GROUP ALGEBRAS IN THE LARGE

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1. Introduction. Let G be a unimodular locally compact group. We form the Hilbert space $L_2(G)$, and the left and right regular representations of G on $L_2(G)$. We shall denote the weakly closed algebras generated by these representations by W and W'. It is known [10] that W and W' are the full commuting algebras of each other.

It is a question of some importance to determine how properties of G are reflected in properties of W. The pioneering investigations of Murray and von Neumann in [8] have been followed by several interesting contributions by Mautner [5], [6], [7]. Mautner's results are phrased in terms of von Neumann's decomposition theory [9].

In this paper we shall prove several theorems about the structure of W "in the large". There are several advantages in this change of point of view. Measure-theoretic questions and "almost everywhere" difficulties disappear; separability becomes irrelevant; and above all, the proofs become simpler.

If the properties of W "in the small" are desired for their own sake, then, in the author's opinion, these are best obtained by using a dictionary for translating properties back and forth. Such a dictionary may be expected in the near future.

2. Definitions. We shall use the terminology introduced in [3], but we collect the definitions for the reader's convenience.

Let W be a weakly closed self-adjoint algebra of operators on a Hilbert space. A non-zero projection e is *abelian* if eWe is commutative; e is *finite* if left and right inverses in eWe coincide. We say that W is finite if its unit element is finite; W is of type I if every direct summand contains an abelian projection; W is of type II if there are no abelian projections and every direct summand contains a finite projection; W is of type III if all projections are infinite; W is of type II_1 if it is finite and of type II. By [3, Th. 4.6] W is uniquely a direct sum of algebras of types I, II and III.

Suppose W is finite and of type I. The structure of W is implicitly given in [3] and may be described as follows: W is a direct sum of algebras W_n ($n = 1, 2, \cdots$), where W_n is an n by n total matrix algebra over a commutative algebra.

We shall now specialize to the case where W, W' are the group algebras of G. There are then two further known facts. (1) W has no part of type III. (2) W and W' decompose in the same manner; more precisely, if h is any projection in the center of W (which is also the center of W')

then hW and hW' are of the same type (they are even isomorphic).

In this paper we shall mostly be concerned with the still more special case where G is discrete. Then it is known that W is finite. We write W as the direct sum of an algebra W_0 of type II, and an algebra which is finite and of type I, and (as above) we split the latter into its summands $W_1 + W_2 + \cdots$. As systematic notation we write e, e_i for the unit elements of W_0, W_i respectively, and we set $f_n = e_1 + \cdots + e_n$. When it is desirable to indicate the dependence on the group G, we write $e_i(G)$, etc.

Finally we shall make use of the fact, shown in [8] and [11], that when G is discrete there exists a certain canonical weight function defined on all of W. It may be conveniently defined by first noting that W is in a natural way a subset of $L_2(G)$; for $x \in W$ we set $T(x)$ equal to the value assumed by x at the unit element of G. We shall take for granted the elementary properties of this function T.

Special interest is attached to the numbers $T(e) = r$, $T(e_i) = r_i$. These constitute a sequence of invariants for the group G, with $r + \sum r_i = 1$.

3. The commutator group. The fact that every discrete group has such a sequence of invariants suggests the desirability of determinig their meaning, as far as possible. Now if G is a finite group of order k, it is known that $r_i = i^2 s_i/k$, where s_i is the number of (inequivalent) irreducible representations of degree i. In particular $r_1 = s_1/k$. But s_1, the number of one-dimensional representations of G, coincides with the number of characters of the abelian group G/C, where C is the commutator subgroup of G, and this number is equal to the order of G/C. Hence r_1 is equal to the reciprocal of the order of C. This statement is also valid for infinite groups.

THEOREM 1. *Let G be any discrete group and C its commutator subgroup. Then $e_1(G)$ is the average of the elements of C, and $r_1(G)$ is equal to the reciprocal of the order of C. In particular, $r_1(G) = 0$ if C is infinite.*

PROOF. We know that $e_1 W$ is in the center of W. Then for any a, b in G we have $(e_1 a)(e_1 b) = (e_1 b)(e_1 a)$, whence $e_1 aba^{-1}b^{-1} = e_1$. It follows that e_1 is constant on the cosets of C. This already proves the last statement of the theorem.

Now suppose that C has finite order k. Write d for the average of the elements of C. Then $T(d) = k^{-1}$, and so the proof of the theorem will be finished if we show that $e_1 = d$. It is plain that d is a central projection in W. The fact that e_1 is constant on cosets of C means that $de_1 = e_1$. Again $daba^{-1}b^{-1} = d$ for any a, b in G. From this it follows that dW is commutative. By the definition of e_1, we have $d \in e_1 W$. Hence $de_1 = d$, and we have proved $d = e_1$.

