ON A BOUNDARY VALUE PROBLEM OF SYSTEMS OF PATHS IN PLANE

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Many results concerning properties in the large of geodesics in Riemannian spaces are known, some of them may be considered in other spaces without the concept of length, but it seems to the author that no general method to deal with such problems is known.

In this paper, we shall investigate the boundary problem of a system of paths in plane which are given by a differential equation of the second order

$$y'' = A(x,y)y'^3 - B(x,y)y'^2 + C(x,y)y' - D(x,y), \tag{1}$$

where $A(x,y)$, $B(x,y)$, $C(x,y)$ and $D(x,y)$ are bounded continuous functions of x and y and dashes denote derivatives with respect to x.

S. Sasaki have investigated the special cases of the problem in which A, B, C and D are constants and obtained the result as follows[1]:

These systems of paths are classified into two types. For one of them any two points can be bound always by a path, and for any ather point can be bound by a path with those and only those point which lie between certain parallel lines at equal distance from the first points. We call these types (α) and (β) respectively.

In Part I, we shall investigate the special cases in another method. Our result may be stated in a more detailed form as follows:

THEOREM 1. *For the system of paths in plane which are given by the differential equation of the second order*

$$y'' = Ay'^3 - By'^2 + Cy' - D, \tag{2}$$

where A, B, C and D are all constants, if, and only if, the conditions

$$\Phi = (27A^2D - 9ABC + 2B^3)^2 + (27AD^2 - 9BCD + 2C^3)^2 \neq 0, \tag{3}$$

$$\Psi = 4(AC^3 + B^3D) - B^2C^2 + 27A^2D^2 - 18ABCD > 0 \tag{4}$$

are satisfied, it is of the type (β) and otherwise it is of the type (α).

Then, in Part II, we shall investigate the general case by means of a fixed point theorem in functional spaces. Our main result may be stated as follows:

THEOREM 2. *Let* (1) *be the differential equation of a system of paths in*

1) S SASAKI, A boundary value problem of some special ordinary differential equations of the second order, Journal of the Mathematical Society of Japan, Vol. 1, No. 2, 1949, pp. 79–90.

plane. For two given points (a_0, b_0), (a_1, b_1) $(a_0 < a_1)$, suppose that there exists a path binding them in the system of paths which are given by the differential equation

$$y'' = A(a_0, b_0)y'^3 - B(a_0, b_0)y'^2 + C(a_0, b_0)y' - D(a_0, b_0).$$

Then, if the differential equations

$$\frac{dz}{dx} = f_1(z), \qquad \frac{dz}{dx} = F_1(z),$$

where $f_1(z)$ and $F_1(z)$ are continuous functions of z such that

$$f_1(z) < f(z) = \text{g. l. b. } P(x, y, z),$$
$$F_1(z) > F(z) = \text{l. u. b. } P(x, y, z),$$
$$P(x, y, z) = A(x, y)z^3 - B(x, y)z^2 + C(x, y)z - D(x, y)$$

are soluble in $|x| \leqq a_1 - a_0$ under the initial condition

$$z(0) = \frac{b_1 - b_0}{a_1 - a_0},$$

there exists a path binding them in the given system of paths.

Part I

§1. Reduction of (2). As is well known, (1) is the equation of paths in an affinely connected space whose parameters of connexion Γ^i_{jk} $(i, j, k = 1, 2)$ are given by $\Gamma^1_{11} = 2g + C$, $\Gamma^1_{12} = \Gamma^1_{21} = f$, $\Gamma^1_{22} = A$, $\Gamma^2_{11} = D$, $\Gamma^2_{12} = \Gamma^2_{21} = g$, $\Gamma^2_{22} = 2f + B$ in the coordinates $x^1 = x$, $x^2 = y$, where f and g are any functions of x^1 and x^2. In the space, the paths are determined also by the differential equations

$$\ddot{x}^i + \Gamma^i_{jk}\, \dot{x}^j \dot{x}^k = 0 \qquad (i = 1, 2), \tag{5}$$

where dots denote derivatives with respect to an affine parameter of path.

