ABSOLUTE CESÀRO SUMMABILITY OF ORTHOGONAL SERIES

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Introduction

For a series $\sum_{n=0}^{\infty} a_n$ denote by σ_n^α the n-th Cesàro mean of order α $(\alpha > -1)$, i.e.,

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^\alpha a_k$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)}\ .$$

If the series

$$\sum_{n=0}^{\infty} |\sigma_{n+1}^\alpha - \sigma_n^\alpha|$$

converges, we say that the series $\sum a_n$ is absolutely Cesàro summable with order α or briefly summable $|C, \alpha|$.

Various theorems concerning this summability of orthogonal series and of Fourier series were obtained by many authors. One of them is the following F. T. Wang's theorem [5]

THEOREM A. (i) *If the series*

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2)(\log n)^{1+\epsilon} \tag{1}$$

is convergent for some $\varepsilon > 0$, *then the trigonometric series*

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{2}$$

is summable $|C, \alpha|$ *for almost every* x, *where* $\alpha > 1/2$;

(ii) *If the series*

$$\sum_{n=1}^{\infty} (a_n^2 + b_n)(\log n)^{2+\epsilon} \tag{3}$$

is convergent for some $\varepsilon > 0$, *then the series* (2) *is summable* $|C, 1/2|$ *for almost every* x.

(iii) *If* $0 < \alpha < 1/2$, *and if the series*

$$(4) \qquad \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\, n^{1-2\alpha}\, (\log n)^{1+\epsilon}$$

is convergent for some $\varepsilon > 0$, *then the series* (2) *is summable* $|C, \alpha|$ *for almost every* x.

F. T. Wang showed by a counter example that one cannot suppress the number $\varepsilon > 0$ in the proposition (iii), but he said nothing about the suppression of ε in the results (i) and (ii).

In Part I of this paper we shall complete this theorem by showing that the number $\varepsilon > 0$ is indispensable in the propositions (i) and (ii).

Further we shall give some sufficient conditions for the summability, from which we may deduce easily the Wang theorem.

In Part II we shall treat the Fourier series. For a function $f(x)$ of $L^1(0, 2\pi)$ the summability $|C, 1|$ of its Fourier series $\mathfrak{S}[f]$ is not in general of local property, that is, the summability $|C, 1|$ of $\mathfrak{S}[f]$ at a point $x = x_0$ is not completely decided by the behaviour of $f(x)$ in the neighbourhood of x_0, but from the total behaviour in the interval $(0, 2\pi)$ (cf. [1]).

On the other hand this summability is of local property under certain restriction of the functions. Prof. Sunouchi and the Author proved the following theorem [3, 4].

THEOREM B. *For a function* $f(x) \in L^p(0, 2\pi), (p > 1)$, *suppose that*

$$(5) \qquad \left(\frac{1}{t}\int_0^t |f(x_0+u) + f(x_0-u) - 2f(x_0)|^p\, du\right)^{1/p} = O\left(\left(\log \frac{1}{|t|}\right)^{-1-\epsilon}\right)$$

as $t \to 0$, *for some* $\varepsilon > 0$. *Then the Fourier series of* $f(x)$ *is summable* $|C, 1|$ *at the point* $x = x_0$.

In this theorem the condition (5) depends only on the behaviour of $f(x)$ in a neighbourhood of x_0, and one see that the summability $|C, 1|$ is of local property for the function of $L^p (p > 1)$.

We shall give an extension of this result and some related theorems.

In what follows we use $K_1, K_2, \dots$ to denote positive constants independent of variable (x, t), (x, n) or (t, n).

Part I. Metric theorems on orthogonal series

1. We shall begin by proving the following theorem.

THEOREM 1. *Let* $\{\varphi_n(x)\}$ *be an orthonormal system defined in the interval* (a, b) *and let* $\alpha > 0$. *If the series*

$$(1.1) \qquad \sum_{n=1}^{\infty} \frac{1}{n^{1+\alpha}} \left[\sum_{k=1}^{n} k^2 (n-k+1)^{2(\alpha-1)} a_k^2\right]^{1/2} + \sum_{n=1}^{\infty} \frac{|a_n|}{n^\alpha}$$

converges, then the orthogonal series

$$\sum_{n=1}^{\infty} a_n \varphi_n(x) \tag{1.2}$$

is summable $|C, \alpha|$ *for almost every* x.

PROOF. Let $\sigma_n^\alpha(x)$ be the n-th Cesàro mean of order α of the series (1.2), that is,

$$\sigma_n^\alpha(x) = \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha \; a_k \; \varphi_k(x).$$

Then easily we have

$$\begin{aligned}
&\sigma_{n+1}^\alpha(x) - \sigma_n^\alpha(x) \\
&= \frac{1}{A_{n+1}^\alpha} \sum_{k=1}^{n+1} A_{n-k+1}^\alpha \; a_k \varphi_k(x) - \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^\alpha \; a_k \; \varphi_k(x) \\
&= \sum_{k=1}^{n} \left(\frac{A_{n-k+1}^\alpha}{A_{n+1}^\alpha} - \frac{A_{n-k}^\alpha}{A_n^\alpha} \right) a_k \; \varphi_k(x) + \frac{1}{A_{n+1}^\alpha} a_{n+1} \varphi_{n+1}(x) \\
&= \sum_{k=1}^{n} \frac{A_{n-k}^\alpha}{A_{n+1}^\alpha} \frac{\alpha k}{(n+1)(n-k+1)} a_k \varphi_k(x) + \frac{1}{A_{n+1}^\alpha} a_{n+1} \varphi_{n+1}(x).
\end{aligned}$$