It does not seem to be possible to give a correspondingly simple interpretation for the higher invariants r_i. However there is a special case where these higher invariants vanish, and in this case we can give a structure theorem for W.

THEOREM 2. *Let G be an infinite discrete group and suppose its commutator subgroup C coincides with its center and is cyclic of prime order n. Then $W(G)$ is the direct sum of n algebras each carrying weight $1/n$; the first of these is commutative and the others are factors of type II_1. In particular, $r_1(G) = 1/n$, $r_0(G) = (n-1)/n$.*

PROOF. Let the elements of C be $1, c, \cdots, c^{n-1}$. Let $\varphi_1 = 1, \varphi_2, \cdots, \varphi_n$ be the n^{th} roots of unity. Write $nh_i = 1 + \varphi_i c + \cdots + (\varphi_i c)^{n-1}$. Then $h_1, \cdots, h_n$ are orthogonal idempotents with sum 1, they are central in $W(G)$, and $T(h_i) = 1/n$. By Theorem 1, $h_1 = e_1(G)$. It therefore only remains to prove that for $i > 1$, $h_i W$ is a factor of type II_1.

Let x be an element in the center of $h_i W$. We claim that x is a scalar multiple of h_i. To prove this, it evidently suffices to show that it is a linear combination of $1, c, \cdots, c^{n-1}$. Suppose on the contrary that x contains a term involving a group element s not in C. Since C is the center of G, there exists an element t in G which does not commute with s. Then $t^{-1}st$ is a conjugate of s not equal to s, hence of the form $c^r s$ with $c^r \neq 1$. Now, since $x \in h_1 W$, the coefficients of x at $c^r s$ and s are in the ratio φ_i^r; at any rate they are unequal. This contradicts the equation $xt = tx$.

We have thus proved that the center of $h_i W$ is just the complex numbers. It is moreover true that $h_i W$ is infinite-dimensional; in fact, if we choose elements $a_1, a_2, \cdots$ lying in distinct cosets of G mod C, then $h_i a_1, h_i a_2, \cdots$ are linearly independent elements of $h_i W$. Hence $h_i W$ is a factor of type II_1.

4. **Locally finite groups.** A group G is said to be locally finite if every finitely generated subgroup is finite. Let $\{H_j\}$ be a *defining* set of finite subgroups, that is, every finite subset of G is contained in some H_j. (Of course we might in particular take all finite subgroups, but it often pays to be more economical). Theorem 3 shows how the invariants of a locally finite group are determined in a simple way by those of a defining set of subgroups. We remind the reader that $f_i(G) = e_1(G) + \cdots + e_i(G)$, as defined in 2.

THEOREM 3. *Let G be a discrete locally finite group and $\{H_j\}$ a defining set of finite subgroups. Then, for $i = 1, 2, \cdots$*

$$f_i(G) = \inf_j f_i(H_j).$$

REMARK. If we apply the weight function T, and take account of the weak continuity of T, we deduce

$$r_1(G) + \cdots + r_i(G) = \inf_j [r_1(H_j) + \cdots + r_i(H_j)].$$

Then by subtracting off successive components we find that $r_i(G)$ is the limit of the directed set $r_i(H_j)$. In case G is countable, this limit over a directed set can be replaced by an ordinary sequential limit.

The proof of Theorem 3 rests in part on the following lemma.

LEMMA 1. *Let G be a discrete group and H a subgroup. Then $f_m(H) \geqq f_m(G)$ for $m = 1, 2, \cdots$.*

PROOF. Let n be any integer greater than m. In the algebra $W(H)$ we know that $e_n(H)$ is the sum of n equivalent orthogonal projections $k_1, \cdots, k_n$. The elements $f_m(G)k_1, \cdots, f_m(G)k_n$ are likewise equivalent orthogonal projections (it is to be noted that $W(H)$ is a subalgebra of $W(G)$, and $f_m(G)$ is central in $W(G)$). By [3, Lemma 4.10], the presence of n such projections is impossible in the algebra $f_m(G)W(G)$, unless all $f_m(G)k_i$ are 0, whence $f_m(G)e_n(H) = 0$. Since this is true for all $n > m$, we find $f_m(G)[1 - f_m(H)] = 0$, that is, $f_m(H) \geqq f_m(G)$.

PROOF OF THEOREM 3. Let us provisionally write p_i for the inf over j of $f_i(H_j)$. By Lemma 1, $f_i(G) \leqq f_i(H_j)$ and hence $f_i(G) \leqq p_i$.