Now, in the following, we assume that A, B, C, D, f, g are all constants. For any pair of constants $(\lambda, \mu) \neq (0, 0)$, we get from (5) the relation

$$(\lambda\ddot{x} + \mu\ddot{y}) + (\lambda(2g + C) + \mu D)\dot{x}\dot{x} + 2(\lambda f + \mu g)\dot{x}\dot{y}$$
$$+ (\lambda A + \mu(2f + B))\dot{y}\dot{y} = 0.$$

In order that the part of the second order with respect to $\dot{x}$ and $\dot{y}$ in the left side of the above relation is proportional to $(\lambda\dot{x} + \mu\dot{y})^2$, it is necessary and sufficient that

$$\frac{\lambda(2g + C) + \mu D}{\lambda^2} = \frac{\lambda f + \mu g}{\lambda\mu} = \frac{\lambda A + \mu(2f + B)}{\mu^2},$$

that is

$$-f\lambda^2 + (g + C)\lambda\mu^2 + Dr^2 = 0,$$
$$A\lambda^2 + (f + B)\lambda\mu - g\mu^2 = 0. \tag{6}$$

Accordingly, f and g have to satisfy the relation

$$-f : g + C : D = A : f + B : -g. \tag{7}$$

Now, we distinguish three cases as follows:

Case I. $A \neq 0$, $D \neq 0$. If we put $y = y_1 + \frac{B}{3A}x$, we get

$$y_1'' = Ay_1'^3 + \frac{3AC - B^2}{3A} y_1'^2 - \frac{1}{27A^2}(27A^2D - 9ABC + 2B^3).$$

Furthermore, if

$$27A^2D - 9ABC + 2B^3 = 27A^2D_1 \neq 0, \tag{8}$$

putting $y_1 = -D_1 y_2$, we get

$$y_2'' = AD_1^2 y_2'^3 + \frac{3AC - B^2}{3A} y_2' + 1.$$

Lastly, putting $y_2 = -(AD_1^2)^{-1} y_3$, $x = -(AD_1^2)^{-\frac{1}{3}} x_3$, we get the canonical form

$$\frac{d^2y_3}{dx_3^2} = -\left(\frac{dy_3}{dx_3}\right)^3 + c\frac{dy_3}{dx_3} + 1, \tag{9}$$

where

$$c = -\frac{3(3AC - B^2)}{(27A^2D - 9ABC + 2B^3)^{\frac{2}{3}}}. \tag{10}$$

In the above process, if $D_1 = 0$, we may treat the case in Case II.

Case II. $A = 0$, $D \neq 0$ or $A \neq 0$, $D = 0$. We may put $A = 0$, $D \neq 0$. Then, if $B = 0$, we can easily see that the case is of the type (α). Hence, we may consider only the case $B \neq 0$. We shall reduce (2) to a canonical form as follows: If we put $y = y_1 + \frac{C}{2B}x$, we get

$$y_1'' = -By_1'^2 - \frac{4BD - C^2}{4B}.$$

Then, if

$$4BD - C^2 \neq 0, \tag{11}$$

putting $y_1 = \mp \frac{1}{B} y_2$, $x = 2|4BD - C^2|^{-\frac{1}{2}} x_2$, we get the canonical form

$$\frac{d^2y_2}{dx_2^2} = \pm\left(\frac{dy_2}{dx_2}\right)^2 + 1, \tag{12}$$

where signs $+$, $-$ correspond to $4BD - C^2 > 0$, < 0 respectively. In the above process, if $4BD - C^2 = 0$, the case may be treated in Case III.

Case III. $A = D = 0$. We can easily see that the case is of the type (α).

§2. Canonical cases. For the system of paths given by the differential equation

$$y'' = -y'^3 + cy' + 1, \tag{9'}$$

(7) becomes

$$-f : (g + c) : 1 = -1 \; ; f : g.$$

Hence, f is a solution of the equation

$$f^3 - cf - 1 = 0, \tag{13}$$

which has at least a real solution $\neq 0$. Conversely, f satisfying the above equation and $g = 1/f$ satisfy (7). Making use of such a pair of f and g, we can easily obtain the conclusions as follows:

i) If $(g+c)^2 - 4f > 0$, (6) has two real solutions with respect to $\lambda : \mu$, hence this case is of the type (α).

ii) If $(g+c)^2 - 4f < 0$, (6) has two conjugate imaginary solutions with each other, hence it is of the type (β).

iii) If $(g+c)^2 - 4f = 0$, f is $4^{\frac{1}{3}}$ and it follows $c^3 = 27/4$. Then, we can take also $f_1 = -2^{-\frac{1}{3}}$ as a solution of (13). For such a pair of f_1 and $g_1 = 1/f_1$, $f_1^2 - 4g_1 > 0$. Accordingly, this case is of the type (α), since (6) has two real solutions.