Hence by the Schwarz inequality we get

$$\begin{aligned}
&\int_a^b \sum_{n=1}^{\infty} |\sigma_{n+1}^\alpha(x) - \sigma_n^\alpha(x)| \, dx \\
&\leqq \sum_{n=1}^{\infty} \sqrt{b-a} \left[\int_a^b \{\sigma_{n+1}^\alpha(x) - \sigma_n^\alpha(x)\}^2 \, dx \right]^{1/2} \\
&\leqq \sqrt{b-a} \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \left(\frac{A_{n-k}^\alpha}{A_{n+1}^\alpha} \frac{\alpha k}{(n+1)(n-k+1)} \right)^2 a_k^2 + \left(\frac{1}{A_{n+1}^\alpha} \right)^2 a_{n+1}^2 \right]^{1/2} \qquad (1.3) \\
&\leqq \sqrt{b-a} \sum_{n=1}^{\infty} \left[\left\{ \sum_{k=1}^{n} \left(\frac{A_{n-k}^\alpha}{A_{n+1}^\alpha} \frac{\alpha k}{(n+1)(n-k+1)} \right)^2 a_k^2 \right\}^{1/2} + \frac{|a_{n+1}|}{A_{n+1}^\alpha} \right] \\
&\leqq K_1 \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \frac{k^2(n-k+1)^{2(\alpha-1)}}{n^{2(\alpha+1)}} a_k^2 \right]^{1/2} + K_2 \sum_{n=1}^{\infty} \frac{|a_n|}{n^\alpha} .
\end{aligned}$$

The last expression is finite by the hypothesis and immediately we get the theorem.

COROLLARY 1. *Let* $1 < p \leqq 2$ *and let* $1/p + 1/q = 1$.

	Hypotheses: ($\varepsilon > 0$)	Conclusions: *For almost every x, the series* (1.2) *is summable*
(1.4)	$\sum_{n=1}^{\infty} \lvert a_n \rvert^p (\log n)^{p-1+\epsilon} < \infty,$	$\lvert C, \alpha \rvert$ $(\alpha > 1/q)$;
(1.5)	$\sum_{n=1}^{\infty} \lvert a_n \rvert^p (\log n)^{p+\epsilon} < \infty,$	$\lvert C, 1/q \rvert$;
(1.6)	$\begin{cases} 0 < \alpha < 1/q, \\ \sum_{n=1}^{\infty} \lvert a_n \rvert^p n^{p-1-\alpha p} (\log n)^{p-1+\epsilon} < \infty, \end{cases}$	$\lvert C, \alpha \rvert$;
(1.7)	$\sum_{n=1}^{\infty} \lvert a_n \rvert^q n^{q/2-1} (\log n)^{q-1+\epsilon} < \infty,$	$\lvert C, \alpha \rvert$ $(\alpha > 1/p)$;
(1.8)	$\sum_{n=1}^{\infty} \lvert a_n \rvert^q n^{q/2-1} (\log n)^{q+\epsilon} < \infty,$	$\lvert C, 1/p \rvert$;
(1.9)	$\begin{cases} 0 < \alpha < 1/p, \\ \sum_{n=1}^{\infty} \lvert a_n \rvert^q n^{q(\alpha-1/2)} (\log n)^{q-1+\epsilon} < \infty, \end{cases}$	$\lvert C, \alpha \rvert$.

PROOF. In (1.3) we use the Hölder inequality and the Hausdorff-Young inequality. Then we obtain

$$\int_a^b \sum_{n=1}^{\infty} |\sigma_{n+1}^{\alpha}(x) - \sigma_n^{\alpha}(x)| \, dx$$

$$\leqq \sum_{n=1}^{\infty} (b-a)^{1/p} \left[\int_a^b |\sigma_{n+1}^{\alpha}(x) - \sigma_n^{\alpha}(x)|^q \, dx \right]^{1/q}$$

$$\leqq (b-a)^{1/p} \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \left(\frac{A_{n-k}^{\alpha}}{A_{n+1}^{\alpha}} \frac{\alpha k}{(n+1)(n-k+1)} \right)^p |a_k|^p \right.$$

$$\left. + \left(\frac{1}{A_{n+1}^{\alpha}} \right)^p |a_{n+1}|^p \right]^{1/p}$$

$$\leqq K_3 \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} \frac{k^p (n-k+1)^{p(\alpha-1)}}{n^{p(\alpha+1)}} |a_k|^p \right]^{1/p} + K_4 \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\alpha}} .$$

By the Hölder inequality, the first term of the last expression is majorated by:

$$K_5 \left(\sum_{n=1}^{\infty} \frac{1}{n(1+\log n)^{1+\epsilon}} \right)^{1/q} \left(\sum_{n=1}^{\infty} \frac{(1+\log n)^{(p-1)(1+\epsilon)}}{n^{1+\alpha p}} \sum_{k=0}^{n} k^p (n-k+1)^{p(\alpha-1)} |a_k|^p \right)^{1/p}.$$

