CENTERING OF AN OPERATOR ALGEBRA

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1. Introduction. The recent progress of the "rings of operators" (or equivalently *W^*-algebra* in the sence of I. E. Segal), through the monumental works by F. J. Murray and J. von Neumann, brought out many algebraic notions, like as trace, direct product, fundamental group, algebraic type, genus, etc. And, more recently, J. Dixmier publishes a remarkable paper [1] and establishes the theory of the generalized trace, without assuming a W^*-algebra being a factor and also the separability of the space. He defines "trace" as a linear operation $\natural$ on the algebra (considered as a Banach space) onto its center with the properties

$$(1)\ (xy)^\natural = (yx)^\natural \qquad (2)\ x \in Z \rightarrow x = x^\natural,$$
$$(3)\ x^{*\natural} = x^{\natural *}, \qquad (4)\ (xx^*)^\natural = 0 \rightarrow x = 0,$$
$$(5)\ x \geqq 0 \rightarrow x^\natural \geqq 0, \qquad (6)\ (x^\natural y)^\natural = x^\natural y^\natural.$$

On the other hand, we have today rapidly growing literatures concerning uniformly closed operator algebra on a Hilbert space, i. e., C^*-algebra in the sense of I. E. Segal [7]. The theories of C^*-algebras and locally compact groups can be interpreted each other (as it is presented in I. E. Segal [7]). Although the study of C^*-algebras developed very much, it has still a wide gap against W^*-algebras at present, and the structures of group algebras are not explained so well by the former.

It is desirable that this gap should be filled in. In the present note we propose, for a trial, to introduce into a C^*-algebra the operation "centering" which is an analogue of the Dixmier trace. We will prove in the following that the existence of such operation is closely connected with an algebraic structure of the algebra. This is done as below by the help of a notion which is introduced recently by I. Kaplansky [4].

Throughout our note, we will use the standard terminologies of C^*-algebras due to I. E. Segal [7]. In the notation, we will use somewhat different symbols to that of Segal as follows: Large roman letters denote C^*-algebras and their subalgebras, small roman letters denote their elements, greek capitals denote the sets of states or primitive ideals, small greeks denote real numbers (α, β, etc.) or indices (γ, δ, etc.) or states (ω, σ, τ, χ etc.), except otherwise explicitly stated. Instead to say that an operator a is non-negative definite, we will call it *positive* and use the symbol $a \geqq 0$.

§ 2 contains some definitions and preliminary considerations, § 3 consists of the statement of main theorem, § § 4–5 are devoted for its proof and § 6 will give an integral representation of a (numerical) trace in the sense of [5] on the character space.

In this occation, one of the authors wishes to withdraw the false § 5 of the previous paper [5]. We also wish to express our thanks to H. Yoshizawa who kindly reported the errors of the proofs of Theorems 4 and 5 of [5].[1)]

2. Definitions. We bigin to repeat some descriptions of definitions concerning the linear functionals defined on a C^*-algebra having the identity (considered as a Banach space over the complex number field). A linear functional ω is called a state if $\omega(xx^*) \geqq 0$ and $\omega(1) = 1$. A state τ is called a trace if $\tau(xy) = \tau(yx)$ holds for any pair of x and y. The set of all traces of an algebra is called the *trace space*. The trace space is a convex, weakly* closed subset in the conjugate space of the algebra, whence it has extreme points by the well-known Krein-Milman Theorem. We will call these extreme points as *characters* (and denote by χ), and call as the *character space* the set X of all characters (including its weak* topology as a subset of the trace space). If there exists for any given x a trace τ such as $\tau(xx^*) > 0$ then we will call that the algebra has *sufficiently many traces*. If an algebra has sufficiently many traces then it has also sufficiently many characters by the Krein-Milman Theorem. The set of all maximal ideals of the algebra is called, following I. E. Segal [6], the *spectrum* with the Stone topology. It is known that the spectrum of an algebra having the identity is compact.

Following I. Kaplansky [4], we will give some notions belonging to the algebraic structures: A C^*-algebra is called *strongly semi-simple* provided that the intersection of all maximal ideals vanishes, and *central* provided that two primitive ideals P and P' coincide if and only if $P \cap Z = P' \cap Z$ where Z denotes the center of the algebra. It is proved by I. Kaplansky [4] that *a central C^*-algebra is strongly semi-simple, and that each primitive ideal is maximal in a central algebra.* It is also proved by him that *the spectrum of a central C^*-algebra is homeomorphic with that of the center by the natural mapping.*

Hereafter, we need, instead of Theorems 4 and 5 of [5] which we have failed to prove, the following

HYPOTHESIS A. *For a character χ, the set of all x such that $\chi(xx^*) = 0$ is a maximal ideal.*

This implies at once

LEMMA B. *A C^*-algebra having sufficiently many traces and satisfying Hypothesis A is strongly semi-simple.*

LEMMA C. *Under same hypothesis, maximal ideals, which are determined as a vanishing points of characters, are dense in the spectrum of the algebra.*

1) Between the first presentation and the correcting (the later is occurred by the advise of H. Yoshizawa) the authors have an oppotunity to see a recent paper of R. Godement [0]. His results are closedly connected with our's. For an example, he shows (among many other things) that the Dixmier trace is the centering in our sense and Hypothesis A (cf. §2) is satisfied for a W^*-algebra of the finite class (*i. e.*, "ring of operators" having sufficiently many traces).

