ON THE ORDER OF THE DERIVATIVE OF A MEROMORPHIC FUNCTION

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1. Whittaker[1] proved the theorem.

THEOREM. *Let $f(z)$ be a meromorphic function for $|z| < \infty$, which is of order ρ ($\leqq \infty$), then $f'(z)$ is of order ρ.*

Whittaker remarked in the addendum inserted in the end of the same journal that the theorem was proved previously by Valiron[2], but in Valiron's paper cited, we find no detail proof, so that we will give a simple proof of it in the following lines.

If $f(z)$ is an integral function, then the theorem follows from relation:

$$\frac{1}{r}(M(r) - |f(0)|) \leqq M_1(r) \leqq \frac{1}{r} M(2r),$$

where $M(r) = \underset{|z|=r}{\text{Max.}} |f(z)|$, $M_1(r) = \underset{|z|=r}{\text{Max.}} |f'(z)|$.

For the proof, of the case, when $f(z)$ has poles, we use the following lemma.

LEMMA. *Let $F(z)$ be an integral function of finite order ρ and $P(z)$ be a canonical product formed with $\{a_n\}$ and of order $\rho' < \rho$. Then*

$$F'(z)P(z) - F(z)P'(z) = G(z) \tag{1}$$

is of order ρ.

PROOF. Since $F'(z)$ is of order ρ and $P(z)$ of order $\rho' < \rho$, $G(z)$ is of order $\leqq \rho$. Hence it suffices to prove that $G(z)$ is of order $\geqq \rho$.

We consider (1) as a differential equation for $F(z)$ and solving it, we have

$$F(z) = \text{const.}\, P(z) + P(z)\int_{z_0}^{z} \frac{G(z)}{(P(z))^2}\, dz. \tag{2}$$

Suppose that $G(z)$ is of order $< \rho$, then

$$|G(z)| < e^{r^{\rho_1}} \quad (|z| = r \geqq r_1), \ (\rho_1 < \rho). \tag{3}$$

Since for the canonical product, its order coincides with the convergence exponent of $\{a_n\}$,

$$\sum_n \frac{1}{|a_n|^{\rho'+\varepsilon}} < \infty \quad (\rho' + \varepsilon < \rho). \tag{4}$$

We draw circles $C_n : |z - a_n| = 1/|a_n|^{\rho'+\varepsilon}$, then outside C_n $(n = 1, 2, \cdots)$,

1) J. M. WHITTAKER: The order of the derivative of a meromorphic function. Journ. London Math. Soc. **11** (1936).

2) G. VALIRON: Sur la distribution des valeurs des fonctions méromorphes. Acta Math. **47** (1926).

$$|P(z)| > e^{-r^{\rho_2}} \ (r \geqq r_2),\ \ \rho' < \rho_2 < \rho. \tag{5}$$

Let E be the set of intervals $I_n: |x - |a_n|| \leqq 1/|a_n|^{\rho'+\epsilon}$ on the positive real axis, then by (5), if R lies outside E,

$$|P(z)| > e^{-R^{\rho_2}} \text{ on } |z| = R\ (R \geqq r_2). \tag{6}$$

Since the sum of radii of C_n is convergent, there exists θ_0, such that the half-line $L: z = re^{i\theta_0}$ $(\max(r_1, r_2) \leqq r_0 < r < \infty)$ lies outside C_n $(n = 1, 2, \cdots)$, so that

$$|P(re^{i\theta_0})| > e^{-r^{\rho_2}} (r_0 \leqq r < \infty). \tag{7}$$

Let R lie outside E and $z = Re^{i\theta}$ be any point on $|z| = R$ $(R \geqq r_0)$. In (2), we first integrate on the segment $z = re^{i\theta_0}$ $(r_0 \leqq r \leqq R)$ and then on the circular arc on $|z| = R$, which is bounded by $Re^{i\theta_0}$ and $Re^{i\theta}$, then by (3), (6), (7), we have

$$\left| \int_{z_0}^{z} \frac{G(z)}{(P(z))^2} dz \right| \leqq \int_{r_0}^{R} \frac{|G(re^{i\theta_0})|}{|P(re^{i\theta_0})|^2} dr + \int_{\theta_0}^{\theta} \frac{|G(Re^{i\varphi})|}{|P(Re^{i\varphi})|^2} R d\varphi < e^{R^{\rho_3}}\ (R \geqq r_3),\ (\rho_3 < \rho),$$

where $z_0 = r_0 e^{i\theta_0}$. Hence from (2),

$$|F(z)| < e^{R^{\rho_4}} \text{ on } |z| = R\ \ (R \geqq r_4),\ \ (\rho_4 < \rho). \tag{8}$$