The p-th power of this expression is not greater than

$$
\begin{aligned}
&K_6 \sum_{k=1}^{\infty} k^p |a_k|^p \sum_{n=k}^{\infty} \frac{(n-k+1)^{\alpha p - p}(\log n)^{(p-1)(1+\epsilon)}}{n^{1+\alpha p}} \\
&\leqq K_6 \sum_{k=1}^{\infty} k^p |a_k|^p \Big(\sum_{n=k}^{2k} + \sum_{n=2k+1}^{\infty} \Big) \frac{(n-k+1)^{\alpha p - p}(\log n)^{(p-1)(1+\epsilon)}}{n^{1+\alpha p}} \\
&\leqq \begin{cases} K_6 \sum_{1=k}^{\infty} k^p |a_k|^p \, O\Big(\frac{(\log k)^{(p-1)(1+\epsilon)}}{k^p} \Big) & \text{if } \alpha > 1/q, \\ K_6 \sum k^p |a_k|^p \, O\Big(\frac{(\log k)^{(p-1)(1+\epsilon)+1}}{k^p} \Big) & \text{if } \alpha = 1/q, \\ K_6 \sum k^p |a_k|^p \, O\Big(\frac{(\log k)^{(p-1)(1+\epsilon)}}{k^{1+\alpha p}} \Big) & \text{if } \alpha < 1/q. \end{cases}
\end{aligned}
$$

On the other hand the series

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^{\alpha}}$$

is not greater than

$$
\begin{cases}
\Big(\sum_{n=1}^{\infty} |a_n|^p (1+\log n)^{p-1+\epsilon} \Big)^{1/p} \Big(\sum_{n=1}^{\infty} \frac{1}{n^{\alpha q}(1+\log n)^{1+\epsilon/(p-1)}} \Big)^{1/q} & \text{if } \alpha > 1/q\,; \\
\Big(\sum_{n=1}^{\infty} |a_n|^p (1+\log n)^{p+\epsilon} \Big)^{1/p} \Big(\sum_{n=1}^{\infty} \frac{1}{n(1+\log n)^{(p+\epsilon)/(p-1)}} \Big)^{1/q} & \text{if } \alpha = 1/q\,; \\
\Big(\sum_{n=1}^{\infty} |a_n|^p \, n^{p-1-\alpha p} (1+\log n)^{p-1+\epsilon} \Big)^{1/p} \Big(\sum_{n=2}^{\infty} \frac{1}{n(1+\log n)^{1+\epsilon(q-1)}} \Big)^{1/q} & \text{if } \alpha < 1/q.
\end{cases}
$$

Consequently we deduce, by the above estimations, the first three propositions of the corollary.

The last three may be obtained from Theorem 1 by the repeated use of the Hölder inequality. q. e. d.

In the corollary, if we put $p = q = 1/2$. we get the Wang theorem A.

For the case of general p and q instead of 2, one may also obtain analogous conditions to that of Theorem 1, which implies Corollary and forms an extension of Theorem 1.

2. Let now $\{r_n(x)\}$ be the Rademacher system. For the series

$$\sum_{n=1}^{\infty} a_n r_n(x). \tag{2.7}$$

instead of general orthogonal series, an inverse of Theorem 1 will be established.

THEOREM 2. *Suppose that, for a set of positive measure $E\subset(0,1)$, the series* (2.1) *is summable* $|C,\alpha|$ $(\alpha>0)$. *Then the series* (1.1) *converges.*

For the proof we need a lemma.

LEMMA 1. *Given a set of positive measure $E\subset(0,1)$, there exist a positive constant $B=B(E)$ and a positive integer $N=N(E)$ such that for any sequence $\{c_{n,k}\}$ it holds the inequality:*

$$\int_E\left|\sum_{k=N}^{n} c_{n,k}r_k(x)\right|dx\geqq B\left(\sum_{k=N}^{n} c_{n,k}^2\right)^{1/2}\qquad(n>N). \tag{2.2}$$

This is essentially known as the Khintchine inequality, but for the sake of completeness we sketch the proof.

By the Hölder inequality

$$\int_E\left|\sum_{k=N}^{n} c_{n,k}\,r_k(x)\right|^2dx\leqq\left(\int_E\left|\sum_{k=N}^{n} c_{n,k}\,r_k(x)\right|dx\right)^{1/2}\left(\int_E\left|\sum_{k=N}^{n} c_{n,k}\,r_k(x)\right|^3dx\right)^{1/2}. \tag{2.3}$$

And by the Khintchine inequality

$$\int_E\left|\sum_{k=N}^{n} c_{n,k}\,r_k(x)\right|^3dx\leqq K_7\left(\sum_{k=N}^{n} c_{n,k}^2\right)^{3/2} \tag{2.4}$$