Let us consider the relations between the above cases i), ii), iii) and c. If we put $H = (g+c)^2 - 4f$, then for a solution f of (13) and $g = 1/f$ it is written as $H = f(cf - 3)$. If $c = 0$, we may have by (13) that $f^3 = 1$, hence $H = -3f < 0$. If $c < 0$, we see easily that $f > 0$, since $f(f^2 - c) = 1$. Hence, $H < 0$. Lastly, if $c > 0$, $H < 0$ for $0 < f < 3/c$. On the other hand, if we put $\varphi(f) = f^3 - cf - 1$, we have the relations $\varphi(0) = -1$ and $\varphi'(0) = -c < 0$. Accordingly, in order that $f^3 - cf - 1 = 0$ has a solution such that $0 < f < 3/c$, it is necessary and sufficient that $\varphi(3/c) = 27/c^3 - 4 > 0$, that is $c^3 < 27/4$. Thus, from these considerations we see that the canonical case (9′) is of the type (β), if, and only if, $c^3 < 27/4$.

For the system of paths given by the differential equation

$$y'' = \varepsilon y'^2 + 1, \tag{12'}$$

where $\varepsilon = \pm 1$, (7) becomes

$$-f : g : 1 = 0 : f - \varepsilon : g.$$

If $\varepsilon = -1$, we may put $f = 0$, $g = \pm 1$ and $(\lambda, \mu) = (1, 0), (1, g)$. Hence, this case is of the type (α). If $\varepsilon = 1$, we may put $f = 1$, $g = 0$. Then, the second equation of (6) vanishes. From the first equation of (6) we get $\lambda : \mu = 1 : \pm\sqrt{-1}$. Accordingly, this case is of the type (β).

§3. Conclusion. By means of the considerations in the preceding paragraphs, we can say that the system of paths given by the differential equation $y'' = Ay'^3 - By'^2 + Cy' - D$ is of the type (β), if, and only if, the constants A, B, C, D satisfy the conditions as follows:

I. $A, D \neq 0$,

$$\Phi = (27A^2D - 9ABC + 2B^3)^2 + (27AD^2 - 9BCD + 2C^3)^2 \neq 0, \tag{3}$$

and

$$\Psi = 4(AC^3 + B^3D) - B^2C^2 + 27A^2D^2 - 18ABCD > 0. \tag{4}$$

For the condition $c^3 < 27/4$ is rewritten by (10) as (4).

II. $A = 0$, B, $D \neq 0$ (or $D = 0, C, A \neq 0$),

$$4BD - C^2 > 0 \quad (\text{or } 4CA - B^2 > 0).$$

Let us now consider the case A, $D \neq 0$ and $\Phi = 0$ which was not discussed in §2. By the argument as in the case I in §1 we get

$$y_1'' = Ay_1'^3 + \frac{3AC - B^2}{3A} y_1'$$

which is equivalent to (2). In order that it is of the type (β), it is necessary and sufficient that $3AC - B^2 > 0$ or $3DB - C^2 > 0$, by means of the above condition II. But, in this case, neither of them occurs. For, if $A,\ D \neq 0$ and $\Phi = 0$, that is $27A^2D - 9ABC + 2B^3 = 0$ and $27AD^2 - 9BCD + 2C^3 = 0$, it follows that B or $C \neq 0$, and $B^3D = C^3A$. Hence, $B, C \neq 0$. Accordingly, we obtain from the first of the above relations $27A^3C^3 - 9AB^4C + 2B^6 = 0$. Therefore, we get the relation $(3AC - B^2)^3 = 27A^3C^3 - 27A^2B^2C^2 + 9AB^4C - B^6 = -3B^2(3AC - B^2)^2 < 0$, hence $3AC - B^2 < 0$. We get also $3DB - C^2 < 0$. Thus, we see that the case is of the type (α).

Let us now arrange the conditions I, II. For the system of conditions $\Phi \neq 0$, $\Psi > 0$, it is clear that $A^2 + D^2 \neq 0$. If $A, D \neq 0$, it becomes the condition I. If $A = 0$, $D \neq 0$, it is equivalent to the system of conditions $\Phi = (2B^3)^2 + (2C^3 - 9BCD)^2 \neq 0$, $\Psi = B^2(4BD - C^2) > 0$, that is $B \neq 0, 4BD - C^2 > 0$, which is the condition II. Thus, we have completed the proof of the Theorem 1.

