NOTES ON FOURIER ANALYSIS (XLIX): SOME NEGATIVE EXAMPLES

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1. It is well known that, if $\varphi(u)$ is integrable, then

$$\int_0^t \varphi(u)\,du = o(1) \qquad \text{as } t \to 0, \tag{1.1}$$

and that this can not be improved, that is, for any given function $\varepsilon(t)$ tending to zero with t, there exists an integrable function $\varphi(u)$ such that the relation

$$\left| \int_0^t \varphi(u)\,du \right| \geqq \varepsilon(t) \tag{1.2}$$

holds for infinitely many values of t tending to zero.

We shall show by example that (1.1) cannot be improved even when the Fourier series of $\varphi(u)$, supposed even, converges at $u = 0$. More precisely we shall prove the following

THEOREM 1. *For any given function $\varepsilon(t)$ tending to zero with t, there exists an integrable function $f(t)$ such that the Fourier series of $f(t)$ converges at $t = x$ and*

$$\left| \int_0^t \varphi_x(u)\,dv \right| \geqq \varepsilon(t) \tag{1.3}$$

for infinitely many values of t tending to zero, where

$$\varphi_x(u) = f(x+u) + f(x-u) - 2f(x).$$

On the other hand it is known [3] that:

(*) *If we denote by $\sigma_n^\beta(x)$ the n-th Cesàro mean of the β-th order of the Fourier series of an integrable function $f(t)$, and if*

$$\sigma_n^\beta(x) - f(x) = o(n^{\gamma-\beta}) \qquad \text{as } n \to \infty, \tag{1.4}$$

where $\beta > \gamma > -1$, then we have

$$\Phi_\alpha(t) = o(t^{\alpha+\beta-\gamma}) \qquad \text{as } t \to 0 \tag{1.5}$$

for $\alpha > 1 + \gamma$, where $\Phi_\alpha(t)$ is the α-th integral of $\varphi_x(t)$.

As a special case of this result we have the following theorem, and we shall give its simple proof.

THEOREM 2. *Let $f(x)$ be an integrable function and let $s_n(x)$ be the n-th* *partial sum of the Fourier series of* $f(x)$. *If*

(1.6) $$s_n(x) - f(x) = o(1/n^\gamma) \qquad as\ n \to \infty$$

for $0 < \gamma < 1$, *then we have*

(1.7) $$\int_0^t \varphi_x(u)\, du = o(t^{1+\gamma}) \qquad as\ n \to \infty.$$

Further we shall show that Theorem 2 is best possible, that is,

THEOREM 3. *Let* $\mathcal{E}(t)$ *be given such that* $\mathcal{E}(t)/t^{1+\gamma} \to 0$ *as* $t \to 0$, *and let* $0 < \gamma < 1$. *Then, there exists an integrable function* $f(t)$ *such that* (1.6) *holds for* $t = x$ *and that*

(1.8) $$\left| \int_0^t \varphi_x(u) du \right| \geqq \mathcal{E}(t)$$

holds for infinitely many t *tending to zero.*

Finally we prove the Theorem (*) is best possible, that is,

THEOREM 4. *Let* $\beta > \gamma > -1$ *and let* $\mathcal{E}(t)$ *be given such that* $\mathcal{E}(t)/t^{\alpha+\beta-\gamma} \to 0$ *as* $t \to 0$. *Then there exists an integrable function* $f(t)$ *such that* (1.4) *holds for* $t = x$ *and that*

(1.9) $$|\Phi_\alpha(t)| \geqq \mathcal{E}(t)$$

holds for infinitely many t, *tending to zero, where* $\alpha > 1 + \gamma$.

2. Proof of Theorem 1. Without loss of generality we can suppose that $x = 0$, and we shall find an even function $f(t) = \varphi_x(t)$.

Let us take a monotone vanishing sequence $\{t_n\}$, $t_n > 0$ $(n = 1, 2, \ldots.)$ and two sequences of positive numbers $\{u_n\}$, $\{v_n\}$ such that

(2.1) $$\sum_{n=1}^\infty \mathcal{E}(t_n) < \infty, \quad v_n/u_n \downarrow 0, \quad u_n/t_n \to 0 \qquad \text{as } n \to \infty$$

and that the intervals

$$(t_n - u_n, t_n + v_n) \qquad (n = 1, 2, \ldots.)$$

are mutually disjoint and contained in $(0, \pi)$.