On the other hand, we get

$$\int_E\left(\sum_{k=N}^{n} c_{n,k}\,r_k(x)\right)^2dx=|E|\sum_{k=N}^{n} c_{n,k}^2+\sum_{\substack{k,j=N\\k\neq j}}^{n} c_{n,k}\,c_{n,j}\int r_k(x)r_j(x)\,dx=I+J \tag{2.5}$$

say. Then we have

$$|J|^2\leqq\left(\sum_{k,j=N}^{n} c_{n,k}\,c_{n,j}\right)\left(\sum_{k,j=N}^{n}\left|\int_E r_k(x)r_j(x)\,dx\right|^2\right).$$

The second factor of the right hand side becomes smaller when we take $N=N(E)$ sufficiently large in virtue of the Bessel inequality. For such n, we get

$$|J|\leqq\frac{1}{2}\,|E|\left(\sum_{k,j=N}^{n} c_{n,k}^2\,c_{n,j}^2\right)^{1/2}\leqq\frac{1}{2}\,|E|\sum_{k=N}^{n} c_{n,k}^2.$$

This and (2.5) imply that

$$\int_E\left|\sum_{k=N}^{n} c_{n,k}\,r_k(x)\right|^2dx\geqq\frac{1}{2}\,|E|\sum_{k=N}^{n} c_{n,k}^2. \tag{2.6}$$

From (2.3), (2.4) and (2.6) we get

$$\left(\int_E\left|\sum_{k=N}^{n} c_{n,k}r_k(x)\right|dx\right)^{1/2}\cdot K_7\left(\sum_{k=N}^{n} c_{n,k}^2\right)^{\frac{3}{2}\cdot\frac{1}{2}}\geqq\frac{1}{2}\,|E|\sum_{k=N}^{n} c_{n,k}^2,$$

from which we may deduce the required. q. e. d.

To prove Theorem 2, we may assume that the series (2. 1) is summable $|C,\alpha|$ uniformly in E, that is, we may put

$$(2.7)\qquad \int\sum_{n=1}^{\infty}\left|\sum_{k=1}^{n}\frac{A_{n-k}^{\alpha}}{A_{n+1}^{\alpha}}\frac{\alpha k}{(n+1)(n-k+1)}a_k r_k(x)+\frac{1}{A_{n+1}^{\alpha}}a_{n+1}r_{n+1}(x)\right|dx<K_8.$$

Consider the integer $N=N(E)$ founded in Lemma 1 and replace a_1, $a_2,\dots,a_{N-1}$ by zeros. This replacement has no influence in the inequality (2.7), if we take another constant K_9 instead of K_8, since the series under the integral sign does not varies but an absolutely and uniformly convergent series. Now applying Lemma 1 we obtain

$$B\sum_{n=1}^{\infty}\left[\sum_{k=N}^{n}\left(\frac{A_{n-k}^{\alpha}}{A_{n+1}^{\alpha}}\frac{\alpha k}{(n+1)(n-k+1)}a_k\right)^2+\left(\frac{a_{n+1}}{A_{n+1}^{\alpha}}\right)^2\right]^{1/2}<K_9.$$

Repeating the same reason we may replace N by 1, and using the asymptotic formulas of A_t^{α} we complete the proof.

3. We shall show that one cannot be allowed the suppression of $\varepsilon>0$ in Theorem A following the Paley and Zygmund argument [cf. 6, p. 125].

LEMMA 2. *Let* $A_n(x)=a_n\cos nx+b_n\sin nx\,(n=0,1,2,\dots)$ *and let* $\alpha>0$. *If the series*

$$(3.1)\qquad \sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}\left[\sum_{k=0}^{n}k^2(n-k+1)^{2(\alpha-1)}A_k(x)^2\right]^{1/2}+\sum_{n=1}^{\infty}\frac{|A_n(x)|}{n^{\alpha}}$$

converges in a set of positive measure E, *then the series*

$$(3.2)\qquad \sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}\left[\sum_{k=0}^{n}k^2(n-k+1)^{2(\alpha-1)}(a_k^2+b_k^2)\right]^{1/2}+\sum_{n=1}^{\infty}\frac{|a_n|+|b_n|}{n^{\alpha}}$$

converges. Conversely the convergence of (3. 2) *implies that of* (3. 1) *for almost every* x.

PROOF. Both series (3. 1) and (3. 2) are of positive terms and hence it is sufficient to prove that

$$\int_E\sum_{k=0}^{n}k^2(n-k+1)^{2(\alpha-1)}A_k(x)^2\,dx\sim\sum_{k=0}^{n}k^2(n-k+1)^{2(\alpha-1)}(a_k^2+b_k^2)$$

as $n\to\infty$, or more simply that

$$\int_E A_k(x)^2dx\sim a_k^2+b_k^2$$

as $k\to\infty$, which will be evident.

LEMMA 3. *If the series* (3. 2) *converges, then almost all series of*

$$(3.3)\qquad a_0+\sum_{n=1}^{\infty}\pm(a_n\cos nx+b_n\sin nx)$$

are summable $|C,\alpha|$ *for almost every* x, *and if the series* (3.2) *diverges, then almost all series* (3.3) *are non-summable* $|C,\alpha|$ *for almost every* x.

PROOF. Since we established Theorem 2 and Lemma 2 it remains to follow the Paley and Zygmund argument. q. e. d.

