(1) this journal.
[2]. T. KOJIMA, On the convergence-abscissa of general Dirichlet series, Tôhoku Math, Jour., Vol. 6 (1914-15).
[3]. V. BERNSTEIN, Leçons sur les progrès récents de la théorie des séries de Dirichlet, Paris, 1933.
[4]. E. LANDAU, Darstellung und Begründung einiger neuerer Ergebnisse Funktionentheorie. Berlin, 1929.
[5]. C. BIGGERI, Sur les singularitiés des fonctions analytiques définies par des séries de Dirichlet, C. R. Acad. Sci., Paris, 209 (1939).
[6]. F. CARLSON-E. LANDAU, Neuer Beweis und Verallgemeinerung des Fabryschen Lückensatzes, Gött. Nachr., 1921.
[7]. O. SZÀSZ, Über Singularitäten von Potenzreihen und Dirichletschen Reihen am Rande des Konvergenzbereiches, Math. Ann., Bd. 85 (1922).

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Theorem B.

2. THEOREM 1. *If*

$$f(t)-f(t')=o\left(1\Big/\log\frac{1}{|t-t'|}\right),\quad (t\to x,\ t'\to x) \tag{2.1}$$

then the Fourier series of $f(t)$ converges uniformly at $t=x$.

PROOF. We can suppose, without loss of generality, that $x=0$ and $f(t)$ is even. Let (x_n) be an arbitrary sequence of positive number tending to zero and $s_n(t)$ be the n-th partial sum of the Fourier series of $f(t)$. Supposing $f(0)=0$, it is sufficient to prove that

$$\lim_{n\to\infty} s_n(x_n)=0. \tag{2.2}$$

For a given $\varepsilon>0$, we can take $\delta>0$ such that if

$$|x_n\pm(t+h)|<\delta,\ \text{and}\ |x_n\pm t|<\delta, \tag{2.3}$$

then

$$|f\{x_n\pm(t+h)\}-f(x_n\pm t)|<\varepsilon/\log(1/h). \tag{2.4}$$

If we put

$$g_n(t)=f(x_n+t)+f(x_n-t), \tag{2.5}$$

then we have

$$\begin{aligned} s_n(x_n)&=\frac{1}{\pi}\int_0^{\pi} g_n(t)\frac{\sin nt}{t}dt+o(1)\\ &=\frac{1}{\pi}\left(\int_0^{\pi/n}+\int_{\pi/n}^{\delta}+\int_{\delta}^{\pi}\right)g_n(t)\frac{\sin nt}{t}dt+o(1)\\ &=\frac{1}{\pi}(I_n+J_n+K_n)+o(1), \end{aligned}$$

say.

Since the point $t=0$ is the Lebesgue point, $I_n=o(1)$ by the well known method and $K_n=o(1)$ by the generalized Riemann-Lebesgue theorem (see Zygmund [5], p. 22). If we put $\pi/n=h$ and using

$$\int_h^{\delta} g_n(t)\frac{\sin nt}{t}dt=\int_{2h}^{\delta+h} g_n(t)\frac{\sin nt}{t}dt+o(1),$$

then we get

$$\begin{aligned} J_n&=\frac{1}{2}\int_h^{\delta}\left\{\frac{g_n(t)}{t}-\frac{g_n(t+h)}{t+h}\right\}\sin nt\,dt+o(1)\\ &=\frac{1}{2}\int_h^{\delta}\frac{g_n(t)-g_n(t+h)}{t+h}\sin nt\,dt+\frac{1}{2}h\int_h^{\delta}\frac{g_n(t)}{t(t+h)}\sin nt\,dt+o(1)\\ &=L_n+M_n+o(1), \end{aligned}$$

say. On the first term L_n, by (2.4) we have

$$L_n = \frac{1}{2}\int_h^\delta \frac{|\{f(x_n+t)-f(x_n-t)\}-\{f(x_n+t+h)-f(x_n-t-h)\}|}{t}\,dt$$

