ON THE UNIFORMIZATION OF AN ALGEBRAIC FUNCTION OF GENUS $p \geqq 2$

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1.

Picard[1] proved the following theorem.

THEOREM 1. *Let F be a closed Riemann surface of genus $p \geqq 2$ spread over the x-plane Then we can not uniformize F by $x = x(t)$, which is one-valued and meromorphic in $0 < |t| \leqq R$ and has an essential singularity at $t = 0$.*

PROOF. Suppose that there exists a function $x = x(t)$, which satisfies the condition of the theorem. Since F is of hyperbolic type, we can map the universal covering surface $F^{(\infty)}$ of F on $|z| < 1$ by $x = \varphi(z)$ and put $z = z(t)$. Then $z(t)$ is not one-valued in $0 < |t| \leqq R$. For, if it is one-valued, then, since $|z(t)| < 1$ in $0 < |t| < R$, $z(t)$ is regular at $t = 0$, so that $x = x(t)$ is meromorphic at $t = 0$, which contradicts the hypothesis. Hence $z(t)$ is many valued, so that to a circle C: $|t| = \rho\ (< R)$, there corresponds a curve L on F, which is not homotop null, hence the image of C in $|z| < 1$ is either (i) a Jordan curve, which has a common point γ with $|z| = 1$ or (ii) a Jordan arc, whose end points α, $\beta\ (\alpha \neq \beta)$ lie on $|z| = 1$. The case (i) does not occur. For, if it does occur, then $\lim_{t \to 0} z(t) = \gamma$, so that $\lim_{t \to 0} x(t) = x_0 = \varphi(\gamma)$, which contradicts the hypothesis, that $t = 0$ is an essential singularity of $x(t)$. Hence the case (ii) occurs. Let α, β be so chosen that

$$\lim_{\theta \to +\infty} z\ (\rho e^{i\theta}) = \alpha, \quad \lim_{\theta \to -\infty} z\ (\rho e^{i\theta}) = \beta. \tag{1}$$

By $u = \log t$, we map $0 < |t| \leqq \rho$ on the half-plane $\Re u \leqq \log \rho$ and then by a linear transformation, we map this half-plane on $|\tau| \leqq 1$, such that $\tau = -1$ corresponds to $u = -\infty$ and put $z(t) = z(\tau)$, then $z(\tau)$ is regular and $|z(\tau)| < 1$ in $|\tau| < 1$. By (1)

$$\lim_{\varphi \to \pi - 0} z(e^{i\varphi}) = \alpha, \quad \lim_{\varphi \to -\pi + 0} z(e^{i\varphi}) = \beta \quad (\alpha \neq \beta).$$

which contradicts Lindelöf's theorem. Hence our theorem is proved.

2.

Let F be a Riemann surface and F^* be its covering surface. If F^* has no branch points relatively to F, then we call F^* a non-ramified (unverzweigt) covering surface of F.

THEOREM 2. *Let F be a closed Riemann surface of genus $p \geqq 2$ spread*

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1) E. PICARD : Démonstration d'un théorème géneral des fonctions uniformes liées par une relation algébrique. Acta. Math. 11(1887).

over the x-sphere. Then there exists no function $x = x(t)$, which is one-valued and meromorphic in a neighbourhood U of a closed set E of logarithmic capacity zero, every point of which is an essential singularity of $x(t)$, such that the Riemdnn surface F^* generated by $x = x(t)$ is a non-ramified covering surface of F.

We remark that in Theorem 1, the Riemann surface F^* generated by $x = x(t)$ is not supposed to be non-ramified relatively to F.

PROOF. Suppose that there exists a function $x = x(t)$, which satisfieds the condition of the theorem, such that the Riemann surface F^* generated by $x = x(t)$ is a non-ramified covering surface of F.

