NOTE ON DIRICHLET SERIES (I)
ON THE SINGULARITIES OF DIRICHLET SERIES (I)

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1. Fundamental theorem I. Put

$$(1.1)\qquad F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s = \sigma + it,\ 0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty).$$

Let (1.1) be simply convergent for $\sigma > 0$. In this present Note, by the systematic method based upon A. Ostrowski's criterion of singularities, we shall study the relation between singularities of (1.1) and coefficients $\{a_n\}$. We begin with some definitions.

DEFINITION I. Let $\{a_n\}$ be real. We say that the sign-change occurs between $\{a_{n_k-1},\ a_{n_k}\}$, provided that

(i) $\qquad a_{n_k} \neq 0,\ a_{n_k-1} \neq 0$ and $a_{n_k} \cdot a_{n_k-1} < 0.$

or

(ii) $\qquad a_{n_k} \neq 0,\ a_{n_k-1} = a_{n_k-2} = a_{n_k-3} = \cdots = a_{n_k-\nu+1} = 0$ and $a_{n_k} \cdot a_{n_k-\nu} < 0.$

DEFINITION II. We call that the sequence of coefficients $\{a_n\}$ has the normal sign-change, provided that the sign-change occurs between $\{a_{n_k-1}, a_{n_k}\}$ $(k = 1, 2, \cdots)$ with $\varliminf_{k\to\infty} (\lambda_{n_k} - \lambda_{n_k-1}) > 0$.

DEFINITION III. We say that the sequence of coefficients $\{a_n\}$ has the normal sign-change in the sequence of intervals $\{I_k\}$ $(I_i \cdot I_j = 0,\ i \neq j)$, provided that the subsequence $\{a_{n_i}\}$ $(i = 1, 2, \cdots)$, whose exponent λ_{n_i} belongs to $\{I_k\}$ $(k = 1, 2, \cdots)$, has the normal sign-change in the sense of Definition 2.

Our fundamental theorem states as follows.

FUNDAMENTAL THEOREM I. *Let* (1.1) *be simply convergent for* $\sigma > 0$. *Then* $s = 0$ *is the singular point for* (1.1), *provided that there exist two sequences* $\{x_k\}$ $(0 < x_k \uparrow \infty)$, $\{\gamma_k\}$ $(\gamma_k$ *: real*) *such that*

(a) $$\varlimsup_{k\to\infty} 1/x_k \cdot \log \left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re(a_n \cdot \exp(-i\gamma_k)) \right| = 0,\text{[1]}$$

(b) $\lim_{k\to\infty} \sigma_k/[x_k] = 0$, *where* σ_k: *the number of sign-changes of* $\Re(a_n \exp(-i\gamma_k))$, $\lambda_n \in I_k[[x_k](1-\omega),\ [x_k](1+\omega)]$ $(0 < \omega < 1)$,

(c) *the sequence* $\Re(a_n \exp(-i\gamma_k))$ $(\lambda_n \in \{I_k\})$ *has the normal sign-change in* $\{I_k\}$ $(k = 1, 2, \cdots)$.

2. Lemmas. For its proof, we need some lemmas.

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1) $[x]$ is the greatest integer contained in x.

singular point, provided that there exist two sequences $\{x_k\}$, $\{\gamma_k\}$ (γ_k : *real*) *such that*

$$(a)\qquad \overline{\lim_{k\to\infty}}\, 1/x_k \cdot \log\left|\sum_{[x_k]\leqq\lambda_n<x_k} a_n\right| = 0,$$

$$(b)\qquad \Re(a_n\exp(-i\gamma_k))\geqq 0 \text{ for } \lambda_n\in[[x_k](1-\omega),[x_k](1+\omega)] \quad (k=1,2,\cdots)\quad (0<\omega<1),$$

$$(c)\qquad \lim_{\substack{n\to\infty\\ \lambda_n\in[[x_k],x_k]\ (k=1,2,\cdots)}} [\cos\{\arg(a_n\exp(-i\gamma_k))\}]^{1/\lambda_n} = 1.$$

PROOF. The assumptions *(b)* and *(c)* of the Fundamental Theorem 1 are evidently satisfied. Hence, it is sufficient to prove that

$$(6.1)\qquad \Delta = \overline{\lim_{k\to\infty}}\, 1/x_k \cdot \log\left|\sum_{[x_k]\leqq\lambda_n<x_k} \Re(a_n\exp(-i\gamma_k))\right| = 0.$$