$$= \frac{\varepsilon}{\log 1/k}\int_h^\delta \frac{dt}{t} \leqq \varepsilon.$$

It is therefore sufficient to prove that

$$M_n = \frac{1}{2}\,h\int_h^\delta \frac{g_n(t)}{t(t+h)}\,\sin nt\,dt = o(1)$$

or equivalently

$$P_n \equiv \int_h^\delta \frac{g_n(t)}{t(t+h)}\,\sin nt\,dt = o(n).$$

Now

$$P_n = \left(\int_h^{2h} + \int_{2h}^\delta\right)\frac{g_n(t)}{t(t+h)}\,\sin nt\,dt = P_1 + P_2,$$

say. Here by the second mean value theorem

$$P_1 = \frac{1}{2h}\int_h^\tau \frac{g\,(t)}{t}\sin nt\,dt \quad (h < \tau < 2h)$$

$$= o(n).$$

Also

$$P_2 = \int_{2h}^\delta \frac{g_n(t)}{t(t+h)}\,\sin nt\,dt$$

$$= -\int_h^{\delta-h} \frac{g_n(t+h)}{(t+h)(t+2h)}\,\sin nt\,dt$$

$$= -\int_h^\delta \frac{g_n(t+h)}{(t+h)(t+2h)}\,\sin nt\,dt + o(1),$$

we obtain

$$P_2 = \frac{1}{2}\int_h^\delta\left\{\frac{g_n(t)}{t(t+h)} - \frac{g_n(t+h)}{(t+h)(t+2h)}\right\}\sin nt\,dt + o(1)$$

$$= \frac{1}{2}\int_h^\delta \frac{g_n(t)-g_n(t+h)}{(t+h)(t+2h)}\,\sin nt\,dt + h\int_h^\delta \frac{g_n(t)}{t(t+h)(t+2h)}\,\sin nt\,dt$$

$$+\,o(1)$$

$$= P_3 + hP_4 + o(1),$$

say. Here

$$|P_3| \leqq \frac{1}{6h}\int_h^\delta \frac{|g_n(t)-g_n(t+h)|}{t}\,dt \leqq \varepsilon\cdot\frac{1}{h} = o(n)$$

and since $t=0$ is the Lebesgue point, we have

$$|P_4| = o(n^2).$$

Thus we can completes the proof of the theorem.

3. THEOREM 2. *If,*

(3.1) $$f(t)-f(t') = o(1/\log_2|t-t'|^{-1}) \quad (t\to x,\ t'\to x),$$

and the n-th Fourier coefficients of $f(t)$ is of order $(\log n)^\alpha/n$, $(\alpha>0)$, *then the Fourier series of $f(t)$ converges uniformly at $t=x$.*

PROOF. We shall adopt the simplification in the proof of Theorem 1. Then we have

$$s_n(x_n) = \frac{1}{\pi}\int_0^\pi g_n(t)\frac{\sin nt}{t}\,dt + o(1)$$

$$= \frac{1}{\pi}\left(\int_0^{\pi/n}+\int_{\pi/n}^{\pi\beta_n}+\int_{\pi\beta_n}^{\pi}\right)g_n(t)\frac{\sin nt}{t}\,dt + o(1)$$

$$= \frac{1}{\pi}(I_n+J_n+K_n)+o(1),$$

say, where

(3.2) $$\beta_n = (\log n)^\beta/n, \quad (\alpha+1<\beta).$$

By the condition (3.1) we have easily $I_n = o(1)$.