For the later use, we shall add here some modifications of the notion of the centrality: A C^*-algebra is ***weakly central*** if and only if the definitive property of the centrality is held for maximal ideals in stead of primitive ideals, and ***hypercentral*** if it is (i) weakly central, (ii) having sufficiently many traces and (iii) the correspondence of the characters and the maximal ideals are one-to-one.

Finally, we will modify the definition of "trace" due to J. Dixmier [2]: With (1)-(5) and, instead of (6),

(6′) $\tau(x) = \tau(x^{\natural})$ *for any trace* τ.

In this case the operation will be called ***centering***, $x^{\natural}$ will be called the *centering of* x.

3. THEOREM. *A C^*-algebra with the identity and satisfying Hypothesis A has centering if and only if it is hypercentral.*

4. Proof of the Necessity. Suppose that a C^*-algebra R has the centering.

Firstly, we will prove that *the algebra has sufficiently many traces.* Considering the centering as a linear operation on the algebra onto the center, there exists as usual the conjugate operation of the centering which maps linearly the conjugate space of the center to that of the algebra. We denote the latter by $\omega^{\natural}$ for convenience. This mapping clearly carries the states of the center to linear functionals of the algebra. Moreover, (5) implies that the image of a state of the center is positive, and (2) implies that it gives the unity for the identity, that is, it is a state of the algebra. Furthermore, (1) implies that it is a trace. On the other hand, by (4), non-zero positive element x is mapped by the centering to a positive element of the center, whence there exists a state ω_0 of the center which gives strictly positive for $x^{\natural}$. Since $\omega_0^{\natural}(x) = \omega_0(x^{\natural})$ by the definition, $\omega_0^{\natural}$ gives strictly positive value for x. This proves the above statement.

Since the condition (6′) tells us that the traces of the algebra are completely determined on the center, two distinct traces are distinct also on the center, that is, the conjugate mapping of the centering carries the state space of the center onto the trace space of the algebra in one-to-one fashion. Since the both spaces are compact in the weak* topologies, and since the natural mapping is the converse of the conjugate of the centering and is continuous, the correspondence is topological. Moreover, since the conjugate operation is linear, the pure states (which are exactly extremes of the state space) of the center are continuously mapped onto the character space. Therefore *the character space is compact* since pure states of the center are compact, and *characters correspond one-to-one to pure states of the center.*

On the other hand, there exists a one-to-one mapping of the character space into the spectrum as described in §2. We will now show that this mapping is continuous. Suppose that M belongs to the closure of $\{M_\delta\}$, i.e., $M \geqq \bigcap_\delta M_\delta$ in the spectrum. Let χ_δ and χ be the corresponding characters of M_δ and M respectively. Clearly the hypothesis implies $M \cap Z \geqq \cap_\delta (M_\delta \cap Z)$, that is, $M \cap Z$ belongs to the closure of $\{M_\delta \cap Z\}$ in the spectrum of the center, whence there exists an δ such as $|\chi_\delta(x^\natural) - \chi(x^\natural)| < \varepsilon$ for any given x and real $\varepsilon > 0$. Since (6′) implies $\chi(x^\natural) = \chi(x)$ we have also $|\chi_\delta(x) - \chi(x)| < \varepsilon$, that is, the mapping is continuous. The image of the character space by such mapping is dense in the spectrum, the compactness of the character space implies at once that the character space is homeomorphic with the spectrum. The later conclusion and that of preceeding paragraph show that the mapping $M \leftarrow\rightarrow M \cap Z$ gives a homeomorphism between the spectra of the algebra and it center. Therefore, *the algebra is weakly central.*

Since two characters vanish on a maximal ideal, they define a same homomorphism of the center, whence they coincide on the center by the assumption. Therefore (iii) is obvious. This proves the neccessity.

