NOTE ON DIRICHLET SERIES (I)
ON THE SINGULARITIES OF DIRICHLET SERIES (I)

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1. Fundamental theorem I. Put

$$(1.1)\qquad F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \quad (s = \sigma + it,\ 0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty).$$

Let (1.1) be simply convergent for $\sigma > 0$. In this present Note, by the systematic method based upon A. Ostrowski's criterion of singularities, we shall study the relation between singularities of (1.1) and coefficients $\{a_n\}$. We begin with some definitions.

DEFINITION I. Let $\{a_n\}$ be real. We say that the sign-change occurs between $\{a_{n_k-1},\ a_{n_k}\}$, provided that

(i) $\qquad a_{n_k} \neq 0,\ a_{n_k-1} \neq 0$ and $a_{n_k} \cdot a_{n_k-1} < 0.$

or

(ii) $\qquad a_{n_k} \neq 0,\ a_{n_k-1} = a_{n_k-2} = a_{n_k-3} = \cdots = a_{n_k-\nu+1} = 0$ and $a_{n_k} \cdot a_{n_k-\nu} < 0.$

DEFINITION II. We call that the sequence of coefficients $\{a_n\}$ has the normal sign-change, provided that the sign-change occurs between $\{a_{n_k-1}, a_{n_k}\}$ $(k = 1, 2, \cdots)$ with $\varliminf_{k\to\infty} (\lambda_{n_k} - \lambda_{n_k-1}) > 0.$

DEFINITION III. We say that the sequence of coefficients $\{a_n\}$ has the normal sign-change in the sequence of intervals $\{I_k\}$ $(I_i \cdot I_j = 0,\ i \neq j)$, provided that the subsequence $\{a_{n_i}\}$ $(i = 1, 2, \cdots)$, whose exponent λ_{n_i} belongs to $\{I_k\}$ $(k = 1, 2, \cdots)$, has the normal sign-change in the sense of Definition 2.

Our fundamental theorem states as follows.

FUNDAMENTAL THEOREM I. *Let* (1.1) *be simply convergent for* $\sigma > 0$. *Then* $s = 0$ *is the singular point for* (1.1), *provided that there exist two sequences* $\{x_k\}$ $(0 < x_k \uparrow \infty)$, $\{\gamma_k\}$ $(\gamma_k :$ *real*) *such that*

(*a*) $\qquad \overline{\lim_{k\to\infty}}\ 1/x_k \cdot \log \left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re(a_n \cdot \exp(-i\gamma_k)) \right| = 0,$[1]

(*b*) $\lim_{k\to\infty} \sigma_k/[x_k] = 0$, *where* σ_k: *the number of sign-changes of* $\Re(a_n \exp(-i\gamma_k))$, $\lambda_n \in I_k[[x_k](1-\omega),\ [x_k](1+\omega)]$ $(0 < \omega < 1)$,

(*c*) *the sequence* $\Re(a_n \exp(-i\gamma_k))$ $(\lambda_n \in \{I_k\})$ *has the normal sign-change in* $\{I_k\}$ $(k = 1, 2, \cdots)$.

2. Lemmas. For its proof, we need some lemmas.

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1) $[x]$ is the greatest integer contained in x.

LEMMA 1. *Under the assumptions* (*b*) *and* (*c*), *we have*

(i) $$\varliminf_{\nu\to\infty} (r_{\nu+1} - r_\nu) > 0, \quad \varliminf_{\nu, n\to\infty} |r_\nu - \lambda_n| > 0,$$

(ii) $$\lim_{\nu\to\infty} \nu/r_\nu = 0,$$

provided that $\{r_\nu\}$ *is the sequence arranged in the order of magnitude of* $\{1/2\cdot(\lambda_n + \lambda_{n-1})\}$, *where between* $\Re(a_n \exp(-i\gamma_k))$ *and* $\Re(a_{n-1}\exp(-i\gamma_k))$ ($\lambda_n, \lambda_{n-1} \in I_k$; $k = 1, 2, \cdots$) *the sign-change occurs.*

PROOF. On account of (*c*), (i) is evident. Taking suitable subsequence, if necessary, we can suppose that

$$[x_{k+1}] < 2[x_k]\cdot(1+\omega)/(1-\omega), \quad (k = 1, 2, \cdots).$$

Accordingly,

(2.1) $$1/2\cdot[x_{k+1}] > [x_k]$$
$$[x_{k+1}](1-\omega) > [x_k](1+\omega), \text{ so that } I_i\cdot I_j = 0, i \neq j.$$