Next we claim that p_i is central in $W(G)$. For $f_i(H_j)$ is central in $W(H_j)$; hence $f_i(H_k)$ commutes elementwise with $W(H_j)$ if $H_k \supset H_j$. Now p_i is a weak limit of the projections $f_i(H_k)$. Hence p_i commutes elementwise with $W(H_j)$, hence with their union, and finally with the weak closure of their union, which is all of $W(G)$.

We know now that p_i and $f_i(G)$ are both central projections, and we have $f_i(G) \leqq p_i$. The desired equality of p_i and $f_i(G)$ can now most easily be obtained by appealing to the theory of polynomial identities, for which we refer the reader to [1] and the references given there. In fact, suppose we succeed in proving that $p_i W(G)$ satisfies the identity that characterizes matrix algebras of degree i or less; then $p_i \leqq f_i(G)$ necessarily follows and we have $p_i = f_i(G)$. Now to verify this identity in $p_i W(G)$ it is enough to verify it in each $p_i W(H_j)$, for $W(G)$ is the weak closure of the union of $W(H_j)$. But the identity is valid in $p_i W(H_j)$, since $p_i \leqq f_i(H_j)$. This concludes the proof of Theorem 3.

We shall need the following lemma on finite groups.

LEMMA 2. *Let G be the direct product of $2k$ finite non-commutative groups. Then $r_m(G) \leqq 1/2^k$ for $m \leqq 2^k$.*

PROOF. We know that $r_m(G) = S_m\, m^2/n$, where n is the order of G and S_m is the number of m-dimensional irreducible representations of G. Now any irreducible representation of G is a Kronecker product of irreducible representations of its factors. In factoring thus a representation of degree $m \leqq 2^k$, it must be the case that at least k of the factors are one-dimensional. Further, the number of one-dimensional representations of a non-commutative group is at most half the order of the group. From these facts we get the estimate given in the lemma.

On putting together Lemma 2 and Theorem 3 (or Lemma 1 would do in place of Theorem 3) one obtains immediately the theorem proved by Mautner in [7].

THEOREM 4. *Let G be the (discrete) direct product of an infinite number* *of non-commutative finite groups. Then $W(G)$ is of Type II_1.*

In concluding this section we give an example which serves to illustrate Theorems 1–3. (This example is due to B. H. Neumann, and was communicated to me by K. A. Hirsch). Let H be a finite group whose center coincides with its commutator subgroup and is of order 2; for example, H may be the quaternion group or the dihedral group of order 8. Let G be direct product of an infinite number of copies of H, with amalgamated centers. Then G again has a center=commutator subgroup of order 2. By Theorem 2, $W(G)$ is the direct sum of a commutative algebra and a factor of type II_1, each carrying weight 1/2. Again G is a locally finite group; if we let H_j be the product of j copies of H with amalgamated centers, we get a set of defining subgroups. We find $r_1(H_j) = r_{2j}(H_j) = 1/2$, and in the limit the invariants of G are obtained: $r_1(G) = r_0(G) = 1/2$.

5. Infinite conjugate classes. The following is the theorem proved by Mautner in [6].

THEOREM 5. *Let G be a discrete group, G_0 its subgroup of finite conjugate classes, and suppose G/G_0 is infinite. Then $W(G)$ is of type II_1.*

PROOF. We follow the same idea as that used by Mautner. Let Z denote the common center of W and W', and Z' the commuting algebra of Z. Now if W has a direct summand, say of type I_m, then the corresponding direct summand of W' is likewise of type I_m, and the corresponding direct summand of Z' is of type I_{m^2}. The existence of such a summand will be ruled out by proving that Z' contains an infinite set of equivalent orthogonal non-zero projections.

Let G_j denote a typical coset of G mod G_0, and let E_j be the operator which projects G on G_j; that is, E_j is the identity on G_j and annihilates elements not in G_j. Then $E_j \in Z'$. For we have merely to prove that E_j commutes with characteristic functions of finite conjugate classes, and this follows from a simple computation. Again let U be the (unitary) operator of left multiplication by any element in G_j. Then $U^{-1}E_jU = E_0$. Moreover U is in Z' since it is even in W. Thus the projections E_j are the desired set of equivalent orthogonal non-zero projections in Z'.

6. CCR-groups. Our objective is to extend Theorem 7.3 of [4] to arbitrary $*$-representations, and we adopt the terminology of that paper without further discussion. We need a preliminary lemma.

LEMMA 3. *Any GCR-algebra A contains a non-zero self-adjoint element x such that xAx is commutative.*

PROOF. Before launching the proof, we collect three remarks for which we shall have repeated use.