After R. Salem, we write, putting $m = [(\log n)^\beta]$,

$$J_n = \int_{\pi/n}^{m\pi/n} g_n(t)\frac{\sin nt}{t}\,dt + o(1)$$

$$= \sum_{k=1}^{m-1}\int_{k\pi/n}^{(k+1)\pi/n} g_n(t)\frac{\sin nt}{t}\,dt + o(1)$$

$$= \sum_{k=1}^{m-1}\int_0^{\pi/n}(-1)^k g_n\left(u+\frac{k\pi}{n}\right)\frac{\sin nu}{u+k\pi/n}\,du + o(1)$$

$$= 2\sum_{k=1}^{m-1}\frac{(-1)^k}{n\theta+k\pi}\,g_n\left(\theta+\frac{k\pi}{n}\right) + o(1),$$

where $0\leqq\theta\leqq\pi/n$. If we replace $n\theta+2k\pi$ and $n\theta+(2k+1)\pi$ by $(2k+1)\pi$ in the last sum, the error is $o(1)$. Thus

$$J_n = 2\sum_{k=1}^{m-1}\frac{(-1)^k}{n\theta+k\pi}\left\{f\left(x_n+\frac{k\pi}{n}+\theta\right)+f\left(x_n-\frac{k\pi}{n}-\theta\right)\right\}+o(1)$$

$$= \frac{2}{\pi}\sum_{k=0}^{[(m-1)/2]}\frac{1}{2k+1}\left\{f\left(x_n+\frac{2(k+1)\pi}{n}+\theta\right)-f\left(x_n+\frac{2(k+1)\pi}{n}+\theta\right)\right\}$$

$$+\left\{f\left(x_n - \frac{2(k+1)\pi}{n} - \theta\right) - f\left(x_n - \frac{(2k+1)\pi}{n} - \theta\right)\right\} + o(1)$$

$$= o\left(\frac{1}{\log_2 n} \sum_{k=0}^{(m-2)/2} \frac{1}{2k+1}\right) = o\left(\frac{\beta \log_2 n}{\log_2 n}\right) = o(1).$$

Finaly, by the second mean value theorem and the order of a_n, we get

$$K_n = 2\sum_{m=1}^{\infty} a_m \cos mx_n \int_{\pi\beta_n}^{\pi} \frac{\cos mt \sin nt}{t}\, dt$$

$$= \sum_{m=1}^{\infty} a_m \cos mx_n \int_{\pi\beta_n}^{\pi} \{\sin(m+n)t - \sin(m-n)t\} \frac{dt}{t}$$

$$= o(1) + \frac{1}{\pi} \sum_{m=1}^{\infty}{}' \frac{n(\log m)^{\alpha}}{(\log n)^{\beta}\, |m-n|\, m}$$

$$= o(1) + \sum_{m=1}^{n-1} + \sum_{m=n+1}^{\infty} = o(1) + K_1 + K_2,$$

where $'$ denotes that the term $m = n$ is omitted. Now

$$\pi K_1 = o\left(\frac{n(\log n)^{\alpha+1}}{\log n)^{\beta} n}\right) = o(1),$$

since $\beta > \alpha + 1$. Analogously we get $K_2 = o(1)$, and this completes the proof.

4. THEOREM 3. *For any x, there is an integrable function f(t) such that*

(4.1) $$f(x+t) - f(x) = O(t) \quad (t \to 0),$$

and the n-th Fourier coefficients of f(t) are of order $n^{-\delta}$ $(0 < \delta < 1)$, *but the Fourier series of f(t) does not converge uniformly at* $t = x$.

PROOF. Let (n_k) be a sequence of integers such that

(4.2) $$n_{k+1} > e^{2n_k} \quad (k = 1, 2, \cdots)$$

and let (Δ_k) and (Δ_k') be sequences of intervals such that

(4.3) $$\Delta_k \equiv \left(\frac{\pi}{\log n_k} - \frac{[n_k^{1-\delta}]\pi}{n_k},\ \frac{\pi}{\log n_k}\right) \qquad (k = 1, 2, \cdots)$$

$$\Delta_k' \equiv \left(\frac{\pi}{\log n_k},\ \frac{\pi}{\log n_k} + \frac{[n_k^{1-\delta}]\pi}{n_k}\right).$$

Then they are disjoint systems.

