

If

$$\varphi(t) = \left[\left(1 - \frac{it}{\alpha}\right)\left(1 - \frac{it}{\beta + i\gamma}\right)\left(1 - \frac{it}{\beta - i\gamma}\right)\right]^{-1}$$

is decomposed into partial fractions it is seen that

$$\varphi(t) = a\Big/\left(1 - \frac{it}{\alpha}\right) + b\Big/\left(1 - \frac{it}{\beta + i\gamma}\right) + \bar{b}\Big/\left(1 - \frac{it}{\beta - i\gamma}\right),$$

where

$$a = \left[\left(1 - \frac{\alpha}{\beta + i\gamma}\right)\left(1 - \frac{\alpha}{\beta - i\gamma}\right)\right]^{-1} = \frac{\beta^2 + \gamma^2}{(\beta - \alpha)^2 + \gamma^2} > 0,$$

and

$$b = \left[\left(1 - \frac{\beta + i\gamma}{\alpha}\right)\left(1 - \frac{\beta + i\gamma}{\beta - i\gamma}\right)\right]^{-1}.$$

We have

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t) e^{-itx} dt \\ &= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T}\left[ae^{-ixt}\left(1 - \frac{it}{\alpha}\right)^{-1} + be^{-ixt}\left(1 - \frac{it}{\beta + i\gamma}\right)^{-1}\right. \\ &\qquad \left. + \bar{b}e^{-ixt}\left(1 - \frac{it}{\beta - i\gamma}\right)^{-1}\right] dt \\ &= \begin{cases} a\alpha e^{-\alpha x} + b(\beta + i\gamma)\, e^{-(\beta+i\gamma)x} + \bar{b}(\beta - i\gamma)\, e^{-(\beta - i\gamma)x} & (x > 0), \\ 0 & (x < 0), \end{cases} \end{aligned}$$

If $x > 0$,

$$\begin{aligned} f(x) &= a\alpha e^{-\alpha x} + 2\Re\{b(\beta + i\gamma)\, e^{-(\beta + i\gamma)x}\} \\ &= a\alpha e^{-\alpha x}\left[1 - e^{-(\beta - \alpha)x}\left\{\cos \gamma x + \frac{\beta - \alpha}{\gamma}\sin \gamma x\right\}\right] \\ &\geqq 0 \end{aligned}$$

where we have used that

$$b(\beta + i\gamma) = -\frac{a\alpha}{2}\left\{1 + i\frac{\beta - \alpha}{\gamma}\right\}.$$

COROLLARY 1. *If a polynomial of degree of* $3n + m$ ($n = 1, 2, \cdots$; $m = 0, 2, \cdots$) *with real coefficients*

$$Q(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_{3n+m} z^{3n+m}$$

has $n + m$ *real roots* α_j ($j = 1, 2, \cdots, n + m$) *and* n *pairs of conjugate complex roots* $\beta_j \pm i\gamma_j$ ($j = 1, 2, \cdots, n$) (*multiple roots being enumerated by its multiplicity*) *and if*

$$\alpha_j \beta_j > 0, \quad |\alpha_j| \leqq |\beta_j| \quad (j = 1, 2, \cdots, n),$$

then, $1/Q(it)$ *is a characteristic function.*

COROLLARY 2. *Under the condition that*

$$n \leqq 4,$$

in order that (3.1) *be a characteristic function, it is necessary and sufficient that* (3.2)—(3.4) *holds.*

NOTICE: If $n = 5$, (3.2)—(3.4) is not sufficient in order that (3.1) be a characteristic function.

Example. Let

$$\varphi(t) = \left[\left(1 - \frac{it}{\alpha}\right)\left(1 - \frac{it}{\beta + i\gamma_1}\right)\left(1 - \frac{it}{\beta - i\gamma_1}\right)\left(1 - \frac{it}{\beta + i\gamma_2}\right)\left(1 - \frac{it}{\beta - i\gamma_2}\right)\right]^{-1}$$

where $0 < \alpha < \beta$, $\gamma_1 = \left(2n + \frac{1}{2}\right)\pi - \theta$, $\gamma_2 = \left(2n + \frac{1}{2}\right)\pi + \theta$,

$$0 < \theta < \frac{\pi}{2} \quad \text{and } n\pi \frac{\sin\theta}{\theta} > 1. \ (n = 1, 2, 3, \cdots\cdots).$$