Part II

§1. A Lemma. For a differential equation of the second order,

$$y'' = A(x,y)y'^3 - B(x,y)y'^2 + C(x,y)y' - D(x,y), \tag{1}$$

where $A(x,y)$, $B(x,y)$, $C(x,y)$ and $D(x,y)$ are bounded continuous functions in plane, let

$$P(x,y,z) = A(x,y)z^3 - B(x,y)z^2 + C(x,y)z - D(x,y),$$

and

$$f(z) = \underset{(x,y)}{\text{g. l. b.}}\ P(x,y,z),$$
$$F(z) = \underset{(x,y)}{\text{l. u. b.}}\ P(x,y,z).$$

Then, we have the following lemma.

LEMMA 1. *$f(z)$ and $F(z)$ are continuous for $-\infty < z < \infty$ and satisfy Lipschitz's condition in any interval.*

Let us put

$$Q(x,y,z) = \frac{\partial P(x,y,z)}{\partial z} = 3A(x,y)z^2 - 2B(x,y)z + C(x,y),$$

$$R(x,y,z) = \frac{1}{2}\frac{\partial^2 P(x,y,z)}{\partial z^2} = 3A(x,y)z - B(x,y),$$

and define the auxiliary functions as follows:

$$f_2(z) = \underset{(x,y)}{\text{g. l. b.}}\ Q(x,y,z), \qquad F_2(z) = \underset{(x,y)}{\text{l. u. b.}}\ Q(x,y,z),$$
$$f_1(z) = \underset{(x,y)}{\text{g. l. b.}}\ R(x,y,z), \qquad F_1(z) = \underset{(x,y)}{\text{l. u. b.}}\ R(x,y,z),$$
$$f_0 = \underset{(x,y)}{\text{g. l. b.}}\ A(x,y), \qquad F_0 = \underset{(x,y)}{\text{l. u. b.}}\ A(x,y).$$

Let $h > 0$, then we can easily obtain the following relations.

$$3hf_0 \leqq f_1(z+h) - f_1(z) \leqq 3hF_0,$$
$$3hf_0 \leqq F_1(z+h) - F_1(z) \leqq 3hF_0,$$
$$2hf_1(z) + 3h^2 f_0 \leqq f_2(z+h) - f_2(z) \leqq 2hF_1(z) + 3h^2(2F_0 - f_0),$$
$$2hf_1(z) + 3h^2(2f_0 - F_0) \leqq F_2(z+h) - F_2(z) \leqq 2hF_1(z) + 3h^2 F_0,$$
$$hf_2(z) + h^2 f_1(z) + h^3 f_0 \leqq f(z+h) - f(z)$$
$$\leqq hF_2(z) + h^2(2F_1(z) - f_1(z)) + h^3(4F_0 - 3f_0),$$
$$hf_2(z) + h^2(2f_1(z) - F_1(z)) + h^3(4f_0 - 3F_0)$$
$$\leqq F(z+h) - F(z) \leqq hF_2(z) + h^2 F_1(z) + h^3 F_0.$$

These relations show that $f(z)$, $F(z)$ are countinuous and there exist two constants $m(a,b)$, $M(a,b)$ such that

$$|f(z_2) - f(z_1)| \leqq m(a,b)|z_2 - z_1|,$$
$$|F(z_2) - F(z_1)| \leqq M(a,b)|z_2 - z_1|$$

for $a \leqq z_1$, $z_2 \leqq b$. Lemma 1 is established.

§2. Functional space $\mathfrak{B}$. Let $f = (y(x), z(x))$ be a pair of two continuous functions in a given closed interval $a_0 \leqq x \leqq a_1$. Then, the set of such elements becomes a vector space, as is well known, if we define two operations such that

$$cf = (cy(x), cz(x)),$$
$$f_1 + f_2 = (y_1(x) + y_2(x), z_1(x) + z_2(x)),$$

where c is a real constant. Furthermore, if we define the norm of f by

$$\|f\| = \max_{a_0 \leqq x \leqq a_1} |y(x)| + \max_{a_0 \leqq x \leqq a_1} |z(x)|,$$

it has the following properties

$$\|cf\| = |c|\,\|f\|, \quad \|f_1 + f_2\| \leqq \|f_1\| + \|f_2\|.$$

Since we can easily verify that the linear normed space is complete, it is a Banach space[2]. We denote the space by $\mathfrak{B}$.