Let us define an even function $f(t)$ by

$$f(t) = (-1)^k t \sin n_k(t - \pi/\log n_k) \quad \text{in } \Delta_k$$
$$= (-1)^{k+1} t \sin n_k(t - \pi/\log n_k) \quad \text{in } \Delta_k'$$

and $f(t) = 0$ in $(0, \pi) - \bigvee_{k=1}^{\infty} (\Delta_k \cup \Delta_k')$. Then we can easily see that $f(t)$ satisfies the condition (4.1) and that the Fourier coefficients of $f(t)$ are of order $n^{-\delta}$.

Let us now consider the n-th partial sum of the Fourier series of $f(t)$ at $t = x_n$ where $\{x_n\}$ will be determined later. Then, as in the proof of Theorem 1,

$$s_n(x_n) = \frac{1}{\pi}\left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi/n^{\delta/2}} + \int_{\pi/n^{\delta/2}}^{\pi}\right)[f(x_n+t)+f(x_n-t)]\frac{\sin nt}{t}\,dt + o(1)$$

$$= \frac{1}{\pi}\int_{\pi/n}^{\pi/n^{\delta/2}}[f(x_n+t)+f(x_n-t)]\frac{\sin nt}{t}\,dt + o(1)$$

$$\equiv \frac{1}{\pi}J_n + o(1),$$

say. Taking $n \equiv n_k$, $m = [n_k^{1-\delta/2}]$ and $x_{n_k} \equiv \pi/\log n_k$, we have

$$J_{n_k} = \sum_{\lambda=1}^{m-1}\int_{\lambda\pi/n_k}^{(\lambda+1)\pi/n_k}\left[f\left(\frac{\pi}{\log n_k}+t\right)+f\left(\frac{\pi}{\log n_k}-t\right)\right]\frac{\sin n_k t}{t}\,dt,$$

$$(-1)^{k+1}J_{n_k} = \frac{2\pi}{\log n_k}\int_0^{\pi/n_k}\sum_{\lambda=1}^{m-1}(-1)^{\lambda}\left[\sin n_k\left(\frac{\lambda\pi}{n_k}+t\right)\right.$$

$$\left. - \sin n_k\left(-\frac{\lambda\pi}{n_k}-t\right)\right]\frac{\sin n_k t}{t+\lambda\pi/n_k}\,dt$$

$$= \frac{2\pi}{\log n_k}\int_0^{\pi/n_k}\sum_{\lambda=0}^{(m-2)/2}\frac{n_k}{\lambda+1}\sin^2 n_k t\,dt + o(1)$$

$$= \frac{\text{const.}}{\log n_k}\sum_{\lambda=0}^{(m-2)/2}\frac{1}{\lambda+1} + o(1).$$

Thus (J_{n_k}) does not converge to zero. Hence $s_n(\pi/\log n)$ does not converge. Thus the theorem is proved.

5. THEOREM 4. *If*

(5.1) $\varphi(t) = f(x+t) + f(x-t) - 2f(x) \to 0$, *as* $t \to 0$,

(5.2) *the function* $\theta(t) = t\varphi(t)$ *is of bounded variation in an interval* $(0, \eta)$, *and*

(5.3) $$\int_0^h |d\theta(t)| \leq Ah,$$

for small h, *where* A *is a constant, then the Fourier series of* $f(x)$ *converges uniformly at the point* x *to the value* $f(x)$.

PROOF. We have

$$s_n(t_n) = \frac{1}{\pi}\int_0^{\pi}\varphi(t_n+t)\frac{\sin nt}{t}\,dt + o(1)$$

$$= \frac{1}{\pi}\left(\int_0^{k/n} + \int_{k/n}^{\eta} + \int_{\eta}^{\pi}\right)\varphi(t_n + t)\,\frac{\sin nt}{t}\,dt + o(1)$$

$$= \frac{1}{\pi}(P_n + Q_n + R_n) + o(1),$$

where k is a fixed number and $t_n \to 0$.