In the remainder of this section we wish to prove that, *for each primitive ideal P, if the quotient algebra R/P satisfies either Hypothesis A or strongly semi-simplicity, then the R becomes central in Kaplansky's sense.*

Suppose now that P is a non-maximal primitive ideal of the algebra. By the existence of the identity, it is easy to prove that P is closed. Hence R/P is a primitive C^*-algebra with the identity. We will prove that R/P *has a centering* which we denote the new operation by $\natural$ without use of a new symbol, by putting $x^{\theta\natural} = x^{\natural\theta}$ where x^θ means the residue class containing x. To prove first that the operation can be well-defined, it needs to show that the centering of an element of P belongs to P too. Suppose that a maximal ideal M contains P and that a character χ corresponds to M by the preceeding argument. Clearly χ vanishes on M, whence its vanishing point on the center is precisely $M \cap Z$. . This and (6′) imply $M \cap Z = M^\natural$ where $M^\natural$ is the set of all centerings of elements of M. On the other hand, it is known that $P \cap Z$ is maximal in the center, whence $P \cap Z = M \cap Z$. This shows, by the above, that $P^\natural$ is contained in P, as was to be proved. Properties (1)–(5) for our new operation on R/P is obvious. Clearly, it leaves the identity invariant. To prove (6′), suppose that τ' is a trace of R/P. Put $\tau(x) = \tau'(x^\theta)$. It is easy to verify that τ is a trace of R. Hence $\tau(x) = \tau(x^\natural)$ implies $\tau'(x^\theta) = \tau'(x^{\theta\natural})$ by the definition. This shows our statement.

Finally, we will prove using the above considerations that *the algebra in question is central.* By the existence of the centering in the quotient algebra R/P. we can prove the first part of our present proof, that is, we can show that the quotient algebra R/P has sufficiently many traces. Hence by Lemma B, R/P is strongly semi-simple. Let now M be a maximal ideal containing P. Then the weak centrality implies the uniqueness of M, that is, R/P has only one maximal ideal M/P which does not vanish by the assumption. This contradicts to the strong semi-simplicity.

5. Proof of the Sufficiency. Suppose that the algebra R is hypercentral. Then, by Kaplansky's theorem, spectra of the algebra and its center are homeomorphic.

Suppose now that a directed set of characters χ_δ converges weakly* to τ. By the compactness of the trace space, τ is a trace. On the other hand, χ_δ converges weakly* on the center to τ. Since the pure states form a compact set in the center, τ gives a pure state on the center. We will prove that τ is a character. Assume the contrary and let I be the kernel of τ. I is an ideal and is not maximal. Let Γ be the set of all traces vanishing on I. Γ is non-void by the assumption and weakly* compact, whence, it has at least two extreme points, for otherwise by the preceeding paragraph τ is unique and τ must be a character (because $\tau = \alpha_1\tau_1 + \alpha_2\tau_2$ implies $\tau_1, \tau_2 \in \Gamma$). This contradicts to the non-maximality of I. Suppose χ and χ' be such two extreme points. Clearly they are characters of the algebra, since the equality $\chi = \alpha\omega + \beta\sigma$ implies ω, $\sigma \in \Gamma$, contrary to the hypothesis. Since distinct characters give distinct maximal ideals as their kernels, there exist two maximal ideals M and M' containing I, or $M \cap Z = M' \cap Z$. This is a contradiction to the centrality. Incidentally we have proved that the character space is compact.

Since the natural mapping of the character space onto the character space of the center is one-to-one continuous, and since the later is homeomorphic to the spectrum of the center, *the character space and the spectrum of the center is homeomorphic.*

Now, we come to the step to construct the centering. Let x be an element of the algebra. Then $\chi(x)$ is a continuous function of the character space. Since the spectrum of the center is homeomorhic, with the character space $\chi(x)$ can be considered as a continuous function of the spectrum of the center. On the other hand, by the well-known theorem due to I. Gelfand, M. Neumark and R. Arens [1], the center of a C^*-algebra is isometrically isomorphic with the algebra of all continuous functions on its spectrum. Hence there exists one and only one element z of the center such that $\chi(z) = \chi(x)$ for all χ. Put $z = x^\natural$.

The linearity of characters implies that of the above operation $^\natural$. (Boundedness follows from the fact $|\chi| = 1$). (2) is immediate, (3) and (5) follow from the reality of characters. (1) is clear since characters are traces. (6′) is immediate for characters by the definition. Since the trace space is generated by characters, (6′) is true for all traces. Hence it remains only to prove (4). Let x be a positive element. Then by the hypercentrality there exists a character χ with $\chi(x) > 0$. This shows that $x^\natural$ is positive and it is non-zero since $x \neq 0$. This proves the theorem.

6. An Application. Suppose that the algebra is hypercentral and τ is a trace. τ is a linear functional on the center which is precisely equivalent to the full continuous function algebra. Hence by Riesz-Markoff-

Kakutani's theorem there is a regular measure μ on the spectrum X_0 of the center such that

$$\tau(x^\natural) = \int_{X_0} \chi(x^\natural) d\mu(\chi).$$

By (6′), the above equality implies that *for any trace τ of the hypercentral algebra there exists a regular measure μ on the character space such that*

$$\tau(x) = \int_{X} \chi(x) d\mu(\chi).$$

It is expected that this integral representation of traces may do some service in proving the Bochner's theorem for a locally compact group whose group algebra is hypercentral. The authors wish to have an oppotunity to discuss this and related problems in the future.

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