If $\beta - \alpha$ is sufficiently small, then $\varphi(t)$ cannot be a characteristic function. In this case, (3.12) can be written

$$f(x) = \begin{cases} A\alpha e^{-\alpha x}[1 - g(x; \beta - \alpha)], & (x > 0), \\ 0, & (x < 0), \end{cases}$$

where

$$A = \left|\left(1 - \frac{\alpha}{\beta + i\gamma_1}\right)\left(1 - \frac{\alpha}{\beta + i\gamma_2}\right)\right|^{-2} > 0,$$

$$g(x; \varepsilon) = \frac{e^{-\varepsilon x}}{\gamma_2^2 - \gamma_1^2}\left\{(\varepsilon^2 + \gamma_2^2)\left(\cos\gamma_1 x + \frac{\varepsilon}{\gamma_1}\sin\gamma_1 x\right) - (\varepsilon^2 + \gamma_1^2)\left(\cos\gamma_2 x + \frac{\varepsilon}{\gamma_2}\sin\gamma_2 x\right)\right\}.$$

Since

$$\lim_{\varepsilon \to 0} g(1; \varepsilon) = g(1, 0) = \frac{\gamma_2^2 \cos\gamma_1 - \gamma_1^2 \cos\gamma_2}{\gamma_2^2 - \gamma_1^2} > n\pi \frac{\sin\theta}{\theta} > 1,$$

if $\beta - \alpha$ is sufficiently small, then

$$g(1; \beta - \alpha) > 1$$

and we have

$$f(1) < 0.$$

Therefore, $\varphi(t)$ cannot be a characteristic function.

4. In this §, we shall consider characteristic functions which satisfy (1.1). Assuming $\omega(t)$ in (1.2) to be a polynomial of t, from (1.3), $\omega(t)$ contains only terms of odd degree. Let

$$\omega(t) = a_1 t - a_3 t^3 + \cdots\cdots + (-1)^n a_{2n+1} t^{2n+1},$$

then

$$\varphi(t) = (1 + i\omega(t))^{-1} = [1 + a_1(it) + a_3(it)^3 + \cdots\cdots + a_{2n+1}(it)^{2n+1}]^{-1},$$

that is

$$\varphi(t) = \frac{1}{Q(it)} \tag{4.1}$$

where $Q(z)$ is a polynomial of degree $2n+1$ of real coefficients, without terms of $z^2, z^4, \cdots, z^{2n}$,

(4.2) $$Q(z) = 1 + a_1 z + a_3 z^3 + \cdots + a_{2n+1} z^{2n+1}.$$

As (4.1) satisfies (1.1), we will consider if (4.1) be a characteristic function.

If $2n+1$ roots $\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}$ of the polynomial $Q(z)$ are all real, then $\varphi(t)$ is rewritten

(4.3) $$\varphi(t) = \prod_{j=1}^{2n+1} \left(1 - \frac{it}{\alpha_j}\right)^{-1},$$

which proves that $\varphi(t)$ is a characteristic function, for the product of characteristic functions is a characteristic function. Conversely, in order that (4.3) satisfies (1.1), it is necessary that $\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}$ are roots of a polynomial of the form (4.2). We have proved the following

THEOREM 4.1. *For any n real numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $\alpha_j \neq 0$* $(j = 1, \cdots, n)$, *in order that the characteristic function*

$$\varphi(t) = \prod_{j=1}^{n} \left(1 - \frac{it}{\alpha_j}\right)^{-1}$$

satisfies (1.1), *it is necessary and sufficient that n is odd and all the elementary symmetric functions of even degree of $\alpha_1, \alpha_2, \cdots, \alpha_n$ vanish.*

It seems to be interesting, if there is a characteristic function (4.1) such that (4.2) has complex roots. If $n \leqq 2$, however, there is no such characteristic functions. Indeed, we have the following

THEOREM 4.2. *In order that*

(4.5) $$\varphi(t) = [1 + a(it) + b(it)^3 + c(it)^5]^{-1}$$

where a, b, c are real numbers, be a characteristic function, it is necessary and sufficient that the polynomial

$$Q(z) = 1 + az + bz^3 + cz^5$$

has no complex roots.