By virtue of (*b*), for any given ε (>0), there exists $k(\varepsilon)$ such that

(2.2) $$\sigma_{k+r} < \varepsilon/4\cdot[x_{k+r}] \quad \text{for } r \geqq 0.$$

Hence by (2.2) and (2.1),

(2.3) $$\sum_{i=0}^{r}\sigma_{k+i} < \varepsilon/4\cdot\sum_{i=0}^{r}[x_{k+i}] < \varepsilon/4\cdot[x_{k+r}]\sum_{i=0}^{r} 1/2^i < \varepsilon/2\cdot[x_{k+r}].$$

For sufficiently large $r \geqq r(\varepsilon)$, we have evidently

(2.4) $$\sum_{i=1}^{k-1}\sigma_i < \varepsilon/2\cdot[x_{k+r}], \quad r \geqq r(\varepsilon).$$

By (2.3) and (2.4).

$$1/[x_m]\cdot\sum_{i=1}^{m}\sigma_i < \varepsilon \qquad \text{for } m \geqq k(\varepsilon) + r(\varepsilon),$$

so that

(2.5) $$\lim_{m\to\infty} 1/[x_m]\cdot\sum_{i=1}^{m}\sigma_i = 0.$$

If $r_\nu \in I_k$, $\nu = \sigma_1 + \sigma_2 + \cdots \sigma_{k-1} + \sigma_k'$, $\sigma_k' \leqq \sigma_k$. Therefore,

$$0 < \nu/r_\nu \leqq 1/[x_k](1-\omega)\cdot\sum_{i=1}^{k}\sigma_i \to 0 \quad \text{as } \nu\to\infty,$$

so that

$$\lim_{\nu\to\infty} \nu/r_\nu = 0. \qquad \text{q.e.d.}$$

LEMMA 2. *Put* $\varphi(z) = \prod_{\nu=1}^{\infty}(1 - z^2/r_\nu^2)$. *For any given* ε (>0), *we kave*

(i) $$|\varphi(z)| < \exp(\varepsilon|z|) \qquad \textit{for } |z| > R_1(\varepsilon),$$

(ii) $$\exp(-\varepsilon\lambda_n) < |\varphi(\lambda_n)| < \exp(\varepsilon\lambda_n) \qquad \textit{for } \lambda_n > R_2(\varepsilon).$$

By Lemma 1 and Carlson-Landau's theorem ([1], [2] p. 271)[2]), we have easily Lemma 2.

LEMMA 3. *Put*

$$G(s)=\sum_{n=1}^{\infty} a_n\varphi(\lambda_n)\exp(-\lambda_n s). \tag{2.6}$$

Then, $G(s)$ is also simply convergent for $\sigma>0$.

O. Szàsz ([3] p. 102) proved this lemma under the condition $\lim_{n\to\infty}\log n/\lambda_n=0$, but S. Izumi ([4] p. 513) showed that this condition is not necessary. Here for the completeness, we shall give its new proof.

PROOF. By T. Kojima's theorem ([5] p. 3), the simple convergence-abscisses σ_F, σ_G of (1.1) and (2.6) are given by

$$\begin{aligned}\sigma_F&=\overline{\lim_{x\to\infty}}\ 1/x\cdot\log\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big|=0,\\ \sigma_G&=\overline{\lim_{x\to\infty}}\ 1/x\cdot\log\Big|\sum_{[x]\leqq\lambda_n<x} a_n\varphi(\lambda_n)\Big|.\end{aligned}\tag{2.7}$$

By (2.7), for any given $\varepsilon\,(>0)$, we have

$$\Big|\sum_{[x]\leqq\lambda_n<x} a_n\Big|<\exp(\varepsilon[x])\qquad\text{for } [x]>N_1(\varepsilon). \tag{2.8}$$