For any $\varepsilon > 0$, we shall take δ and k such that if $|t_n| < \delta$ and $|t| < k/n$, then $|\varphi(t_n + t)| < \varepsilon$.

Then $|P_n| \leqq \varepsilon \int_0^{k/n} n\,dt = o(1)$, and $|R_n| \leqq \int_\eta^\pi \frac{\varphi(t_n + t)}{t}\sin nt\,dt = o(1)$

uniformly for t_n by the generalized Riemann-Lebesgue theorem.

Put

$$\begin{aligned} \xi_n(t) &= \varphi(t_n + t)/t, \quad \text{for } k/n \leqq t \leqq \eta \\ &= 0, \quad \text{otherwise,} \end{aligned}$$

and if we prove that the total variation of $\xi_n(t)$ in $[0, \pi]$ is εn, then we shall have

$$|Q_n| \leqq \int_{k/n}^{\eta} \frac{\varphi(t_n + t)}{t}\ \sin nt\,dt = \frac{\varepsilon n}{n} = \varepsilon.$$

Since $\xi_n(k/n) = (n/k)\,\varphi(t_n + k/n) = o(n)$, it is enough to prove the same thing for the variation of $\xi_n(t)$ in $[k/n, \eta]$.

Let us write the total variation of $f(x)$ in $[a, b]$ by $V_a^b\,[f]$, then

$$V_{k/n}^{\eta}\left[\theta(t_n + t)\frac{1}{t(t_n + t)}\right] \leqq \int_{k/n}^{\eta} \frac{1}{t(t_n + t)}\,|d\theta(t_n + t)| + \int_{k/n}^{\eta} |\theta(t_n + t)|\left|d\frac{1}{t(t_n + t)}\right|$$

$$= S + T,$$

say. Since

$$|\theta(t + t_n)| = |\theta(t_n + t) - \theta(0)| \leqq V_0^{t_n + t}[\theta(t)] \leqq A(t_n + t),$$

we have

$$|T| \leqq \int_{k/n}^{\eta} |\theta(t + t_n)|\frac{t_n + 2t}{t^2(t_n + t)^2}\,dt \leqq A\int_{k/n}^{\eta} \frac{t_n + 2t}{t^2(t_n + t)}\,dt$$

$$\leqq 2A\int_{k/n}^{\eta}\frac{dt}{t^2} = \frac{\eta A n}{k} = \varepsilon n$$

for large k. Now

$$S = \left[\frac{V_0^t[\theta(t_n + t)]}{t(t_n + t)}\right]_{k/n}^{\eta} + \int_{k/n}^{\eta} \frac{V_0^t[\theta(t_n + t)]\,(t_n + 2t)}{t^2(t_n + t)^2}\,dt,$$

and

$$|S| \leqq A\varepsilon n + 2\int_{k/n}^{\eta} \frac{A(t_n + t)}{t^2(t_n + t)}\, dt = A\varepsilon n + \int_{k/n}^{\eta} \frac{2A}{t^2}\, dt$$

$$\leqq 4A\varepsilon n.$$

Thus we get the theorem.

REFERENCES

[1]. G. H. HARDY AND J. E. LITTLEWOOD, Some new convergence criteria for Fourier series, Annali di Pisa, 3 (1934), 41-62.

[2]. S. IZUMI, Notes on Fourier analysis (XVI), Tôhoku Math. Journ., 1 (1950), 144-166; Notes on Fourier analysis (XXVI), ibid. 2 (1950), 74-95.

[3]. R. SALEM, Sur un test général pour la convergence uniforme de séries de Fourier, Comptes Rendus Paris, 207 (1938), 662-665.

[4]. O. SZÁSZ, On uniform convergence of Fourier series, Bull. Amer. Math. Soc., 50 (1944), 587-595.

[5]. A. ZYGMUND, Trigonometrical series, Warszawa, 1935.

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