Now, let $H(x,\tau)$ be the function of x, τ defined by

$$H(x,\tau) = \begin{cases} (a_1 - x)(\tau - a_0)/(a_1 - a_0) & (x \geqq \tau), \\ (a_1 - \tau)(x - a_0)/(a_1 - a_0) & (x < \tau) \end{cases} \tag{14}$$

and $\bar{f} = (\bar{y}(x), \bar{z}(x)) = \varphi(f)$, $f \in \mathfrak{B}$ be the transformation of $\mathfrak{B}$ into $\mathfrak{B}$ defined by

$$\bar{y}(x) = -\int_{a_0}^{a_1} H(x,\tau) P(\tau, y(\tau), z(\tau)) d\tau + \frac{(a_1 - x)b_0 + (x - a_0)b_1}{a_1 - a_0}, \tag{15}$$

$$\bar{z}(x) = -\int_{a_0}^{a_1} \frac{\partial}{\partial x} H(x,\tau) P(\tau, y(\tau), z(\tau)) dz + \frac{b_1 - b_0}{a_1 - a_0}.$$

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2) S. BANACH, Théorie des Opérations Linéaires, 1932, Ch. V.

In fact, $\bar{f} = \varphi(f) \in \mathfrak{B}$. For we can easily obtain by means of (14), (15) the following relations

$$\frac{d}{dx}\bar{y}(x) = \bar{z}(x), \quad \frac{d}{dx}\bar{z}(x) = P(x, y(x), z(x)), \tag{16}$$

hence

$$\int_{x_2}^{x_1} f(z(\tau))d\tau \leqq \bar{z}(x_2) - \bar{z}(x_1) \leqq \int_{x_2}^{x_1} F(z((\tau))dz.$$

These relations show that $\bar{f} = (\bar{y}(x), \bar{z}(x)) \in \mathfrak{B}$. We get also $\bar{y}(a_0) = b_0$, $y(a_1) = b_1$.

(16) shows that if f is invariant under φ, then we have

$$\frac{d}{dx}y(x) = z(x), \quad \frac{d^2}{d^2x}y(x) = \frac{d}{dx}z(x) = P(x, y(x), \frac{d}{dx}y(x)).$$

Thus we have obtained the following lemma.

LEMMA 2. *An invariant element $f = (y(x), z(x))$ under φ determines a solution of* (1) *$y = y(x)$ binding (a_0, b_0), (a_1, b_1) such that $z(x) = y'(x)$. The converse is also true.*

LEMMA 3. *φ is continuous and compact*[3].

Let $f_0 = (y_0(x), z_0(x)) \in \mathfrak{B}$. On the closed domain Δ in the coordinate space (x,y,z) defined by $a_0 \leqq x \leqq a_1$, $|y| \leqq \max|y_0(x)| + 1 = M$, $|z| \leqq \max|z_0(x)| + 1 = N$, for any $\varepsilon > 0$, there exists a $0 < \delta < 1$ such that if $|y_1 - y_2| < \delta$, $|y_1|$, $|y_2| \leqq M$, $|z_1 - z_2| < \delta$, $|z_1|$, $|z_2| \leqq N$, then $|P(x, y_2, z_2) - P(x,y_1,z_1)| < \varepsilon$. For any $f \in \mathfrak{B}$ such that $\|f - f_0\| < \delta$, since we have $|y(x) - y_0(x)| < \delta$, $|z(x) - z_0(x)| < \delta$, $a_0 \leqq x \leqq a_1$, it follows that $|y(x)| < M$, $|z(x)| < N$. Accordingly, we get

$$|\bar{y}(x) - \bar{y}_0(x)| \leqq \int_{a_0}^{a_1} |H(x,\tau)|\,|P(\tau,y(\tau),z(\tau)) - P(\tau,y_0(\tau),z_0(\tau))|\,d\tau$$
$$< (a_1 - a_0)^2\varepsilon,$$
$$|\bar{z}(x) - \bar{z}_0(x)| \leqq \int_{a_0}^{a_1} \left|\frac{\partial}{\partial x}H(x,\tau)\right|\,|P(\tau, y(\tau), z(\tau) - P(\tau, y_0(\tau), z_0(\tau))|\,d\tau$$
$$< (a_1 - a_0)\varepsilon,$$