1. If A has a non-zero projection e, then the lemma need only be proved in eAe. For $x \in eAe$ implies $xAx = x(eAe)x$; and by [4, Th. 7.4], eAe is

again a GCR-algebra.

2. Suppose I is a closed ideal in A. Then the problem can be reduced from A to I; in fact, if x is a self-adjoint non-zero element in I with xIx commutative, then x^2Ax^2 is commutative, and $x^2 \neq 0$.

3. Let B be a CCR-algebra with a Hausdorff structure space X; let U be a non-void open subset of X with closure Y; and let J be the intersection of the primitive ideals comprising Y. We shall show how the problem of proving the lemma can be reduced from B to B/J. Suppose then that we have a non-zero self-adjoint element y in B/J such that $y(B/J)y$ is commutative. Let z be any self-adjoint element in B mapping on y. Suppose for definiteness that z does not vanish at the point P of U. Select a real-valued continuous function f on X which satisfies $f(P) = 1$ and vanishes outside a tiny neighborhood of P. By [4, Th. 3. 3], the element $x = fz$ is a well defined element of B. It satisfies the condition that xBx is commutative, since it does so modulo every primitive ideal. Finally x is non-zero since it is non-zero at P.

The proof of Lemma 3 will now be carried out in several successive steps.

Case I. A satisfies a polynomial identity, say the one for n by n matrices. By [4, p. 237], A contains a non-zero closed ideal which is homogeneous of degree n. By remark 2, we may therefore assume that A itself is homogeneous of degree n. We note that by [4, Th. 4. 2], A has a Hausdorff structure space X. Take any P in X, and let $t \in A$ be a self-adjoint element mapping into a projection of rank one in A/P. Let f be any continuous real-valued function of a real variable, which vanishes in a neighborhood of 0, and takes the value 1 in a neighborhood of 1. Then the element $s = f(t)$ maps into a projection in an entire neighborhood of P. There must be a suitable smaller neighborhood U where s maps into a projection $\neq 1$, for otherwise $s(P)$ would be 1. Let Y be the closure of U, let J be the intersection of the primitive ideals comprising Y, let $C = A/J$, and let e be the image of s mod J. By remarks 1 and 3, our problem can be reduced to the algebra eCe. Since the latter satisfies a polynomial identity for matrices of degree less than n, we may cite an induction on n.

Case II. A is a CCR-algebra with a non-zero projection e. By remark 1, we reduce the problem to the algebra eAe, which by [4, Th. 7. 4] is again CCR. Now by [4, Th. 6. 1], the algebra eAe has a non-zero ideal satisfying a polynomial identity. By remark 2, this returns us to Case I.

Case III. A is a CCR-algebra with a Hausdorff structure space X. The construction in Case I may be repeated verbatim, and leads us to a suitable homomorphic image containing a projection. To this we apply Case II.

Case IV. A is any GCR-algebra. By [4, Th. 6. 2] there is a non-zero ideal which is a CCR-algebra with a Hausdorff structure space. We apply Case III. This concludes the proof of Lemma 3.

We now prove the desired generalization of [4, Th. 7. 3].

THEOREM 6. *Any $*$-representation of a GCR-algebra is of type I.*

PROOF. We may suppse that we have a faithful $*$-representation of the GCR-algebra A. In other words, A is a GCR-algebra of operators, and we have to prove that the weak closure A_1 is of type I. It is enough to prove that A_1 contains a single abelian projection, for then transfinite induction will complete the proof. By Lemma 3, A contains a non-zero self-adjoint element x such that xAx is commutative; xA_1x is likewise commutative. Now A_1 contains a non-zero projection which is a multiple of x. This is the desired abelian projection.

As suggested in [4], we define a CCR-group to be a locally compact group G with the property that for any irreducible unitary representation, the extension to the L_1-algebra of G consists of completely continuous operators. The importance of this class of groups has increased since Harish-Chandra [2] has proved that *any connected semi-simple Lie group is a CCR-group.*

THEOREM 7. *Any unitary representation of a CCR-group is of type I (that is, the weakly closed algebra generated by the representing operators is of type I). In particular, any unitary representation of a connected semi-simple Lie group is of type I.*

PROOF. Theorem 7 follows from Theorem 6 and some known facts which we sketch. Let G be a locally compact group, A its L_1-algebra, B the result of re-norming A by assigning to every element the sup of its norms in all possible $*$-representations, and C the completion of B in this new norm. Then C is a $C*$-algebra, and it is known that there is a 1-1 correspondence between $*$-representations of C and (strongly continuous) unitary representations of G. The statement that G is a CCR-group is precisely equivalent to saying that C is a CCR-algebra. We now apply Theorem 6.

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