Since E is a closed set of logarithmic capacity zero, by Evans' theorem[2], we can distribute a positive mass $d\mu(a)$ on E of total mass 1, such that

$$u(t) = \int_E \log \frac{1}{|t-a|}\, d\mu(a), \qquad \left(\int_E d\mu(a) = 1\right) \tag{1}$$

tends to $+\infty$, if t tends to any point of E. Let C_r be the niveau curve: $u(t) = r$, then C_r consists of a finte number of Jordan curves, which cluster to E as $r \to \infty$.

Let $\theta(t)$ be conjugate harmonic function of $u(t)$, then since the total mass is 1,

$$\int_{C_r} d\theta(t) = 2\pi. \tag{2}$$

We put

$$\tau = e^{u+i\theta} = r(t)e^{i\theta(t)}, \quad x = x(t) = x(\tau).$$

Let $A(r)$ be the area on the x-sphere of the image of the domain D_r, which is bounded by C and C_r, where C is the boundary of U and $L(r)$ the length of the image of C_r, then

$$A(r) = \int_{r_0}^{r}\int_{C_r}\left(\frac{|x'(\tau)|}{1+|x(\tau)|^2}\right)^2 r\,dr\,d\theta + \text{const.},$$

$$L(r) = \int_{C_r}\frac{|x'(\tau)|}{1+|x(\tau)|^2}\, r\,d\theta,$$

where we write $r = r(t)$, $\theta = \theta(t)$. Then by (2),

$$(L(r))^2 \leqq 2\pi r\int_{C_r}\left(\frac{x'(\tau)}{1+|x(\tau)|^2}\right)^2 r\,d\theta = 2\pi r\,\frac{dA(r)}{dr}. \tag{3}$$

Since $x(t)$ has an essential singularities on E, $x(t)$ takes in U any value

2) G. C. EVANS: Potentials and positively infinite singularities of harmonic functions. Monatshefte f. Math. u. Phys. **43**(1936).

infinitely often, except a set of values of logarithmic capacity zero[3], so that

$$\lim_{r\to\infty} A(r) = \infty. \tag{4}$$

Let $L(r) > (A(r))^{\frac{3}{4}}$ in a set of intervals $I_\nu = [r_\nu, r_\nu']$ $(\nu = 1, 2, \cdots)$, then from (3)

$$\sum_\nu \int_{I_\nu} \frac{dr}{r} \leqq 2\pi \int^\infty \frac{dA(r)}{(A(r))^{\frac{3}{2}}} < \infty,$$

hence there exists $r_1 < r_2 < \cdots < r_n \to \infty$, such that $L(r_n) \leqq (A(r_n))^{\frac{3}{4}}$, so that

$$\frac{L(r_n)}{A(r_n)} \leqq \frac{1}{(A(r_n))^{\frac{1}{4}}} \to 0 \text{ as } r_n \to \infty. \tag{5}$$

Hence F^* is regularly exhaustible in Ahlfors' sense[4].

Let C_r consists of $n = n(r)$ Jordan curves $C_r = C_r^{(1)} + \cdots + C_r^{(n)}$ and let $L_r^{(i)}$ be the length of the image $\Lambda_r^{(i)}$ of $C_r^{(i)}$ on the x-sphere, then $L(r) = L_r^{(1)} + \cdots + L_r^{(n)}$.

Since by the hypothesis, F^* is non-ramified relatively to F and $x(t)$ has essential singularitites on E, we see easily that $\Lambda_r^{(i)}$ is not homotop null on F, so that $L_r^{(i)} \geqq L_0 > 0$, where L_0 is a fixed constant, hence

$$L(r) \geqq L_0 n(r). \tag{6}$$

Let F_r^* be the image of D_r on the x-sphere, then F_r^* is a covering surface of F, so that by Ahlfors' theorem on covering surfaces[5],

$$\overset{+}{\rho}(r) \geqq \rho_0 S(r) - h(L(r) + \lambda_0), \quad \overset{+}{\rho} = \mathrm{Max}(\rho, 0), \tag{7}$$

where $S(r) = A(r)/n\pi$, n being the number of sheets of F, $\rho(r)$ is the Euler's characteristic of F_r^*, $\rho_0 = 2(p-1) > 0$ is that of F and λ_0 is the length of the image of C and h is a constant.