hence

$$|\varphi(f) - \varphi(f_0)| < ((a_1 - a_0) + (a_1 - a_0))^2\varepsilon.$$

Thus, the continuity of φ has been proved. In order to prove that φ is compact, it is clearly sufficient to prove that for any bounded set $G \subset \mathfrak{B}$, the component functions of the elements $\in \varphi(G)$ are uniformly bounded and equi-continuous. Let G_1 be a set of elements $f = (y(x), z(x))$ of $\mathfrak{B}$ such that $|y(x)| < M_1$, $|z(x)| < N_1$ $(a_0 \leqq x \leqq a_1)$, and let L be the maximum of $P(x, y, z)$ for $a_0 \leqq x \leqq a_1$, $|y| \leqq M_1$, $|z| \leqq N_1$. Then, we get easily the relations

3) S. BANACH, *loc. cit.*, p. 96.

$$|y(x)| \leqq \int_{a_0}^{a_1} |H(x,\tau)|\,|P(\tau, y(\tau), z(\tau)|d\tau + \left|\frac{(a_1 - x)b_0 + (x - a_0)b_1}{a_1 - a_0}\right|$$
$$\leqq (a_1 - a_0)^2 L + \max\{|b_0|, |b_1|\}.$$
$$|\bar{z}(x)| \leqq (a_1 - a_0)L + \left|\frac{b_1 - b_0}{a_1 - a_0}\right|.$$

By virtue of (15), for $a_0 \leqq x_1 < x_2 \leqq a_1$ we get

$$|\bar{y}(x_2) - \bar{y}(x_1)| \leqq \frac{x_2 - x_1}{a_1 - a_0}\Big\{((x_1 - a_0)^2 + (x_2 - a_0)(x_2 - x_1)$$
$$+ (a_1 - x_1)(x_2 - x_1) + (a_1 - x_2)^2)L + |b_1 - b_0|\Big\},$$

hence

$$|\bar{y}(x_2) - \bar{y}(x_1)| \leqq |x_2 - x_1| \left\{2(a_1 - a_0)L + \left|\frac{b_1 - b_0}{a_1 - a_0}\right|\right\}$$

and analogously

$$|\bar{z}(x_2) - \bar{z}(x_1)| \leqq L|x_2 - x_1|.$$

Thus, the lemma is established.

Now, let $P(x, y, z\,;\,\varepsilon)$ be the function defined by

$$P(x, y, z\,;\,\varepsilon) = P((1 - \varepsilon)a_0 + \varepsilon x,\ (1 - \varepsilon)b_0 + \varepsilon y, z),$$

then $P(x,y,z;0) = P(a_0, b_0, z)$, $P(x,y,z\,;1) = P(x,y,z)$. Let $\varphi_\varepsilon : \mathfrak{B} \to \mathfrak{B}$ be a transformation defined by $\varphi_\varepsilon((y(x), z(x)) = (\bar{y}(x), \bar{z}(x))$, such that

$$\bar{y}(x) = -\int_{a_0}^{a_1} H(x,\tau)P(\tau, y(\tau), z(\tau)\,;\,\varepsilon)d\tau + \frac{(a_1 - x)b_0 + (x - a_0)b_1}{a_1 - a_0,},$$

$$\bar{z}(x) = -\int_{a_0}^{a_1} \frac{\partial}{\partial x} H(x,\tau)P(\tau, y(\tau), z(\tau)\,;\,\varepsilon)d\tau + \frac{b_1 - b_0}{a_1 - a_0}.$$

We can easily see that φ_ε has the same properties as φ. Furthermore, we may prove the following lemma, with slight modifications of the argument in the proof of Lemma 3.

LEMMA 4. *$\varphi_\varepsilon(f)$, as a transformation $\mathfrak{B} \times I \to \mathfrak{B}$, is continuous and compact.*

In the lemma, I denotes the interval $0 \leqq \varepsilon \leqq 1$.