Now we are in a position to prove the following

THEOREM 3. (i) *There exists a trigonometric series*

$$(3.4)\qquad \sum_{n=1}^{\infty}c_n\cos nx$$

such that the series

$$(3.5)\qquad \sum_{n=1}^{\infty}c_n^2\log n$$

converges, and that the series (3.4) *is non-summable* $|C,\alpha|$ *for almost every* x *with any large* α.

(ii) *There exists a series* (3.4) *which is non-summable* $|C,1/2|$ *for almost every* x, *such that the following series converges*:

$$(3.6)\qquad \sum_{n=1}^{\infty}c_n^2(\log n)^2.$$

PROOF. (i) Put

$$a_n=\frac{1}{\sqrt{n}\,\log n\log\log n}\qquad(n=0,1,2,\dots)$$

where we understand a_n to be zero if the right hand side is negative or lose its sense. Then

$$\sum_{n=1}^{\infty}a_n^2\log n=\sum_{n=1}^{\infty}\frac{1}{n\log n(\log\log n)^2}<\infty.$$

The first term of (3.2), suppose $\alpha>1$ and $b_n=0$ $(n=1,2,\dots)$, is equal to

$$\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}\left[\sum_{k=0}^{n}\frac{k^2(n-k+1)^{2(\alpha-1)}}{k(\log k)^2(\log\log k)^2}\right]^{1/2}$$

$$\geq\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}\left[\sum_{k=[n/2]}^{n}\cdots\cdots\cdots\cdots\right]^{1/2}$$

$$\geq\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}\left[\frac{n}{2(\log n\log\log n)^2}\sum_{k=1}^{[n/2]}k^{2(\alpha-1)}\right]^{1/2}$$

$$\geq K_{10}\sum_{n=1}^{\infty}\frac{1}{n^{1+\alpha}}\,\frac{n^{\alpha}}{\log n\log\log n}$$

$$= K_{10}\sum_{n=1}^{\infty}\frac{1}{n\log n\log\log n} = \infty.$$

Hence, by Lemma 3, with a suitable choice of a sequence of signs, putting

$$c_n = \pm a_n \qquad (n = 0, 1, 2, \dots\dots),$$

we conclude the existence of the required series.

(ii) Similarly as above we put

$$a_n = \frac{1}{n^{1/2}(\log n)^{3/2}(\log\log n)} \qquad (n = 0, 1, 2, \dots\dots).$$

Then

$$\sum a_n^2(\log n)^2 = \sum\frac{1}{n\log n(\log\log n)^2} < \infty,$$

and the first term of (3.2) with $\alpha = 1/2$ is equal to

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\left[\sum_{k=1}^{n}\frac{k^2(n-k+1)^{-1}}{k(\log k)^3(\log\log k)^2}\right]^{1/2}$$

$$\geqq \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\left[\sum_{k=[n/2]}^{n}\dots\right]^{1/2}$$

$$\geqq \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\left[\frac{n}{2(\log n)^3(\log\log n)^2}\sum_{k=1}^{[n/2]}\frac{1}{k}\right]^{1/2}$$

$$\geqq K_{11}\sum_{n=1}^{\infty}\frac{1}{n\log n\log\log n} = \infty.$$

Therefore we get the conclusion by the similar reason as in the proof of (i). q. e. d.

Part II. Summability $|C, \alpha|$ of Fourier series at a point

4. We first establish an extension of Theorem B. In the sequel we write, for an integrable function $f(x)$,

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

and

$$\Phi_x^{(p)}(t) = \int_0^t |\varphi_x(u)|^p du.$$

We see by the Hölder inequality that

$$\left|\frac{1}{t}\Phi_x^{(r)}(t)\right|^{1/r} \leqq \left|\frac{1}{t}\Phi_x^{(s)}(t)\right|^{1/s}$$

if $0 < r < s$.

THEOREM 4. *For a function* $f(x) \in L^p(0, 2\pi)$ $(p > 1)$, *suppose that*

$$\Phi_x^{(p)}(t) = O\left(|t| \left(\log\left(\frac{1}{|t|}\right)\right)^{-p\varepsilon}\right) \qquad (\varepsilon > 0) \tag{4.1}$$

as $t \to 0$. *Then, if* $1/p - 1 < \alpha < 0$ *and if* $k < 1/|\alpha|$, *we have*

$$\sum_{m=1}^{n} |\sigma_m^\alpha(x) - f(x)|^k = O\left(n\,(\log n)^{-k\varepsilon}\right) \tag{4.2}$$

as $n \to \infty$, *where* $\sigma_n^\alpha(x)$ *is the n-th* (C,α) *mean of the Fourier series of* $f(x)$.

The case $\alpha = 0$ is Theorem B and if $\alpha > 0$ the theorem will be evident with any $k > 0$.

LEMMA 4. *Denote by* $G_n(x)$ *the n-th* (C,α) *mean of the series*

$$\frac{1}{\pi} + \frac{2}{\pi}\sum_{m=1}^{\infty} \cos mx.$$

Then

$$G_n^\alpha(x) = g_n^\alpha(x) + h_n^\alpha(x),$$

$$g_n^\alpha(x) = \frac{2}{\pi}\frac{1}{A_n^\alpha}\frac{\sin\left\{\left(n + \frac{1}{2} + \frac{\alpha}{2}\right)x - \frac{1}{2}\alpha\pi\right\}}{\left(2\sin\frac{x}{2}\right)^{1+\alpha}}$$

and $|G_n(x)| \leqq K_{12}n$, $|h_n^\alpha(x)| \leqq K_{12}n^{-1}x^{-2}$.