Consider the sequence of sets

(2.2) $$\Delta_n = (t_n - u_n, t_n - v_n) \cup (t_n + v_n, t_n + u_n) \qquad (n = 1, 2, \ldots.)$$

which are mutually disjoint. Let us define an even function $f(t)$ as follows:

(2.3) $$f(t) = \frac{c_n t}{t_n - t} \quad \text{if } t \in \Delta_n \qquad (n = 1, 2, \ldots.)$$

and $f(t) = 0$ elsewhere in $(0, \pi)$, where $\{c_n\}$ is a sequence of positive numbers which will be determined later.

We have

$$\int_0^\pi |f(t)| dt = \sum_{n=1}^\infty c_n \int_{\Delta_n} \frac{t}{t_n - t} dt$$

$$= 2\sum_{n=1}^{\infty} c_n\, t_n \log \frac{u_n}{v_n}.$$

Hence if

$$\sum_{n=1}^{\infty} c_n\, t_n \log \frac{u_n}{v_n} < \infty, \tag{2.4}$$

the function $f(t)$ defined above is integrable.

Now, we have

$$\int_0^{\pi} f(t) \frac{\sin mt}{t}\, dt = \sum_{n=1}^{\infty} c_n \int_{\Delta_n} \frac{\sin mt}{t_n - t}\, dt \tag{2.5}$$

$$= \sum_{n=1}^{\infty} c_n \int_{\Delta_n} \{\sin mt_n \cos m(t-t_n) + \cos mt_n \sin m(t-t_n)\} \frac{dt}{t_n - t}$$

$$= \sum_{n=1}^{\infty} c_n \cos mt_n \int_{\Delta_n} \frac{\sin m(t-t_n)}{t_n - t}\, dt$$

$$= 2\sum_{n=1}^{\infty} c_n \cos mt_n \int_{v_n}^{u_n} \frac{\sin mt}{t}\, dt.$$

If we suppose that

$$\sum_{n=1}^{\infty} c_n < \infty, \tag{2.6}$$

then the last sum of (2.5) tends to zero as $m \to \infty$, by the uniform convergence of the sum and the Riemann-Lebesgue theorem. Hence the Fourier series of $f(t)$ converges to zero at $t = 0$ under the condition (2.6).

On the other hand we have

$$\int_{\Delta_n} f(t)\, dt = c_n \int_{t_n - u_n}^{t_n - v_n} \frac{t}{t_n - t}\, dt - c_n \int_{t_n + v_n}^{t_n + u_n} \frac{t}{t_n - t}\, dt = 2c_n(u_n - v_n),$$

and

$$\left| \int_{t_n - u_n}^{t_n} f(t)\, dt \right| = c_n \int_{t_n - u_n}^{t_n - v_n} \frac{t}{t_n - t}\, dt$$

$$= c_n \left\{ t_n \log \frac{u_n}{v_n} - (u_n - v_n) \right\}$$

$$\geqq c_n t_n \log \frac{u_n}{v_n}.$$

Hence, if

$$(2.7) \qquad \sum_{i=n+1}^{\infty} c_i(u_i - v_i) < \frac{1}{4} c_n t_n \log \frac{u_n}{v_n}$$

and if

$$(2.8) \qquad \frac{1}{2} c_n t_n \log \frac{u_n}{v_n} \geqq \mathcal{E}(t_n)$$

then we have

$$\begin{aligned}\left|\int_0^{t_n} f(u)\, du\right| &= \left|\sum_{i=n+1}^{\infty} \int_{\Delta_i} f(t)\, dt + \int_{t_n-u_n}^{t_n} f(t)\, dt\right| \\ &\geqq c_n\, t_n \log \frac{u_n}{v_n} - 2 \sum_{i=n+1}^{\infty} c_i(u_i - v_i) \\ &\geqq \frac{1}{2} c_n\, t_n \log \frac{u_n}{v_n} \\ &\geqq \mathcal{E}(t_n).\end{aligned}$$

After the sequences $\{t_n\}$ and $\{c_n\}$ are determined such that $\sum \mathcal{E}(t_n) < \infty$ and that (2.6) holds, we may suppose that the sequences $\{u_n\}$ and $\{v_n\}$ satify the additional relations (2.7) and

$$\frac{1}{2} c_n\, t_n \log \frac{u_n}{v_n} = \mathcal{E}(t_n),$$

that is,

$$\frac{u_n}{v_n} = \exp\left(\frac{2\mathcal{E}(t_n)}{c_n\, t_n}\right).$$

Then the conditions (2.1), (2.4), (2.6), (2.7) and (2.8) are satisfied. The theorem is thus completely proved.