§3. Lemmas for a differential equation of the second order. Let us assume that the differential equation of the second order

$$\frac{d^2y}{dx^2} = P\left(a_0, b_0, \frac{dy}{dx}\right)$$

has a solution $y = H(x)$ such that $H(a_0) = b_0$, $H(a_1) = b_1$. By means of Theorem 1 in Part I, we can easily see whether there exists such a solution or not.

Let $f_1(z)$, $F_1(z)$ be two continuous functions such that

$$f(z) > f_1(z),\ \ F(z) < F_1(z) \qquad (-\infty < z < \infty), \tag{17}$$

and consider the differential equations as follows:

$$\frac{d^2y}{dx^2} = f_1\left(\frac{dy}{dx}\right),\quad \frac{d^2y}{dx^2} = F_1\left(\frac{dy}{dx}\right). \tag{18}$$

Let $g(x)$ and $G(x)$ be the solutions of them which satisfy the initial conditions $y(a_0) = b_0$, $y'(a_0) = m$, where $m = (b_1 - b_0)/(a_1 - a_0)$. Now, we assume that $g(x), G(x)$ are defined at least on $[a_0, a_1]$. Then, we shall prove the following lemma.

LEMMA 5. $G(x) > g(x) \qquad (a_0 < x \leqq a_1)$.

Since $g'(a_0) = G'(a_0) = m$, $g''(a_0) = f_1(m) < F_1(m) = G''(a_0)$, it follows that $g(x) < G(x)$ $(x \neq a_0)$ in a sufficiently small right neighborhood of a_0. Suppose that there exist such values of x that $g(x) = G(x)$ $(a_0 < x \leqq a_1)$ and that x_1 be the minimum of them. Let x_2 be a value of x, where $G(x) - g(x)$ becomes its maximum in the interval $a_0 < x < x_1$. Then, $G'(x_2) = g'(x_2)$ and $F_1(G'(x_2)) = G''(x_2) \leqq g''(x_2) = f_1(g'(x_2))$, which is a contradiction. The lemma is established.

Let now be $g(x)$ and $G(x)$ determined in the interval $2a_0 - a_1 \leqq x \leqq a_1$ and put

$$\begin{aligned} l &= \min\{\min_{2a_0 - a_1 \leqq x \leqq a_1} g'(x),\ \min_{2a_0 - a_1 \leqq x \leqq a_1} G'(x)\}, \\ L &= \max\{\max_{2a_0 - a_1 \leqq x \leqq a_1} g'(x),\ \max_{2a_0 - a_1 \leqq x \leqq a_1} G'(x)\}. \end{aligned} \tag{19}$$

Since $g'(a_0) = G'(a_0) = m$, $g''(a_0) = f_1(m) < F_1(m) = G''(a_0)$, we have $l < L$, $l \leqq m \leqq L$.

Take a suitable positive number $\delta_1 < 0$ such that

$$G_1(x) = G(x) - \delta_1 < H(x) < g(x) + \delta_1 = g_1(x),$$

in the interval $a_0 \leqq x \leqq a_1$. Then, we shall prove the following lemma.

LEMMA 6. *Let $y = \psi(x)$, $z = \psi'(x)$ be a solution of the differential equations*

$$\frac{dy}{dx} = z,\quad \frac{dz}{dx} = P(x, y, z\,; \varepsilon)$$

such that $\psi(a_0) = b_0$, $\psi(a_1) = b_1$, and $G_1(x) \leqq \psi(x) \leqq g_1(x)$, $l \leqq \psi'(x) \leqq L$ for $a_0 \leqq x \leqq a_1$, then none of the signes of equality in the above relations holds good.

We suppose in the first place that $\psi(x_1) = g_1(x_1)$ $(a_0 \leqq x_1 \leqq a_1)$. Since $G_1(a_0) < H(a_0) = b_0 = \psi(a_0) < g_1(a_0)$, $G_1(a_1) < H(a_1) = b_1 = \psi(a_1) < g_1(a_1)$, it follows that $a_0 < x_1 < a_1$. Then we have $g'(x_1) = g_1'(x_1) = \psi'(x_1)$, $g''(x_1) = g_1''(x_1) \geqq \psi''(x_1)$. On the other hand, we have $g_1''(x_1) = g''(x_1) = f_1(g'(x_1)) < f(g'(x_1))$, hence $g_1''(x_1) < f(g_1'(x_1)) \leqq P(x_1, \psi(x_1), \psi'(x_1) : \varepsilon) = \psi''(x_1)$. This contradicts to the above relation. Therefore, we see that $\psi(x) < g_1(x)$. We get similarly $G_1(x) < \psi(x)$.