By T. Kojima's theorem,

$$\Delta \leqq \overline{\lim_{k\to\infty}}\, 1/x_k \cdot \log\left|\sum_{[x_k]\leqq\lambda_n<x_k} a_n\right| \leqq \overline{\lim_{x\to\infty}}\, 1/x\cdot\left|\sum_{[x]\leqq\lambda_n<x} a_n\right| = 0,$$

so that

$$(6.2)\qquad \Delta\leqq 0.$$

On other hand, by *(b)* we have

$$1/x_k\cdot\log\left|\sum_{[x_k]\leqq\lambda_n<x_k}\Re(a_n\exp(-i\gamma_k))\right| = 1/x_k\cdot\log\left\{\sum_{[x_k]\leqq\lambda_n<x_k}|a_n|\cos(\theta_n-\gamma_k)\right\}$$

$$\geqq 1/x_k\cdot\log\left\{\cos(\theta_{n_k}-\gamma_k)\sum_{[x_k]\leqq\lambda_n<x_k}|a_n|\right\}\qquad (\theta_n=\arg a_n),$$

where $\cos(\theta_{n_k}-\gamma_k)=\underset{[x_k]\leqq\lambda_n<x_k}{\mathrm{Min}}[\cos(\theta_n-\gamma_k)]$. Hence, by *(a)* and *(c)*,

$$\Delta\geqq\lim_{k\to\infty}\lambda_{n_k}/x_k\cdot 1/\lambda_{n_k}\cdot\log\{\cos(\theta_{n_k}-\gamma_k)\}+\overline{\lim_{k\to\infty}}\,1/x_k\cdot\log\left|\sum_{[x_k]\leqq\lambda_n<x_k}a_n\right|=0,$$

so that

$$(6.3)\qquad \Delta\geqq 0.$$

By (6.2) and (6.3), $\Delta=0$. q. e. d.

Putting $\gamma_k\equiv 0$ ($k=1,2,\cdots$) in Theorem 2, we get

COROLLARY III. *Let* (1.1) *be simply convergent for* $\sigma>0$. *Then,* $s=0$ *is the singular point, provided that there exist two sequences* $\{x_k\}$, $\{\gamma_k\}$ *such that*

$$(a)\qquad \overline{\lim_{k\to\infty}}\,1/x_k\cdot\log\left|\sum_{[x_k]\leqq\lambda_n<x_k}a_n\right|=0,$$

$$(b)\qquad \Re(a_n)\geqq 0 \text{ for } \lambda_n\in[[x_k](1-\omega),[x_k](1+\omega)]\ (k=1,2,\cdots;0<\omega<1),$$

$$(c)\qquad \lim_{\substack{n\to\infty\\ \lambda_n\in[[x_k],x_k](k=1,2,\cdots)}}[\cos\{\arg(a_n)\}]^{1/\lambda_n}=1.$$

COROLLARY IV (C. Biggeri, [5.] pp. 979–980). *Let* (1.1) *be simply convergent for* $\sigma > 0$. *If* $\Re(a_n) \geqq 0$ *for* $n \geqq N$ *and* $\lim_{n\to\infty} [\cos\{\arg a_n\}]^{1/\lambda_n} = 1$, *then* $s = 0$ *is the singular point.*

By T. Kojima's theorem, the simple convergence-abscissa of (1.1) is determined by

$$\varlimsup_{x\to\infty} 1/x \cdot \log \left| \sum_{[x]\leqq\lambda_n<x} a_n \right| = 0.$$

Hence, there exists at least one sequence $\{x_k\}$ such that

$$\varlimsup_{k\to\infty} 1/x_k \cdot \log \left| \sum_{[x_k]\leqq\lambda_n<x_k} a_n \right| = 0.$$

Taking this sequence $\{x_k\}$, the assumptions of Corollary 3 are all satisfied, so that Corollary 4 follows immediately from Corollary 3.

COROLLARY V (M. Fekete, [3] p. 81). *Let* (1.1) *be simply convergent for* $\sigma > 0$. *If* $|\arg a_n| \leqq \theta < \pi/2$ *for* $n \geqq N$, *then* $s = 0$ *is the singular point.*

By $|\arg a_n| \leqq \theta < \pi/2$, we have evidently

$$\Re(a_n) \geqq 0, \ \cos\theta \leqq \cos\{\arg(a_n)\} \leqq 1 \quad \text{for } n \geqq N,$$

so that

$$\lim_{n\to\infty} [\cos\{\arg a_n\}]^{1/\lambda_n} = 1.$$

Hence, we obtain Corollary 5 from Corollary 4.