3. Proof of theorem 2. We may suppose that $f(t)$ is even and $x = 0$. Then $f(t) = \varphi_x(t)$ and put $s_n = s_n(0)$. Let the Fourier series of $f(t)$ be

$$\sum_{n=1}^{\infty} a_n \cos nt$$

supposing $a_0 = 0$. Then we have

$$(3.1) \qquad \Phi(t) = \int_0^t f(u)\, du = \sum_{n=1}^{\infty} a_n \frac{\sin nt}{n} = \sum_{n=1}^{\infty} s_n \Delta \frac{\sin nt}{n}.$$

Now,

$$(3.2) \qquad \begin{aligned}\Delta \frac{\sin nt}{n} &= \frac{(n+1)\sin nt - n \sin (n+1)t}{n(n+1)} \\ &= \frac{\sin nt - \sin (n+1)t}{n+1} + \frac{\sin nt}{n(n+1)};\end{aligned}$$

therefore we have easily

(3.3) $$\left|\Delta \frac{\sin nt}{n}\right| \leqq Ctn^{-1} \qquad (C\text{: constant})$$

for all t in $(0, \pi)$ and for all n.

On the other hand, for $0 \leqq nt \leqq \frac{\pi}{4}\left(<\frac{\pi}{2}\right)$ we have

$$\begin{aligned}
&|(n+1)\sin nt - n\sin(n+1)t| \\
&= |n\sin nt + \sin nt - n\sin nt\cos t - n\cos nt\sin t| \\
&= |n\sin nt(1-\cos t) + (\sin nt - n\sin t) + n\sin t(1-\cos nt)| \\
&\leqq C_1\{n\cdot nt\cdot t^2 + (n^3t^3 + nt^3) + nt\cdot n^2t^2\} \\
&\leqq C_2 n^3t^3,
\end{aligned}$$

where C_1 and C_2 are constants independent of n and t. It holds then

(3.4) $$\left|\Delta \frac{\sin nt}{n}\right| \leqq C_2 nt^3 \qquad \left(0 \leqq nt \leqq \frac{\pi}{4}\right).$$

Dividing the last sum of (3.1) we write

$$\Phi(t) = \left(\sum_{n=1}^{[\pi/4t]} + \sum_{n=[\pi/4t]+1}^{\infty}\right) s_n \Delta \frac{\sin nt}{n} \equiv I + J.$$

Then we have by (3.4)

$$I = o\left(\sum_{n=1}^{[\pi/4t]} \frac{nt^3}{n}\right) = o(t^3\cdot t^{-(2-\alpha)}) = o(t^{1+\alpha}),$$

and by (3.3)

$$J = o\left(\sum_{n=[\pi/4t]+1}^{\infty} \frac{1}{n^\alpha}\frac{t}{n}\right) = o(t^{1+\alpha}).$$

Combining these results we get

$$\Phi(t) = o(t^{1+\alpha})$$

which is the required.

4. Proof of Theorem 3. Let $\{\varepsilon_n\}$ be a positive monotone vanishing sequence such that the relations

(4.1) $$\frac{\varepsilon_n}{24\, n^{1+\gamma}} = \varepsilon\left(\frac{1}{n}\right), \quad \varepsilon_{n+1} \leqq \frac{\gamma}{120}\varepsilon_n$$

hold for infinitely many integers n. This definition may be conceived by the condition on the given function $\varepsilon(t)$.

Let

(4.2) $$f(t) = \sum_{k=1}^{\infty} \varepsilon_k \frac{\cos kt}{k^{1+\gamma}} \qquad (0<\gamma<1).$$

The series (4.2) converges uniformly, and then we have

$$s_n(0) - f(0) = \sum_{k=n+1}^{\infty} \frac{\varepsilon_k}{k^{1+\gamma}} = o\left(\frac{1}{n^\gamma}\right) \qquad \text{as } n\to\infty$$

which is one of the required condition.

On the other hand we have, substituting (4.1),

$$(4.3)\qquad \int_0^t \varphi_0(u)du = 2\int_0^t \{f(u)-f(0)\}\,du$$
$$= 2\sum_{k=1}^{\infty} \frac{\varepsilon_k}{k^{2+\gamma}}(\sin kt - kt)$$
$$= 2\Big(\sum_{k=1}^{[1/t]} + \sum_{k=[1/t]+1}^{\infty}\Big) \equiv I + J$$

say. In the sum I, since $kt - \sin kt \geqq k^3t^3/12$,

$$(4.4)\qquad |I| = 2\sum_{k=1}^{[1/t]} \varepsilon_k \frac{kt-\sin kt}{k^{2+\gamma}} \geqq \frac{1}{6}\sum_{k=1}^{[1/t]} \varepsilon_k k^{1-\gamma}$$
$$\geqq \frac{1}{6}t^3\varepsilon_{[1/t]}\frac{1}{2}\frac{1}{t^{2-\gamma}} = \frac{1}{12}\varepsilon_{[1/t]}t^{1+\gamma}.$$

