A NOTE ON A RIEMANN SURFACE WITH NULL BOUNDARY

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1. Let F be an open abstract Riemann surface and Γ be its ideal boundary. Suppose that $F_n\,(n = 0, 1, \cdots)$ is the compact subdomain of F satisfying the following conditions:

1°) F_0 is simply and $F_n\,(n \neq 0)$ is finitely connected,

2°) the boundary Γ_n of F_n consists of a finite number of analytic closed curves,

3°) $\overline{F}_n \subset F_{n+1}\,(n = 0, 1, \cdots)$ and

4°) $\bigcup_{n=0}^{\infty} F_n = F$.

Such a sequence of domains

$$F_0,\ F_1, \cdots,\ F_n, \cdots \tag{1}$$

is called an exhaustion of F.

Putting $R_n = F_n - F_0$, R_n is a compact subdomain of F and the boundary of R_n consists of Γ_0 and Γ_n. Let u be the harmonic function in R_n such that

$$u = \begin{cases} 0 & \text{on } \Gamma_0 \\ \log \mu_n & \text{on } \Gamma_n \end{cases} \tag{2}$$

and

$$\int_{\Gamma_n} dv = 2\pi,$$

where v is the conjugate function of u and the integral is taken in the positive sense with respect to R_n. Then we call μ_n the modulus of R_n.

2. We shall prove an extension of the maximum principle.

THEOREM 1. *Let F' be any non-compact subdomain of F, Γ' be the relative boundary of F' and U be a single-valued bounded harmonic function on F' which equals to zero on Γ'. If F has a null boundary, then the function U equals identically to zero throughout F'.*

PROOF. First we choose an exhaustion (1) of F such that F_0 is contained in F'. Let u be the harmonic function (2) which defines the modulus μ_n of R_n and v be its conjugate function. Denote by Δ_λ the domain defined by $0 < u < \lambda\ (0 < \lambda \leq \log \mu_n)$ and by Γ'_λ the part of the niveau curve $\Gamma_\lambda : u = \lambda$ contained in F'. Then the Dirichlet integral $D(\lambda)$ of U taken over the compact domain $F'_\lambda = F' \cap (\Delta_\lambda \cup \overline{F}_0)$ equals to

$$D(\lambda) = \int_{\Gamma'_\lambda} U \frac{\partial U}{\partial u}\, dv,$$

where the integral is taken in the positive sense with respect to F'_λ. Since we may suppose that $|U| < M$, we have, from the Schwarz inequality,

$$D^2(\lambda) \leqq \int_{\Gamma'_\lambda} U^2 dv \int_{\Gamma'_\lambda} \left(\frac{\partial U}{\partial u}\right)^2 dv \leqq 2\pi M^2 \frac{dD(\lambda)}{d\lambda}$$

or

$$d\lambda \leqq 2\pi M^2 \frac{dD(\lambda)}{D^2(\lambda)},$$

whence, by integrating from $\lambda = 0$ to $\lambda = \log \mu_n$, it follows

$$\log \mu_n \leqq 2\pi M^2 \left[\frac{1}{D(0)} - \frac{1}{D(n)}\right] \leqq 2\pi M^2 \frac{1}{D(0)},$$

where $D(n)$ is the Dirichlet integral of U taken over $F' \cap F_n$.

On the other hand, it has shown by T. Kuroda [1] that F has a null boundary if and only if $\lim_{n\to\infty} \mu_n = \infty$.

Hence $D(0)$ equals to zero and so the function U must be identically equal to zero throughout F'. q. e. d.

As a corollary of the above theorem we get the following

THEOREM 2. *Let F' be any non-compact subdomain of a Riemann surface F with null boundary and Γ' be the relative boundary of F'. If U is a single-valued bounded harmonic function in F', then for any point of F'*

$$\underline{\lim}_{\Gamma'} U \leqq U \leqq \overline{\lim}^{\Gamma'} U.$$

PROOF. Let us suppose that there exists an inner point p_0 of F' such that

$$U(p_0) > \overline{\lim} U = M.$$

Then we can find a suitable number M_1 satisfying

$$(3) \qquad U(p_0) > M_1 > M.$$

Denote by F'' the set of all points p of F' such that

$$U(p) > M_1.$$

It is easy to see that F'' is non-compact. Since the relative boundary of F'' consists of an enumerable number of analytic arcs or analytic closed curves and $U(p) = M_1$ on Γ'', Γ' and Γ'' are disjoint each other. The function U is single-valued, bounded and harmonic in F'' and equals to M_1 on Γ''. Hence, from Theorem 1, U must be the constant M_1 throughout F'', which contradicts to (3). Thus we have $\overline{\lim}_{\Gamma'} U \geqq U$. Similarly as the above we can show $\underline{\lim}_{\Gamma'} U \leqq U$ which is the required.