If R lies in E, then since the sum of radii of C_n is convergent, we can choose R_1 outside E, such that $R \leqq R_1 \leqq R + 1$, then

$$|F(z)| \leqq \operatorname*{Max.}_{|z|=R_1} |F(z)| < e^{R_1^{\rho_4}} < e^{R^{\rho_5}} \text{ on } |z| = R\ (R \geqq r_5),\ \ (\rho_5 < \rho). \tag{9}$$

Hence from (8), (9), we see that $F(z)$ is of order $< \rho$, which contradicts the hypothesis, so that $G(z)$ is of order $\geqq \rho$. q. e. d.

2. Now we will prove the theorem, when $f(z)$ has poles $\{a_n\}$ and first we suppose that $\rho < \infty$.

Let $P(z)$ be the canonical product formed with $\{a_n\}$, then since the convergence exponent of a_n is $\leqq \rho$, $P(z)$ is of order $\leqq \rho$ and

$$f(z) = \frac{F(z)}{P(z)}, \tag{10}$$

$$f'(z) = \frac{F'(z)P(z) - F(z)P'(z)}{(P(z))^2} = \frac{G(z)}{(P(z))^2}, \tag{11}$$

where $F(z)$ is an integral function of order $\leqq \rho$.

Let ρ' be the order of $f'(z)$, then since $G(z)$, $(P(z))^2$ are of order $\leqq \rho$, we have $\rho' \leqq \rho$. Hence it suffices to prove that $\rho' \geqq \rho$.

Let ρ_1 $(\leqq \rho)$ be the convergence exponent of $\{a_n\}$, then since a_n are poles of $f'(z)$, we have from Nevanlinna's relation $T(r, f') = m(r, \infty, f') + N(r, \infty, f')$,

$$\rho' \geqq \rho_1. \tag{12}$$

Hence if $\rho_1 = \rho$, then $\rho' \geqq \rho$.

If $\rho_1 < \rho$, then $P(z)$ is of order $\rho_1 < \rho$, so that $F(z)$ is of order ρ, hence by the lemma, $G(z)$ is of order ρ, so that from $G(z) = (P(z))^2 f'(z)$, we have $\rho' \geqq \rho$. Hence

$$\rho' = \rho, \text{ if } \rho < \infty. \tag{13}$$

Next we suppose that $\rho = \infty$ and we will prove $\rho' = \infty$. Suppose that $\rho' < \infty$. Let a_n be the poles of $f(z)$, then since a_n are poles of $f'(z)$, in the Nevanlinna's relation,

$$T(r, f) = m(r, \infty, f) + N(r, \infty, f), \tag{14}$$

we have

$$N(r, \infty, f) = O(r^{\rho_1}) \quad (\rho' < \rho_1 < \infty). \tag{15}$$

Let $P(z)$ be the canonical product formed with poles an of $f'(z)$, then

$$f'(z) = \frac{G(z)}{P(z)},$$

where $G(z)$, $P(z)$ are integral functions of order $\leqq \rho'$ and

$$f(z) = \int_{z_0}^{z} \frac{G(z)}{P(z)} dz + \text{const.} \tag{16}$$

Let E be the set defined in the proof of the lemma, then if R lies outside E, we can prove similarly as before,

$$|f(Re^{i\theta})| = O(e^{R^{\rho_1}}), \qquad \text{on } |z| = R \ (\rho_1 < \infty)$$

so that $m(R, \infty, f) = O(R^{\rho_1})$. Hence by (14), (15),

$$T(R, f) = O(R^{\rho_1}).$$

If R lies in E, then we choose R_1 outside E, such that $R \leqq R_1 \leqq R+1$, then

$$T(R, f) \leqq T(R_1, f) = O(R_1^{\rho_1}) = O(R^{\rho_1}).$$

Hence for any R, $T(R, f) = O(R^{\rho_1})$, which contradicts the hypothesis, $\rho = \infty$, so that $\rho' = \infty$.

Hence our theorem is proved.

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