By $|\sin kt - kt| \leqq 2kt$ in the sum J, we get

$$(4.5)\qquad |J| \leqq 2\sum_{k=[1/t]+1}^{\infty} \frac{2\varepsilon_k t}{k^{1+\gamma}} \leqq \frac{5}{\gamma}\varepsilon_{[1/t]+1}t^{1+\gamma}.$$

Hence if $[1/t]$ is an integer which fulfills the conditions (4.1), we have from (4.3), (4.4) and (4.5)

$$\int_0^t \varphi_0(u)\,du \geqq |I| - |J| \geqq \Big(\frac{1}{12}\varepsilon_{[1/t]} - \frac{5}{\gamma}\varepsilon_{[1/t]+1}\Big)t^{1+\gamma}$$
$$\geqq \frac{1}{24}\varepsilon_{[1/t]}t^{1+\gamma} = \varepsilon(t).$$

Thus the condition (1.8) holds for infinitely many t with $t \to 0$.

5. **Proof of Theorem 4.** We shall begin by the case $\beta \geqq 0$. Let $\eta(t) = \varepsilon(t)/t^{\alpha+\beta-\gamma}$, and we may suppose without loss of generality that $\eta(t)\downarrow 0$ as $t \to 0$, and that $x = 0$. By the inequality $\beta - \gamma \geqq 0$, there is an integer M not smaller than $\beta - \gamma$. Put $\beta - \gamma = \delta$ and

$$(5.1)\qquad \eta^*(t) = \frac{4^{M+1}(M+1-\delta)\Gamma(\alpha+M+2)}{\Gamma(M+2)}\eta(t),$$

$$(5.2)\qquad \eta^*_n = \eta^*\Big(\frac{1}{n}\Big).$$

Since $\eta^*_n \to 0$ as $n \to \infty$, we can find a sequence of integers $\{n_k\}$ such that

$$(5.3)\qquad n_1 = 1,\ \eta^*_{n_{k+1}} \leqq \frac{\alpha\delta\Gamma(\alpha)\Gamma(M+2)}{2\cdot 4^{M+2}(M+1-\delta)\Gamma(\alpha+M+2)}\eta^*_{n_k}$$
$$(k = 1, 2, \ldots.).$$

Let $\{\eta'_\nu\}$ be a sequence such that

$$(5.4)\qquad \eta'_\nu = \eta^*_{n_k} \qquad \text{for } n_{k-1} < \nu \leqq n_k\ (k = 1, 2, \ldots.),$$

then obviously $\eta'_\nu \downarrow 0$ $(\nu \to \infty)$.

We shall define a function $f(t)$ by ·

$$f(t) = \sum_{\nu=1}^{\infty} \eta_\nu' \frac{\cos \nu t}{\nu^{1+\delta}}, \tag{5.5}$$

and we shall prove that this function is the required.

We have

$$f(0) - s_n(0) = \sum_{\nu=n+1}^{\infty} \eta_\nu' \frac{1}{\nu^{1+\delta}} = \eta_n' O\left(\frac{1}{n^\delta}\right) = o\left(\frac{1}{n^\delta}\right) \tag{5.6}$$

as $n \to \infty$, and hence immediately

$$f(0) - \sigma_n^\beta(0) = o(n^{-\delta}) = o(n^{\gamma-\beta})$$

as $n \to \infty$, which is one of the required conditions.

Now we get

$$\Phi_\alpha(t) = \frac{2}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \left(\sum_{\nu=1}^{\infty} \frac{\eta_\nu'}{\nu^{1+\delta}} (\cos \nu u - 1) \right) du$$

$$= \frac{2}{\Gamma(\alpha)} \sum_{\nu=1}^{\infty} \frac{\eta_\nu'}{\nu^{1+\delta}} \int_0^t (t-u)^{\alpha-1} (\cos \nu u - 1)\, du.$$