Theorem 2 can be also obtained using the harmonic measure as stated in the famous R. Nevanlinna's monograph: Eindeutige analytische Funktionen (1936).

2. Let $w = f(p)$ be a single-valued meromorphic function on an open

Riemann surface F. The topological space constructed by the elements $q = [p, f(p)]$ $(p \in F)$ defines a covering surface Φ spread over the w-plane. The correspondence $p \leftarrow\rightarrow q$ gives a topological mapping between F and Φ.

We shall state some definitions.

If the set of the values taken by $w = f(p)$ on F is everywhere dense in the w-plane, we say that the function $w = f(p)$ has Weierstrass' property.

Next let Φ_Δ be any connected piece of Φ lying on the disc $|w - w_0| < \rho$, where w_0 is any point on the w-plane and ρ is any positive number. Denote by Δ the domain on F corresponding to Φ_Δ by $p \leftarrow\rightarrow q$. If $f(p) \neq w_0$ in Δ and there exists a sequence of points $\{p_\nu\}$ $(\nu = 1, 2, \cdots)$ in Δ such that $\lim_{\nu\to\infty} f(p^\nu) = w_0$ where p_ν tends to the ideal boundary Γ of F as $\nu \to \infty$, then we may define that the function $w = f(p)$ has Lindelöf's property.

On the other hand if either there exists a path in Δ tending to Γ such that $\lim f(p) = w_0$ along the path or there exists an inner point of Δ such that $f(p) = w_0$, then we may define that the covering surface Φ has Iversen's property.

By a slight modification of K. Noshiro's method [3], we can easily prove the following

THEOREM 3. *If the function has Weierstrass' property and Lindelöf's property, then Φ has Iversen's property.*

Using the above, we can prove the following theorem.

THEOREM 4. (S. Stoïlow [5]).* *Let $w = f(p)$ be a non-constant single-valued meromorphic function on F with null boundary. Then Φ has Iversen's property.*

PROOF. First we shall prove that the function $w = f(p)$ has Weierstrass' property. If not so, we can find a point w_0 such that the function $\varphi(p) = 1/(f(p) - w_0)$ is bounded and regular on F. Hence the real part of this function $\varphi(p)$ is single-valued, bounded and harmonic on F. Since F has a null boundary, such a function must be a constant (c.f. R. Nevanlinna [2]), which contradicts to our assumption. Thus $w = f(p)$ has Weierstrass' property.

Next we shall show that $w = f(p)$ has Lindelöf's property. We construct any connected piece Φ_Δ of Φ lying above the disc $|w - w_0| < \rho$ and the domain Δ corresponding to Φ_Δ by $p \leftarrow\rightarrow q$ as already stated. Let us suppose that $f(p) \neq w_0$ in Δ. It is immediately seen that Δ is non-compact in F and $|f(p) - w_0| < \rho$ in Δ and $|f(p) - w_0| = \rho$ on the relative boundary of Δ. If we suppose that there exist no sequence of points $\{p_\nu\}$ $(\nu = 1, 2, \cdots)$ in Δ such that $\lim_{\nu\to\infty} f(p_\nu) = w_0$ and p_ν tends to Γ as $\nu \to \infty$, then the function $\varphi(p) = 1/(f(p) - w_0)$ is bounded and $|\varphi(p)| = 1/\rho$ on the relative boundary of Δ. Hence, from Theorem 2, $|\varphi(p)| \leqq 1/\rho$ and so $|f(p) - w_0| \geqq \rho$ in Δ, which

*) This stoïlow's paper is refered to only through the Mathematical Reviews.

is absurd. Thus $w = f(p)$ has Lindelöf's property.

From Theorem 3, our assertion is proved.

REMARK. Recently Prof. K. Noshiro [4] proved that, under the same conditions as in the above theorem, Φ has Gross' property. Theorem 4 is contained in his theorem. T. Kuroda gave also a similar proof as the author's independently.

REFERENCES

[1] T. KURODA: On the type of an open Riemann surface, Proc. Jap. Acad. 27 (1951), pp. 57-60.

[2] R. NEVANLINNA: Sur l'existence de certaines classes de différentieles analytiques, C. R. 228 (1949), pp. 2002-2004.

[3] K. NOSHIRO: On the singularities of analytic functions, Jap. Journ. Math., 17 (1940), pp. 37-96.

[4] K. NOSHIRO: Open Riemann surface with null boundary, Nagoya Math. Journ. 3 (1951), pp73-79.

[5] S. STOÏLOW: Sur les singularités des fonctions analytiques multiformes dont la surface de Riemann a sa frontière de mesure harmonique nulle, Mathematica, Timisoara, 19 (1943), pp. 126-138.

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