This lemma is due to Hardy and Littlewood [2].

PROOF OF THEOREM 4. Suppose in the proof, as we may, that $p \leqq 2$, and hence by $1/p - 1 < \alpha < 0$ we have

$$2 \leqq \frac{p}{p-1} < \frac{1}{|\alpha|}.$$

If (4.2) is proved for some exponent k, then, as we see by the Hölder inequality, it remains true for any positive exponent less than k. Therefore we may assume that

$$2 \leqq \frac{p}{p-1} \leqq k < \frac{1}{|\alpha|}. \tag{4.3}$$

Since

$$\sigma_m^\alpha(x) - f(x) = \int_0^\pi \varphi_x(t)\, G_m^\alpha(t)\, dt,$$

we deduce by the Minkowski inequality that

$$I_n \equiv \left(\sum_{m=n}^{2n} |\sigma_m^\alpha(x) - f(x)|^k\right)^{1/k}$$

$$= \left(\sum_{m=n}^{2n}\left|\int_0^\pi \varphi_x(t)\, G_n^\alpha(t)\, dt\right|^k\right)^{1/k}$$

$$\leqq \left(\sum_{m=n}^{2n}\left|\int_0^{1/n}\cdots\right|^k\right)^{1/k}+\left(\sum_{m=n}^{2n}\left|\int_{1/n}^{1/n^\beta}\cdots\right|^k\right)^{1/k}$$
$$+\left(\sum_{m=n}^{2n}\left|\int_{1/n^\beta}^{\pi}\cdots\right|^k\right)^{1/k}$$
$$= I_n^{(1)}+I_n^{(2)}+I_n^{(3)}$$

say, where $\beta<1$ is a fixed positive constant.

By Lemma 4, we have

$$(I_n^{(1)})^k \leqq \sum_{m=n}^{2n}\left(\int_0^{1/n}|\varphi_x(t)|K_{12}\, m\, dt\right)^k$$
$$= O\left(\Phi_x^{(1)}\left(\frac{1}{n}\right)^k\sum_{m=n}^{2n} m^k\right)$$
$$= O(n^{-k}(\log n)^{-k\epsilon}n^{k+1}) = O(n(\log n)^{-k\epsilon}).$$

To estimate $I_n^{(2)}$ we divide it into two parts:

$$I_n^{(2)} \leqq \left(\sum_{m=n}^{2n}\left|\int_{1/n}^{1/n^\beta}\varphi_x(t)g_m^\alpha(t)\,dt\right|^k\right)^{1/k}+\left(\sum_{m=1}^{2n}\left|\int_{1/n}^{1/n^\beta}\varphi_x(t)h_m^\alpha(t)\,dt\right|^k\right)^{1/k}$$
$$= P_n^{(2)}+Q_n^{(2)}$$

say. In $P_n^{(2)}$, observing that

$$g_m^\alpha(t)=\frac{2}{\pi}\frac{1}{A_n^\alpha(2\sin t/2)^{1+\alpha}}\left[\cos\left\{\left(\frac{1}{2}+\frac{\alpha}{2}\right)t-\frac{\alpha}{2}\pi\right\}\sin mt\right.$$
$$\left.+\sin\left\{\left(\frac{1}{2}+\frac{\alpha}{2}\right)t-\frac{\alpha}{2}\pi\right\}\cos mt\right],$$

we have easily by the Hausdorff-Young inequality

$$P_n^{(2)} = O\left(\frac{1}{n^{\alpha k'}}\int_{1/n}^{1/n^\beta}\frac{|\varphi_x(t)|^{k'}}{t^{(1+\alpha)k'}}\,dt\right)^{1/k'} \qquad \left(\frac{1}{k}+\frac{1}{k'}=1\right),$$

since $k\geqq 2$. From (4.3), $k'\leqq p$ and $(1+\alpha)k'>1$, hence by integration by parts we have

$$P_n^{(2)} = O\left(\frac{1}{n^{\alpha k'}}\left[\Phi_x^{(k')}(t)t^{-(1+\alpha)k'}\right]_{1/n}^{1/n^\beta}+\frac{1}{n^{\alpha k'}}\int_{1/n}^{1/n^\beta}\frac{\Phi_x^{(k')}(t)}{t^{(1+\alpha)k'+1}}\,dt\right)^{1/k'}$$
$$= O\left(\frac{1}{n^{\alpha k'}}\frac{(\log n)^{-k'\epsilon}}{n}n^{(1+\alpha)k'}+\frac{1}{n^{\alpha k'}}\int_{1/n}^{1/n^\beta}\frac{dt}{t^{(1+\alpha)k'}(\log 1/t)^{k'\epsilon}}\right)^{1/k'}$$
$$= O\left(n^{(k'-1)}(\log n)^{-k'\epsilon}+n^{k'-1}(\log n)^{-k'\epsilon}\right)^{1/k'}$$
$$= O\left(n^{1/k}(\log n)^{-\epsilon}\right).$$