7. Theorem III. In this section, we shall prove some theorems of Fabry's type concerning the singularities of Dirichlet series, by virtue of Fundamental Theorem 2 established in the previous Note ([1]). Put

$$(1.1) \qquad F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s = \sigma + it, \ 0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty).$$

First we shall prove

THEOREM III. *Let* (1.1) *be simply convergent for* $\sigma > 0$. *Then* $\sigma = 0$ *is the natural boundary for* (1.1), *provided that there exists a subsequnce* $\{\lambda_{n_k}\}$ $(k = 1, 2, \cdots)$ *such that*

(*a*) $\varlimsup_{k\to\infty} 1/\lambda_{n_k} \cdot \log |a_{n_k}| = 0$,

(*b*) $\lim_{k\to\infty} s_k/\lambda_{n_k} = 0$, *where* s_k : *the number of* $a_n \neq 0$,

$\lambda_n \in I_k [[\lambda_{n_k}](1-\omega), \ [\lambda_{n_k}](1+\omega)] \quad (k = 1, 2, \cdots; 0 < \omega < 1)$,

(*c*) $\lim_{\substack{n\to\infty \\ \lambda_n, \lambda_{n+1} \in I_k (k=1,2,\cdots)}} (\lambda_{n+1} - \lambda_n) > 0$.

PROOF. Putting $\gamma_k = \arg(a_{n_k})$ $(k = 1, 2, \cdots)$, by (*a*) we get

$$(7.2) \qquad \varlimsup_{k\to\infty} 1/\lambda_{n_k} \cdot \log |\Re(a_{n_k} \exp(-i\gamma_k))| = 0.$$

Denote by σ_k the number of sign-change of $\Re(a_n \exp(-i\gamma_k))$ $(\lambda_n \in I_k)$. Since

$0 \leqq \sigma_k/[\gamma_{n_k}] \leqq s_k/[\gamma_{n_k}]$, by *(b)* we have

$$\lim_{k\to\infty} \sigma_k/[\lambda_{n_k}] = 0. \tag{7.3}$$

By *(c)*, the sign-change of $\Re(a_n \exp(-i\gamma_k))$ $(\lambda_n \in I_k)$ is evidently normal. Thus, all the assumptions of Fundamental Theorem 2 are satisfied. Hence $s = 0$ is the singular point for (1.1). By the transformation $s = s' + it$ and the same arguments as above, $s = it$ is singular for (1.1). This proves our theorem.

As a consequence of Theorem 1, we can prove

COROLLARY VI (F. Carlson-E. Landau, O. Szàsz, [6], [7], [3] p. p. 140–141). *Let* (1.1) *be simply convergent for* $\sigma > 0$. *If* $\underline{\lim}_{n\to\infty} (\lambda_{n+1} - \lambda_n) > 0$ *and* $\lim_{n\to\infty} n/\lambda_n = 0$, *then* $\sigma = 0$ *is the natural boundary.*

PROOF. Since evidently $\lim_{n\to\infty} \log n/\lambda_n = 0$, by G. Valiron's theorem ([4] p. 4) the simple convergence-abscissa of (1.1) is determined by $\overline{\lim}_{n\to\infty} 1/\lambda_n \cdot \log|a_n| = 0$. Denoting by $N(r)$ the number of λ_n's contained in $[0, r]$, by $\lim_{n\to\infty} n/\lambda_n = 0$ we have

$$N(r) = o(\lambda_{N(r)}) = o(r).$$

Hence,

$$0 \leqq s_n/[\lambda_n] \leqq N([\lambda_n](1+\omega))/[\lambda_n](1+\omega) \cdot (1+\omega) \to 0$$

as $n \to \infty$, where s_n: the number of $a_i \neq 0$, $\lambda_i \in I_n[[\lambda_n](1-\omega), [\lambda_n](1+\omega)]$. Thus, all assumptions of the theorem are satisfied, so that $\sigma = 0$ is the natural boundary. q. e. d.