By Abel's transformation,

$$\sum_{[x]\leqq\lambda_n<x} a_n\varphi(\lambda_n)=\sum_{n=n_1}^{n_2} a_n\varphi(\lambda_n)=\sum_{n=n_1}^{n_2-1}\{\varphi(\lambda_n)-\varphi(\lambda_{n+1})\}\Big(\sum_{i=n_1}^{n} a_i\Big)+\varphi(\lambda_{n_2})\Big(\sum_{i=n_1}^{n_2} a_i\Big),$$

so that by (2.8) and Lemma 2,

$$\begin{aligned}\Big|\sum_{[x]\leqq\lambda_n<x} a_n\varphi(\lambda_n)\Big|&\leqq\exp(\varepsilon[x])\Big\{\sum_{n=n_1}^{n_2-1}\int_{\lambda_n}^{\lambda_{n+1}}|\varphi'(t)|\,dt+|\varphi(\lambda_{n_2})|\Big\}\\ &<\exp(\varepsilon[x])\Big\{\int_{[x]}^{x}|\varphi'(t)|\,dt+\exp(\varepsilon x)\Big\}\quad\text{for } [x]>\mathrm{Max}\,\{R_2,N_1\}.\end{aligned}\tag{2.9}$$

Since $\varphi'(z)$ has the same order and type as $\varphi(z)$, for any given $\varepsilon\,(>0)$,

$$|\varphi'(z)|<\exp(\varepsilon|z|)\qquad\text{for } |z|>R_3(\varepsilon). \tag{2.10}$$

Hence, by (2.9) and (2.10)

$$\Big|\sum_{[x]\leqq\lambda_n<x} a_n\varphi(\lambda_n)\Big|<2\exp(\varepsilon([x]+x))\leqq 2\exp(2\varepsilon x),$$

so that by (2.7) $\sigma_G\leqq 2\varepsilon$. Letting $\varepsilon\to 0$,

$$\sigma_G\leqq 0. \tag{2.11}$$

2) Vide references placed at the end.

By (2.7), for any given $\varepsilon\ (>0)$, we obtain

$$\left|\sum_{[x]\leqq\lambda_r<x} a_n\varphi(\lambda_n)\right| < \exp((\sigma_G+\varepsilon)[x]) \quad \text{for } [x] > N_2(\varepsilon). \tag{2.12}$$

By Abel's transformation,

$$\sum_{[x]\leqq\lambda_n<x} a_n = \sum_{n=n_1}^{n_2} a_n\varphi(\lambda_n)\cdot 1/\varphi(\lambda_n) = \sum_{n=n_1}^{n_2-1}\{1/\varphi(\lambda_n)-1/\varphi(\lambda_{n+1})\}$$
$$\cdot\left(\sum_{i=n_1}^{n} a_i\varphi(\lambda_i)\right) + 1/\varphi(\lambda_{n_2})\cdot\left(\sum_{i=n_1}^{n_2} a_i\varphi(\lambda_i)\right),$$

so that by (2.12), (2.10) and Lemma 2,

$$\left|\sum_{[x]\leqq\lambda_n<x} a_n\right| \leqq \exp((\sigma_G+\varepsilon)[x])\left\{\exp(2\varepsilon x)\int_{[x]}^{x}|\varphi'(t)|\,dt + \exp(\varepsilon x)\right\}$$
$$< 2\exp((\sigma_G+\varepsilon)[x]+3\varepsilon x).$$

Hence,

$$0=\sigma_F\leqq\sigma_G+4\varepsilon.$$

Letting $\varepsilon\to 0$,

$$0\leqq\sigma_G. \tag{2.13}$$

By (2.11) and (2.13), $\sigma_G=\sigma_F=0$. q. e. d.

LEMMA 4 (A. Ostrowski, [6], [2] pp. 12–16). *For* $s=0$ *to be singular point for* (1.1), *it is necessary and sufficient that we have*

$$\overline{\lim_{m\to\infty}}\,|O_m(\sigma,\,\omega\,;F)|^{1/m}\geqq 1,$$

where

$$O_m(\sigma,\omega\,;\,F) = \sum_{\frac{m}{\sigma}(1-\omega)\leqq\lambda_n\leqq\frac{m}{\sigma}(1+\omega)} a_n\cdot(\lambda_n\sigma e/m)^m\cdot\exp(-\lambda_n\sigma)$$

$$(\sigma>0,\ \ 0<\omega<1)$$

LEMMA 5. *If* $s=0$ *is the regular point of* (1.1), *then* (2.6) *is regular at* $s=0$.

From Lemma 2 and Cramer-Ostrowski's theorem ([2] pp. 49–52, [7], [8]), immediately follows Lemma 5.