Applying Lemma 4 again, we get

$$(Q_n^{(2)})^k = O\left(\sum_{m=n}^{2n}\left|\int_{1/n}^{1/n^\beta}\frac{|\varphi_x(t)|}{mt^2}\,dt\right|^k\right)$$

$$= O\left(\frac{1}{n^k}\sum_{m=n}^{2n}\left|\left[\Phi_r^{(1)}(t)t^{-2}\right]_{1/n}^{1/n^\beta} + \int_{1/n}^{1/n^\beta}\frac{\Phi_r^{(1)}(t)}{t^3}\,dt\right|^k\right)$$

$$= \frac{1}{n^k}\sum_{m=n}^{2n} O\left(n^{-1}(\log n)^{-\epsilon}n^2 + (\log n)^{-\epsilon}\int_{1/n}^{1/n^\beta} t^{-2}dt\right)^k$$

$$= \frac{1}{n^k}\sum_{m=n}^{2n} O\left(n^k(\log n)^{-k\epsilon}\right) = O\left(n(\log n)^{-k\epsilon}\right).$$

For $I_n^{(3)}$ we repeat the similar estimation:

$$I_n^{(3)} = \left(\sum_{m=n}^{2n}\left|\int_{1/n^\beta}^{\pi}\varphi_x(t)g_m^\alpha(t)\,dt\right|^k\right)^{1/k} + \left(\sum_{m=n}^{2n}\left|\int_{1/n^\beta}^{\pi}\varphi_x(t)h_m^\alpha(t)\,dt\right|^k\right)^{1/k}$$

$$= P_n^{(3)} + Q_n^{(3)}$$

say, then

$$P_n^{(3)} = O\left(\frac{1}{n^{\alpha k'}}\int_{1/n^\beta}^{\pi}\frac{|\varphi_x(t)|^{k'}}{t^{(1+\alpha)k'}}\,dt\right)^{1/k'}$$

$$= O\left(n^{-\alpha k'}n^{-\beta}(\log n)^{-k'\epsilon}n^{\beta(1+\alpha)k'} + n^{-\alpha k'}n^{\beta\{(1+\alpha)k'-1\}}(\log n)^{-k'\epsilon}\right)^{1/k}$$

$$= O\left(n^{-\alpha+\alpha\beta+\beta/k}(\log n)^{-\epsilon}\right)$$

$$= o\left(n^{1/k}(\log n)^{-\epsilon}\right),$$

since $-\alpha+\alpha\beta+\beta/k < 1/k$ by $\beta < 1$;

$$(Q_n^{(3)})^k = O\left(n^{-k}\sum_{m=n}^{2n} n^{-\beta}(\log n)^{-\epsilon}n^{2\beta} + (\log n)^{-\epsilon}\int_{1/n^\beta}^{\pi} t^{-2}dt\right)^k$$

$$= O\left(n^{-k}\sum_{m=n}^{2n} n^{\beta}(\log n)^{-\epsilon}\right)^k$$

$$= O\left(n^{1-k(1-\beta)}(\log n)^{-\epsilon}\right)^k$$

$$= o\left(n(\log n)^{-k\epsilon}\right).$$

Combining the above estimations we obtain easily

$$(I_n)^k = \sum_{m=n}^{2n}|\sigma_m^\alpha(x) - f(x)|^k = O\left(n(\log n)^{-k\epsilon}\right).$$

Therefore, if n is any integer, supposing $2^N \leqq n < 2^{N+1}$, we have

$$\sum_{m=1}^{n} |\sigma_m^\alpha(x) - f(x)|^k \leqq \sum_{s=0}^{N} \sum_{m=2^s}^{2^{s+1}} |\sigma_m^\alpha(x) - f(x)|^k$$

$$= \sum_{s=0}^{N} O(2^s s^{-k\epsilon}) = O\left(2^N N^{-k\epsilon}\right) = O\left(n (\log n)^{-k\epsilon}\right).$$

This completes the proof of Theorem.

In the condition (4.1) the function $\log(1/|t|)$ may be replaced by other decreasing functions of suitable order. For example, we state the following result without proof, since the proof may be done along the same line as that of Theorem 4.

COROLLARY 2. *For a function $f(x) \in L^p(p > 1)$, if $1/p - 1 < \alpha < 0$ and if $k < 1/|\alpha|$, then, at every point x where $\Phi_x^{(p)}(t) = o(t)$ we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} |\sigma_m^\alpha(x) - f(x)|^k = 0. \tag{4.4}$$

We shall add the following result which is a boundary case of Theorem 4.