8. Theorem IV. Here we shall prove

THEOREM IV. *Let* (1. 1) *be simply convergent for* $\sigma > 0$. *Then,* $s = 0$ *is the singular point, provided that there exists a subsequence* $\{\lambda_{n_k}\}$ $(k = 1, 2, \cdots)$ *such that*

(*a*) $\overline{\lim}_{k\to\infty} 1/\lambda_{n_k} \cdot \log|a_{n_k}| = 0$,

(*b*) $\lim_{\substack{n\to\infty \\ \lambda_n, \lambda_{n+1} \in I_k (k=1,2,\cdots)}} (\varphi_{n+1} - \varphi_n) = 0$,

where $\varphi_n = \arg(a_n)$, *and* $I_k[[\lambda_{n_k}](1-\omega), [\lambda_{n_k}](1+\omega)]$ $(k = 1, 2, \cdots)$,

(*c*) $\lim_{\substack{n\to\infty \\ \lambda_n, \lambda_{n+1} \in I_k (k=1,2,\cdots)}} (\lambda_{n+1} - \lambda_n) > 0$.

From this theorem immediately follows

COROLLARY VII. *Let* (1. 1) *be simply convergent for* $\sigma > 0$. *If* $\underline{\lim}_{n\to\infty} (\lambda_{n+1} - \lambda_n) > 0$ *and* $\lim_{n\to\infty} (\varphi_{n+1} - \varphi_n) = 0$, $\varphi_n = \arg(a_n)$, *then* $s = 0$ *is the singular point.*

In fact, by G. Valiron's theorem, we have $\overline{\lim}_{n\to\infty} 1/\lambda_n \cdot \log|a_n| = 0$. Hence, all hypotheses of Theorem 4 are satisfied, so that we get Corollary 7 by Theorem 4. In the case of Taylor series, Theorem 4 was proved by E. Fabry ([4] p. 84).

PROOF OF THEOREM 4. Taking account of the Fundamental Theorem 2, it is sufficient to prove the existence of the sequence $\{\gamma_k\}$ (γ_k: real) such that

$$\lim_{k\to\infty} 1/\lambda_{n_k} \cdot \log |\Re(a_{n_k} \exp(-i\gamma_k))| = 0, \quad \lim_{k\to\infty} \sigma_k/[\lambda_{n_k}] = 0,$$

where σ_k: the number of sign-changes of $\Re(a_n \exp(-i\gamma_k))$, $\lambda_n \in I_k [[\lambda_{n_k}] \cdot (1-\omega), [\lambda_{n_k}](1+\omega)]$ $(k = 1, 2, \cdots)$. By hypothesis (b), there exists a positive integer $\mu(k)$ such that

(8.1) $$\operatorname*{Max}_{\lambda_n, \lambda_{n+1} \in I_k} |\varphi_{n+1} - \varphi_n| \leqq 1/\mu(k), \quad \lim_{k\to\infty} \mu(k) = \infty.$$

Let us divide the periphery of the unit-circle into $4\mu(k)$ equal parts in such manner that each dividing point does not coincide with $\exp(i\varphi_n)$ $(\lambda_n \in I_k)$. Since $2\pi/4\mu > 1/\mu$, each arc $(\exp(i\varphi_n), \exp(i\varphi_{n+1}))$ $(\lambda_n, \lambda_{n+1} \in I_k)$ contains at most one dividing point. By (c) there exists h such that

$$\lambda_{n+1} - \lambda_n > h > 0 \quad \text{for } \lambda_n, \lambda_{n+1} \in I_k \quad (k = 1, 2, \cdots).$$

Hence, the number of arcs $(\exp(i\varphi_n), \exp(i\varphi_{n+1}))$ $(\lambda_n, \lambda_{n+1} \in I_k)$ is at most

(8.2) $$2\omega [\lambda_{n_k}]/h.$$

Since $\mu \times 2[\lambda_{n_k}]/\mu h > 2\omega[\lambda_{n_k}]/h$, by (8.2) among $\mu(k)$ quadrates we have one quadrate R_k, whose summits touch at most $2[\lambda_{n_k}]/\mu h$ arcs $(\exp(i\varphi_n), \exp(i\varphi_{n+1}))$ $(\lambda_n, \lambda_{n+1} \in I_k)$. Then we can choose a suitable summit $\exp(i\gamma_k)$ such that