3. Proof of fundamental theorem I. Since $\Re(a_n\exp(-i\gamma_k))\,\varphi(\lambda_n)$, $\lambda_n\in I_k$ $(k=1,2,\cdots)$ has the same sign by Lemma 1 and Lemma 2, we have easily

$$|O_{[r_k]}(1,\,\omega\,;G)| = |\exp(-i\gamma_k)\,O_{[r_k]}(1,\,\omega\,;G)|$$
$$\geqq\left|\sum_{\lambda_n\in I_k}\Re(a_n\exp(-i\gamma_k))\,\varphi(\lambda_n)\,(\lambda_n e/[x_k])^{[r_k]}\exp(-\lambda_n)\right| \tag{3.1}$$
$$\geqq\left|\sum_{[r_k]\leqq\lambda_n<r_k}\Re(a_n\exp(-i\gamma_k))\varphi(\lambda_n)\,(\lambda_n e/[x_k])^{[r_k]}\exp(-\lambda_n)\right|$$

$$> \left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re (a_n \exp(-i\gamma_k)) \varphi(\lambda_n) \right| e^{-1}.$$

On the other hand, by Lemma 2,

$$\left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re (a_n \exp(-i\gamma_k)) \right| = \left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re (a_n \exp(-i\gamma_k)) \varphi(\lambda_n) \cdot 1/\varphi(\lambda_n) \right|$$

$$\leqq \left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re (a_n \exp(-i\gamma_k)) \varphi(\lambda_n) \right| \exp(\varepsilon x_k),$$

so that

$$(3.2) \quad \left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re(a_n \exp(-i\gamma_k))\varphi(\lambda_n) \right| \geqq \exp(-\varepsilon x_k) \left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re(a_n \exp(-i\gamma_k)) \right|.$$

By (3.1) and (3.2),

$$|O_{[x_k]}(1, \omega; G)| \geqq \exp(-1-\varepsilon x_k) \left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re (a_n \exp(-i\gamma_k)) \right|.$$

Hence, by the assumption (a),

$$\overline{\lim_{k\to\infty}} |O_{[x_k]}(1, \omega; G)|^{1/[x_k]} \geqq \overline{\lim_{k\to\infty}} \left| \sum_{[x_k] \leqq \lambda_n < x_k} \Re(a_n \exp(-i\gamma_k)) \right|^{1/[x_k]}$$

$$\cdot \underline{\lim}_{k\to\infty} \exp(-(1+\varepsilon x_k)/[x_k]) = 1 \cdot \exp(-\varepsilon).$$

Letting $\varepsilon \to 0$, $\overline{\lim\limits_{k\to\infty}} |O_{[x_k]}(1, \omega; G)|^{1/[x_k]} \geqq 1$. Therefore,

$$\overline{\lim_{m\to\infty}} | O_m(1, \omega; G)|^{1/m} \geqq \overline{\lim_{k\to\infty}} | O_{[x_k]}(1, \omega; G)^{1/[x_k]} \geqq 1.$$

Thus, by Lemma 4, $s=0$ is singular for (2.6), so that by Lemma 5, $s=0$ is also singular for (1.1). q.e.d.

4. Fundamental theorem II. The next theorem is more suitable for the application than the Fundamental Theorem 1.

FUNDAMENTAL THEOREM II. *Let* (1.1) *be simply convergent for* $\sigma > 0$. *Then* $s=0$ *is the singular point for* (1.1), *provided that there exist two sequences* $\{\lambda_{n_k}\}$, $\{\gamma_k\}$, (γ_k : *real*) *such that*

(a) $\overline{\lim\limits_{k\to\infty}}\, 1/\lambda_{n_k} \cdot \log |\Re (a_{n_k} \exp(-i\gamma_k)| = 0$,

(b) $\lim\limits_{k\to\infty} \sigma_k/[\lambda_{n_k}] = 0$, *where* σ_k: *the number of sign-changes of* $\Re (a_n \exp(-i\gamma_k))$, $\lambda_n \in I_k\,[[\lambda_{n_k}](1-\omega), [\lambda_{n_k}](1+\omega)]$ $(0<\omega<1)$.

(c) *the sequence* $\Re (a_n \exp(-i\gamma_k))$ $(\lambda_n \in I_k)$ *has the normal sign-change in* $\{I_k\}$ $(k=1, 2, \cdots)$.