COROLLARY 3. *If $f(x) \in L^p(p > 1)$ and if*

$$\Phi_x^{(p)}(t) = O\left(|t| \left(\log \frac{1}{|t|}\right)^{-p\epsilon}\right), \qquad (1 > \varepsilon > 0)$$

then, for $\max\left(\frac{1}{p} - 1, \varepsilon - 1\right) \leqq \alpha < 0$, we have

$$\sum_{m=1}^{n} |\sigma_m^\alpha(x) - f(x)|^{1/|\alpha|} = O\left(n (\log n)^{(1-\epsilon)/|\alpha| - 1}\right). \tag{4.5}$$

PROOF. In the proof of Theorem 4, we must only estimate $P_n^{(2)}$ and $P_n^{(3)}$. In the estimation of $P_n^{(2)}$, observing $(1 + \alpha)k' = 1$, we get

$$P_n^{(2)} = O\left(n^{-k'-1}(\log n)^{-k'\epsilon} + n^{-\alpha} n^{k'-1} (\log n)^{-k'\epsilon+1}\right)^{1/k'}$$

$$= O\left(n^{1/k} (\log n)^{-\epsilon+1/k'}\right) = O\left(n^{|\alpha|} (\log n)^{-\epsilon+1+\alpha}\right),$$

from which we get

$$(P_n^{(2)})^{1/|\alpha|} = O\left(n (\log n)^{(1-\epsilon)/|\alpha| - 1}\right),$$

The similar equality holds for $P_n^{(3)}$ remembering $\frac{1-\varepsilon}{|\alpha|} - 1 \geqq 0$, and we complete the proof.

5. We shall now study the absolute summability.

THEOREM 5. *If $f(x) \in L^p(p > 1)$ and if, for some $\varepsilon > 0$.*

$$\Phi_x^{(p)}(t) = O\left(|t| \left(\log \frac{1}{|t|}\right)^{-p-\epsilon}\right) \tag{5.1}$$

as $t \to 0$, at a point x, then the Fourier series of $f(x)$ is summable $|C, r|$ *($r > 1/p$) at the point.*

PROOF. Let us assume (5.1). By the well known formula we get

$$\sum_{n=1}^{\infty} |\sigma_n^r(x) - \sigma_{n-1}^r(x)| = \sum_{n=1}^{\infty} \frac{|\sigma_n^r(x) - \sigma_n^{r-1}(x)|}{n}$$

$$\leqq \sum_{n=1}^{\infty} \frac{|\sigma_n^r(x) - f(x)|}{n} + \sum_{n=1}^{\infty} \frac{|\sigma_n^{r-1}(x) - f(x)|}{n}$$

$$= S_1 + S_2$$

say. We may assume that $1/p < r < 1$. From (5.1) we deduce by the usual calculation that

$$\sigma_n^r(x) - f(x) = O((\log n)^{-1-\epsilon/p}),$$

from which we have

$$S_1 = O\left(\sum_{n=1}^{\infty} \frac{1}{n(\log n)^{1+\epsilon/p}}\right) = O(1).$$

Hence it remains us to prove the convergence of S_2. By the Hölder inequality we have

$$S_2 = \sum_{n=1}^{\infty} \frac{|\sigma_n^{r-1}(x) - f(x)|}{n} \tag{5.3}$$

$$\leqq \left(\sum_{n=1}^{\infty} \frac{1}{n(\log n)^{\delta k'}}\right)^{1/k'} \left(\sum_{n=1}^{\infty} \frac{|\sigma_n^{r-1}(x) - f(x)|^k (\log n)^{\delta k}}{n}\right)^{1/k}$$

where $\delta > 0$ and $k > 1$ will be determined later $(1/k + 1/k' = 1)$.

By the Abel transformation we obtain

$$\sum_{n=1}^{\infty} \frac{|\sigma_n^{r-1}(x) - f(x)|^k (\log n)^{\delta k}}{n} \tag{5.4}$$

$$\leqq \sum_{n=1}^{\infty} \Delta \frac{(\log n)^{\delta k}}{n} \sum_{m=1}^{n} |\sigma_m^{r-1}(x) - f(x)|^k + \lim_{n\to\infty} \frac{(\log n)^{\delta k}}{n} \sum_{m=1}^{n} |\sigma_m^{r-1}(x) - f(x)|^k$$

$$= \sum_{n=1}^{\infty} O\left(\frac{(\log n)^{\delta k}}{n^2}\right) O\left(n(\log n)^{-k(1+\epsilon/p)}\right) + \lim_{n\to\infty} \frac{(\log n)^{\delta k}}{n} O\left(n(\log n)^{-k(1+\epsilon/p)}\right),$$

by Theorem 4, if $1/p - 1 < r - 1 < 0$ and if $k < 1/(1-r)$. Now we take δ and k such as

$$k\delta - k(1 + \varepsilon/p) < -1, \quad k < 1/(1-r) \text{ and } \delta k' > 1. \tag{5.5}$$

For example we can set, as easily checked,

$$k = \frac{2}{2-r}, \quad \delta = \frac{1}{2}\left(r + \frac{\varepsilon}{p}\right).$$

Then the first factor in the last side of (5.3) is finite, and the second is, by (5.4),

$$O\Big(\sum_{n=1}^{\infty}\frac{1}{n(\log n)^{k(1+\epsilon/p)-k\delta}}\Big)+O\Big(\lim_{n\to\infty}(\log n)^{k\delta-k(1+\epsilon/p)}\Big)$$
$$=O(1)+o(1)=O(1),$$

and we obtained $S_2<\infty$. This proves the Theorem.

From Theorem 5 we see that, for any function $\in L^p(p>1)$ the summability $|C,r|$ $(r>1/p)$ is of local property. It will be remarkable that, for a function L^1, the summability $|C,1|$ is not necessarily of local property [1].

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