(8.3) $$|\varphi_{n_k} - \gamma_k| \leqq \pi/4.$$

Denoting by σ_k the number of sign-changes of $\Re(a_n \exp(-i\gamma_k))$ $(\lambda_n \in I_k)$, we have evidently

$$0 \leqq \sigma_k \leqq 2[\lambda_{n_k}]/\mu(k)\, h.$$

Therefore, by (8.1)

$$\lim_{k\to\infty} \sigma_k/[\lambda_{n_k}] = 0.$$

On the other hand, by (8.3)

$$\Re(a_{n_k} \exp(-i\gamma_k)) = |a_{n_k}| \cos(\varphi_{n_k} - \gamma_k) \geqq 1/\sqrt{2} \cdot |a_{n_k}|,$$

so that

(8.4) $$\varlimsup_{k\to\infty} 1/\lambda_{n_k} \cdot \log |\Re(a_{n_k} \exp(-i\gamma_k))| \geqq \varlimsup_{k\to\infty} 1/\lambda_{n_k} \cdot \log |a_{n_k}| = 0.$$

By $|\Re(a_{n_k} \exp(-i\gamma_k))| \leqq |a_{n_k}|$, we get evidently

$$\varlimsup_{k\to\infty} 1/\lambda_{n_k} \cdot \log |\Re(a_{n_k} \exp(-i\gamma_k))| \leqq \varlimsup_{k\to\infty} 1/\lambda_{n_k} \cdot \log |a_{n_k}| = 0.$$

Hence, by (8.4)

$$\varlimsup_{k\to\infty} 1/\lambda_{n_k} \cdot \log |\Re(a_{n_k} \exp(-i\gamma_k))| = 0.$$

Thus, $\{\gamma_k\}$ is the desired one. q. e. d.

(to be continued)

REFERENCES

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NOTES ON FOURIER ANALYSIS (XLVIII): UNIFORM CONVERGENCE OF FOURIER SERIES

SHIN-ICHI IZUMI AND GEN-ICHIRÔ SUNOUCHI

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1. G. H. Hardy and J. E. Littlewood [1] proved that

THEOREM A. *If*

$$f(x+t)-f(x)=o\left(1\Big/\log\frac{1}{|t|}\right)\quad(t\to 0) \tag{1.1}$$

and the n-th Fourier coefficients of $f(t)$ *are of order* $n^{-\delta}$ $(0<\delta<1)$, *then the Fourier series of* $f(t)$ *converges at* $t=x$.

Recently O. Szász [4] proved that

THEOREM B. *If* $f(t)$ *is even (or odd) and continuous, and if*

$$\lim_{\lambda\downarrow 0}\limsup_{n\to\infty}\sum_{n}^{\lambda n}(|a_\nu|-a_\nu)=0, \tag{1.2}$$

a_ν *being the n-th Fourier cosine (or sine) coefficient of* $f(t)$, *then the Fourier series of* $f(t)$ *converges uniformly at* $t=x$. *Especially if* a_n *is of order* n^{-1}, *then* (1,2) *is valid.*

In the assumption of these theorems, the first is the continuity condition and the second is the Tauberian condition. We shall prove that the assumption of Theorem A is not sufficient to the uniform convergence of the Fourier series of $f(t)$ at $t=x$. Further, even if (1.1) is replaced by the condition

$$f(x+t)-f(x)=O(|t|)\quad(t\to 0) \tag{1.3}$$

in the assumption of Theorem A, the Fourier series of $f(t)$ does not converge uniformly at $t=x$ in general. But we can prove that, if, instead of (1.1)

$$f(t)-f(t')=o\left(1\Big/\log\frac{1}{|t-t'|}\right)\ (t\to x,\ t'\to x) \tag{1.4}$$

or, if

$$f(t)-f(t')=o\left(1\Big/\log_2\frac{1}{|t-t'|}\right)\ (t\to x,\ t'\to x) \tag{1.5}$$

and the n-th Fourier coefficients of $f(t)$ is of order $(\log n)^\alpha/n$ $(\alpha>0)$, then the Fourier series of $f(t)$ converges uniformly at $t=x$. The condition (1.4) is the type of Dini-Lipschitz test, and (1.5) links Theorem B and this test.

To prove the negative theorem, we construct an example of the type used by one of the authors [2]. For the proof of the positive theorem we use the method due to H. Lebesgue and R. Salem [3].

On the other hand, we can prove that Young's convergence test implies the uniform convergence of Fourier series at a point. This is a dual of