PROOF. Taking account of the Fundamental Thorem 1, it suffices to prove the existence of a sequence $\{x_k\}$ such that

$$(4.1)\qquad \begin{cases} \text{(i)}\quad [x_k] = [\lambda_{n_k}] & (k = 1, 2, \cdots) \\ \text{(ii)}\quad \varlimsup_{k\to\infty} 1/x_k \cdot \log \left| \sum_{[x_k]\leqq\lambda_n<x_k} \Re(a_n \exp(-i\gamma_k)) \right| = 0. \end{cases}$$

Let us put

$$(4.2)\qquad \Delta = \varlimsup_{\substack{x\to\infty \\ [\lambda_{n_k}]\leqq x<[\lambda_{n_k}]+1 (k=1,2,\cdots)}} 1/x \cdot \log \left| \sum_{[x]\leqq\lambda_n<x} \Re(a_n \exp(-i\gamma_k)) \right|.$$

Then, by T. Kojima's theorem ([5]), we have

$$\Delta \leqq \varlimsup_{x\to\infty} 1/x \cdot \log \left| \sum_{[x]\leqq\lambda_n<x} a_n \exp(-i\gamma_k) \right| = \varlimsup_{x\to\infty} 1/x \cdot \log \left| \sum_{[x]\leqq\lambda_n<x} a_n \right| = 0,$$

so that

$$(4.3)\qquad \Delta \leqq 0.$$

On account of (4.2), for any given $\varepsilon\,(>0)$, there exists $N(\varepsilon)$ such that

$$(4.4)\qquad \left| \sum_{[\lambda_{n_k}]\leqq\lambda_n<[\lambda_{n_k}]+1} \Re(a_n \exp(-i\gamma_k)) \right| < \exp((\Delta+\varepsilon)[\lambda_{n_k}]) \text{ for } [\lambda_{n_k}] > N(\varepsilon).$$

Now, if $[\lambda_{n_k}] \leqq \lambda_{n_k-1} < \lambda_{n_k}$ we have

$$\Re(a_{n_k} \exp(-i\gamma_k)) = \sum_{[\lambda_{n_k}]\leqq\lambda_n\leqq\lambda_{n_k}} \Re(a_n \exp(-i\gamma_k)) - \sum_{[\lambda_{n_k}]\leqq\lambda_n\leqq\lambda_{n_k-1}} \Re(a_n \exp(-i\gamma_k)).$$

If $\lambda_{n_k-1} < [\lambda_{n_k}] < \lambda_{n_k}$, we get

$$\Re(a_{n_k} \exp(-i\gamma_k)) = \sum_{[\lambda_{n_k}]\leqq\lambda_n\leqq\lambda_{n_k}} \Re(a_n \exp(-i\gamma_k)).$$

In any case, by (4.4)

$$|\Re(a_{n_k} \exp(-i\gamma_k))| < 2\exp((\Delta+\varepsilon)[\lambda_{n_k}]) \qquad \text{for } [\lambda_{n_k}] > N(\varepsilon).$$

Hence by the assumption (*a*),

$$0 = \varlimsup_{k\to\infty} 1/\lambda_{n_k} \cdot \log |\Re(a_{n_k} \exp(-i\gamma_k))| \leqq \Delta + \varepsilon.$$

Letting $\varepsilon \to 0$,

$$(4.5)\qquad 0 \leqq \Delta$$

By (4.3) and (4.5)

$$\Delta = 0,$$

from which (4.1) immediately follows. q. e. d.

(to be continued)

REFERENCES

[1] F. CARLSON-E. LANDAU, Neuer Beweis und Verallgemeinerung des Fabryschen Lückensatzes, Gött. Nachr., 1921.

[2] V. BERNSTEIN, Leçons sur les progrès récents de la théorie des séries de Dirichlet, Paris, 1933.

[3] O. SZÀSZ, Über Singularitäten von Potenzreihen und Dirichletschen Reihen am Rande des Konvergenzbereiches, Math. Ann., Bd. 85 (1922).

[4] S. IZUMI, On the singularities of functions defined by Dirichlet series, Proc. Imp. Acd., Jap., Bd. 4 (1928).
[5] T. KOJIMA, On the convergence-abscissa of general Dirichlet series, Tôhoku Math. Jour., Vol. 6 (1914-15).
[6] A. OSTROWSKI, Über das Hadamardsche Singularitätskriterien in der Theorie der Taylorschen und Dirichletsche Reihe, Sitz. Berl. Math. Ges., Bd. 27 (1928).
[7] H. CRAMER, Un théorème sur les séries de Dirichlet et son application, Arkiv. för Math. och Tys., 13 (1918).
[8] A. OSTROWSKI, Einige Bemerkungen über Singularitäten Taylorscher und Dirichletscher Reihen, Berl. Sitz., 1923.

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