PROOF. *Sufficiency*: evident. *Necessity*: we shall prove that $\varphi(t)$ cannot be characteristic functions if $Q(z)$ has complex roots. From Theorem 2.1 and its Corollary, it is sufficient to prove in case when $c \neq 0$ and $Q(z)$ has three real roots and a pair of complex roots which are not pure imaginary. If $\varphi(t)$ is a characteristic function, a being real number, $\varphi(at)$ is also a characteristic function. This shows that we can suppose that $c = 1$. $Q(z)$ has then the following form

$$Q(z) = (z-\alpha)(z-\beta)(z-\gamma)\{z^2 + (\alpha+\beta+\gamma)z - (\alpha\beta\gamma)^{-1}\}.$$

Write

$$s_1 = \alpha + \beta + \gamma, \quad s_2 = \alpha\beta + \alpha\gamma + \beta\gamma, \quad s_3 = \alpha\beta\gamma.$$

α, β and γ must satisfy the following conditions

(4.6) $$\alpha\beta\gamma(\alpha+\beta+\gamma) \neq 0, \qquad \alpha, \beta, \gamma \text{ are real},$$

$$(4.7)\qquad s_1 = s_3(s_3 - s_1s_2),$$

$$(4.8)\qquad s_1^2 < -4s_3^{-1}.$$

(4.7) is derived from the condition that the coefficient of z^2 vanishes. From (4.8), we have $s_3 = \alpha\beta\gamma < 0$. We may suppose that $\alpha \leqq \beta \leqq \gamma$.

(a) Case when $\alpha \leqq \beta \leqq \gamma < 0$. The real part $-(\alpha+\beta+\gamma)/2$ of complex roots is positive. $Q(z)$ does not satisfy (3.4).

(b) Case when

$$(4.9)\qquad \alpha < 0 < \beta \leqq \gamma.$$

In this case we have

$$(4.10)\qquad s_1 > 0.$$

In fact, if $\alpha + \beta \geqq 0$ then $s_1 = \alpha+\beta+\gamma > 0$, and if $\alpha+\beta < 0$ then $s_2 = (\alpha+\beta)\gamma + \alpha\beta < 0$, $s_3 < 0$ and so $s_1 = s_3^2/(1+s_2s_3) > 0$. From (4.7) $\times$ (4.8), we have

$$(4.11)\qquad s_1^3 < -4(s_3 - s_1 s_2).$$

From (4.9), (4.10) and (4.11), we will prove that

$$(4.12)\qquad \alpha < -(\alpha+\beta+\gamma)/2 < 0$$

which completes the proof. These relations are invariant when we devide α, β and γ by a same positive number, hence we can suppose that

$$s_1 = \alpha+\beta+\gamma = 1.$$

Substituting 1 for s_1 in (4.11)

$$1 < -4(s_3 - s_2).$$

We have

$$\begin{aligned} 1/4 < s_2 - s_3 &= \alpha(\beta+\gamma) + \beta\gamma - \alpha\beta\gamma \\ &= \alpha(1-\alpha) + \beta\gamma(1-\alpha) \\ &\leqq \alpha(1-\alpha) + (1-\alpha)^3/4 \\ &= (1/4)(1+\alpha-\alpha^2-\alpha^3). \end{aligned}$$

$$\therefore \qquad \alpha^3 + \alpha^2 - \alpha < 0,$$

$$\therefore \qquad \alpha^2 + \alpha - 1 > 0.$$

α being negative, we have

$$\alpha < -\frac{1}{2} = -\frac{\alpha+\beta+\gamma}{2} < 0.$$

Since (3.4) is not satisfied, $\varphi(t)$ cannot be a characteristic function.

COROLLARY. *In order that*

$$\varphi(t) = [1 + a(it) + b(it)^3]^{-1}, \qquad (a,\ b\ \textit{real},\ b \neq 0)$$

is a characteristic function, it is necessary and sufficient that there exists a real number α *such that*

$$(4.13)\qquad \frac{a}{\sqrt[3]{b}} = \frac{1}{\alpha} - \alpha^2,$$

$$(4.14)\qquad \alpha \geqq \sqrt[3]{4}\ \textit{ or }\ \alpha < 0.$$

If there is such α, $\varphi(t)$ *satisfies* (1.1).

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