Since $\psi(a_0) = b_0$, $\psi(a_1) = b_1$, we have at least an x_0 such that

$$\psi'(x_0) = \frac{b_1 - b_0}{a_1 - a_0} = m, \qquad a_0 < x_0 < a_1.$$

$y = G_2(x) = G(x - x_0 + a_0)$ is also a solution of the second of (18) and $G_2'(x_0) = G'(a_0) = m = \psi'(x_0)$. Hence we get

$$\psi''(x_0) = P(x_0, \psi(x_0), m; \varepsilon) \leqq F(m) < F_1(m) = G_2''(x_0).$$

It follows from the above relation that if we take a small $\delta_2 > 0$, $\psi(x) < G_2'(x)$ for $x_0 < x < x_0 + \delta_2$. But this inequality holds good in the interval $x_0 < x \leqq a_1$. For otherwise, let x_2 be the minimum of x such that $\psi'(x) = G_2'(x)$, $x_0 < x < a_1$, then $\psi''(x_2) = P(x_2, \psi'(x_2), \psi'(x_2); \varepsilon) \leqq F(\psi'(x_2)) < F_1(\psi'(x_2)) = G_2''(x_2)$. This shows that $G_2(x) < \psi'(x)$ in a sufficiently small left neighborhood of x_2, which contradicts to the definition of x_2. Making use of $g_2(x) = g(x - x_0 + a_0)$, we see also that $g_2'(x) < \psi'(x)$ $(x_0 < x \leqq a_1)$. Hence, we get the inequality $g_2'(x) < \psi'(x) < G_2'(x)$ $(x_0 < x \leqq a_1)$. By means of the same method as above, the inequality $G_2'(x) < \psi'(x) < g_2'(x)$ holds good for $a_0 \leqq x < x_0$.

By means of (19) we get easily $l < \psi'(x) < L$.

§4. Proof of Theorem 2. Using the same notations as in the previous pargraphs, let M be the subset of elements $f = (y(x), z(x))$ in $\mathfrak{B}$ such that

$$G_1(x) < y(x) < g_1(x), \quad l < z(x) < L \qquad (a_0 \leqq x \leqq a_1).$$

Then, M is a bounded open set in $\mathfrak{B}$ and its closure $\bar{M}$ is the set of elements $f = (y(x), z(x))$ such that

$$G_1(x) \leqq y(x) \leqq g_1(x), \quad l \leqq z(x) \leqq L, \quad (a_0 \leqq x \leqq a_1).$$

By Lemma 4, there exists a compact set K such that $\varphi_\varepsilon(\bar{M}) \subset K$ for $0 \leqq \varepsilon \leqq 1$. Let Φ_ε be the translation defined by $\Phi_\varepsilon(f) = f - \varphi_\varepsilon(f)$, then the null element $0 = (0, 0)$ of $\mathfrak{B}$ does not become images of elements in the boundary of M under any Φ_ε. For otherwise, there exist an ε, f such that $f = \varphi_\varepsilon(f)$, $f \in \bar{M} - M$, that is, $f = (y(x), z(x))$, $y'(x) = z(x)$, $y''(x) = P(x, y(x), y'(x); \varepsilon)$, $y(a_0) = b_0$, $y(a_1) = b_1$, which contradict to Lemma 6, since for the elements of $\bar{M} - M$, at least one of the signs of equality in the relations $G_1(x) \leqq y(x) \leqq g_1(x)$, $l \leqq z(x) \leqq L$ have to hold good. Therefore, according to the theory of the degrees of mappings in functional spaces[4], the degree of Φ_ε at the point 0 with respect to M is constant for $0 \leqq \varepsilon \leqq 1$. By the assumption in § 3, the degree of Φ_0 at the point 0 is not zero. Hence the same is also true for Φ_1. Thus, we see that there exists an $f \in M$ such that $f = (y(x), z(x)) = \varphi_1(f)$, that is

$$\frac{dy(x)}{dx} = z(x), \quad \frac{dz(x)}{dx} = P(x, y(x), z(x)), y(a_0) = b_0, y(a_1) = b_1.$$

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4) LERAY-SCHAUDER, Topologie et équations fonctionelles. Annales de l'École Norm. Sup., 51(1934).