Since $\overset{+}{\rho}(r) = n(r) - 2 \leqq n(r)$, we have from (6),

$$\frac{L(r)}{L_0} \geqq \rho_0 S(r) - h(L(r) + \lambda_0), \quad \text{or}$$

$$S(r) \leqq \frac{1}{\rho_0}\left(\frac{1}{L_0} + h\right)(L(r) + \lambda_0),$$

which contradicts (5). Hence our theorem is proved.

3

Let G be a group of linear transformations: $S_\nu = \dfrac{a_\nu z + b_\nu}{c_\nu z + d_\nu}$ $(\nu = 0, 1, 2, \cdots)$

3) S. KAMETANI : The exceptional values of functions with the set of capacity zero of essential singularities. Proc. Imp. Acad. **17** (1941). M. TSUJI : Theory of meromorphic functions in a neighbourhood of a closed set of capacity zero. Jap. Journ. Math. **19** (1944-1948).

4) K. NOSHIRO : Contribution to the theory of the singularities of analytic functions Jap. Journ. Math. **19** (1944-1948).

5) L. AHLFORS : Zur Theorie der Überlagerungsflächen. Acta Math. **65** (1935)

of Schottky type, which contains at lest two generators. We call such a group a general linear group of Schottky type. Let D_0 be the fnndamental domain of G, which is bounded by p $(2 \leqq p \leqq \infty)$ pairs of Jordan curves C_i, C_i' $(i = 1, 2, \cdots, p)$, where C_i, C_i' are equivalent by G. If we apply all transformations of G to D_0, then its equivalents D_ν cluster to a non-dense perfect set E, which we call the singular set of G. Then Myrberg[6] proved the following theorem.

RHEOREM 3. *The singular set E of a general linear group of Schottky type is of positive logarithmic capacity.*

PROOF. Let D_0' be the domain bounded by C_1, C_1', C_2, C_2' and T_1, T_2 be the transformations of G, such that $C_1' = T_1(C_1)$, $C_2' = T_2(C_2)$ and let G' be the group generated by $\{T_1, T_2\}$. If we apply all transformations of G' to D_0', then its equivalents D_ν' cluster to a non-dense perfect set $E' \subset E$. Now we consider D_0' as a closed Riemann surface F of genus $p = 2$, where equivalent point on C_i, C_i', $(i = 1, 2)$ are considered as the same point of F. Then we have a non-ramified covering surface F^* of F, where an equivalent point z_ν of $z_0 \in D_0'$ corresponds to the point z_0 of F. Hence by Theprem 2, E' and hence E is of positive logarithmic capacity.

Similarly we can prove the following theorem[7].

THEOREM 4. *Let $C_1 \cdots, C_p$ $(3 \leqq p \leqq \infty)$ be p circles on the z-plane, which lie outside each other. We invert C_i inot C_j and we perform inversions indefinitely, then we obtain infinitely many circles clustering to a non-dense perfect set E. Then E is of positive logarithmic capacity.*

PROOF. We take three circles C_1, C_2, C_3 and D_0 be the domain bounded by these circles. We perform indefinitely inversions starting from C_1, C_2, C_3, then we have infinitely many circles clustering to a non-dense perfect set $E' \subset E$. We take two same samples D_0, D_0' as D_0 and conncet them along C_i $(i = 1, 2, 3)$, then we have a closed surface F of genus $p = 2$. Any point outside E' is equivalent to a point of F by inversion, so that we have a non-ramified covering surface F^* of F, hence by Theorem 2, E' and hence E is of positive logarithmic capacity.

REMARK BY A. MORI.

The idea of the proof of Theorem 2 can be formulated in the following form, which is an analogue of Ahlfors's theorem for simply connected covering surfaces[8].

Any non-ramified and unbounded (unberandet) open covering surface F^* of planar character (or, more generally, of finite genus) of a closed basic surface F of genus $\geqq 2$ is not regularly exhaustible in Ahlfors's sense.

PROOF. Let $F_1 \subset F_2 \subset \cdots$ be an exhaustion of F^*. We assume that

6) P. J. MYRBERG: Die Kapazität der singulären Menge der linearen Gruppen. Annales Acad. Fenn. Series A. Math- Phys. **10** (1941).

7) Myrberg. l. c. 6).

8) Ahlfors. l. c 5).

the boundary of F_ν consists of n_ν rectifiable closed Jordan curves $\Lambda_\nu^{(i)}$ $(i = 1, 2, \cdots, n_\nu)$ on F^*. Let A_0, A_ν be the area of F and F_ν, and $L_\nu^{(i)}$ be the length of $\Lambda_\nu^{(i)}$ (both measured in a metric defined on F). We put $S_\nu = A_\nu/A_0$, $L_\nu = \sum_{i=1}^{n_\nu} L_\nu^{(i)}$. Since F^* is open and unbounded, we see easily that $S_\nu \to \infty$, and we have to prove that L_ν/S_ν is bounded from zero.

If one of $\Lambda_\nu^{(i)}$ is null-homotop on F^*, it bounds a compact simply connected domain $\Delta(\Lambda_\nu^{(i)})$ on F^*. We add to F_ν all such domains and put $\overline{F}_\nu = F_\nu + \sum \Delta(\Lambda_\nu^{(i)})$. Then, $\overline{F}_1 \subset \overline{F}_2 \subset \cdots$ is also an exhaustion of F^*, and if $\overline{\Lambda}_\nu^{(j)}$ $(j = 1, 2, \cdots, \overline{n}_\nu)$, $\overline{L}_\nu^{(j)}$, $\overline{L}_\nu$ and $\overline{S}_\nu$ denote the corresponding curves and quantities, we have $\overline{S}_\nu \leqq S_\nu$, $L_\nu \geqq \overline{L}_\nu$. Further, let $\overline{\rho}_\nu$ denote the Euler's characteristic of $\overline{F}_\nu$.

Suppose that one of $\overline{\Lambda}_\nu^{(j)}$ is null-homotop on F^*. Then we see easily that $\overline{F}_\nu$ coincides with $\Delta(\Lambda_\nu^{(j')})$ so that $\delta_\nu{}^+ = 0$.

If none of $\Lambda_\nu^{(j')}$ is null-homotop on F^*, their projections on F are not null-homotop on F, so that $\overline{L}_\nu^{(j)} \geqq \text{const.} = L_0 > 0$. Then, we have $\overline{L}_\nu \geqq \overline{n}_\nu L_0$ and, since F^* is of finite genus,

$$\overline{\rho}_\nu{}^+ \leqq \overline{n}_\nu + \text{const.} \leqq \frac{\overline{L}_\nu}{L_0} + \text{const.}$$

Hence, in any case, Ahlfors' fundamental theorem gives ($\rho_0 > 0$ benig the characteristic of F)

$$\frac{\overline{L}_\nu}{L_0} + \text{const.} \geqq \overline{\rho}_\nu^+ \geqq \rho_0 \overline{S}_\nu - h\, \overline{L}_\nu,$$

so that

$$\frac{L_\nu}{S_\nu} \geqq \frac{\overline{L}_\nu}{\overline{S}_\nu} \geqq \frac{\rho_0 L_0}{1 + hL_0} - O\left(\frac{1}{\overline{S}_\nu}\right).$$

Since $S_\nu \geqq \overline{S}_\nu \to \infty$, we have $\varliminf \dfrac{L_\nu}{S_\nu} > 0$, q. e. d.

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