For $\nu \leqq t^{-1}$ we have

$$-\int_0^t (t-u)^{\alpha-1} (\cos \nu u - 1)\, du \tag{5.7}$$

$$= 2\int_0^t (t-u)^{\alpha-1} \sin^2 \frac{\nu u}{2}\, du$$

$$\geqq 2\int_0^t (t-u)^{\alpha-1} \left(\frac{\nu u}{4}\right)^{M+1} du$$

$$= \frac{2\nu^{M+1}}{4^{M+1}} \int_0^t (t-u)^{\alpha-1} u^{(M+2)-1}\, du$$

$$= \frac{2\Gamma(\alpha)\Gamma(M+2)}{4^{M+1}\Gamma(\alpha+M+2)} \nu^{M+1} t^{\alpha+M-1},$$

and we have also

$$\left| \int_0^t (t-u)^{\alpha-1} (\cos \nu u - 1)\, du \right| \leqq 2 \int_0^t v^{\alpha-1}\, dv = \frac{2}{\alpha} t^\alpha. \tag{5.8}$$

If we put

$$\Phi_\alpha(t) = \frac{2}{\Gamma(\alpha)} \left(\sum_{\nu=1}^{[1/t]} + \sum_{\nu=[1/t]+1}^{\infty} \right) \frac{\eta_\nu'}{\nu^{1+\delta}} \int_0^t (t-u)^{\alpha-1} (\cos \nu u - 1)\, du \equiv K + L,$$

then by (6.7) we get for small t

(5.9) $$|K| \geqq \frac{2}{\Gamma(\alpha)} \sum_{\nu=1}^{[1/t]} \frac{\eta'_\nu}{\nu^{1+\delta}} \frac{2\Gamma(\alpha)\Gamma(M+2)}{4^{M+1}\Gamma(\alpha+M+2)} \nu^{M+1} t^{\alpha+M-1},$$
$$\geqq \frac{\Gamma(M+2)}{4M\Gamma(\alpha+M+2)} \eta'_{[1/t]} t^{\alpha+M+1} \sum_{\nu=1}^{[1/t]} \nu^{M-\delta}$$
$$\geqq \frac{\Gamma(M+2)}{2\cdot 4^M(M+1-\delta)\Gamma(\alpha+M+2)} \eta'_{[1/t]} t^{\alpha+\delta},$$

and by (5.8) we get for small t

(5.10) $$|L| \leqq \frac{2}{\Gamma(\alpha)} \sum_{\nu=[1/t]+1}^{\infty} \frac{\eta'^{\nu}}{\nu^{1+\alpha}} \frac{2}{\alpha} t^\alpha$$
$$\leqq \frac{4}{\alpha\Gamma(\alpha)} \eta_{[1/t]+1} t^\alpha \sum_{\nu=[1/t]+1}^{\infty} \frac{1}{\nu^{1+\delta}}$$
$$\leqq \frac{8}{\alpha\delta\Gamma(\alpha)} \varepsilon'_{[1/t]+1} t^{\alpha+\delta}.$$

If we put $t = 1/n_k$, we have, by (6.1)—(6.4),
$$\eta'_{[1/t]} = \eta'_{n_k} = \eta^*_{n_k} = \eta^*(1/n_k) = \eta^*(t)$$
$$= \frac{4^{M+1}(M+1-\delta)\Gamma(\alpha+M+2)}{\Gamma(M+2)} \eta(t),$$
$$\eta'_{[1/t]+1} = \eta'_{n_k+1} = \eta^*_{n_k+1}$$
$$\leqq \frac{\alpha\delta\Gamma(\alpha)\Gamma(M+2)}{2\cdot 4^{M+2}(M+1-\delta)\Gamma(\alpha+M+2)} \eta^*_{n_k}$$
$$= \frac{\alpha\delta(\alpha)}{8} \eta(t),$$

and hence we get easily from (6.9) and (6.10)
$$|K| \geqq 2\eta(t) t^{\alpha+\delta}$$
and
$$|L| \leqq \eta(t) t^{\alpha+\delta}$$
for sufficiently small t. Thus we conclude that
$$|\Phi_\alpha(t)| \geqq |K| - |L| \geqq \eta(t) t^{\alpha+\delta} = \varepsilon(t)$$
for $t = 1/n_k$, k being sufficiently large.

In the case $\beta \geqq 0$ Theorem was thus proved.

We shall now consider the case $\beta < 0$. We have $0 < \beta - \gamma = \delta < 1$. As in the former case we get easily
$$f(0) - s_k(0) \equiv r_{k+1}(0) \equiv r_{k+1} = o((k+1)^{-\delta}).$$
Remembering $r_1 = f(0)$, we have

$$\sigma_n^\beta(0) - f(0) = \frac{1}{A_n^\beta}\left\{\sum_{k=1}^{n} A_{n-k}^\beta \frac{\eta_k'}{k^{1+\delta}} - A_n r_1\right\}$$

$$= \frac{1}{A_n^\beta}\left\{\sum_{k=1}^{[n/2]} A_{n-k}^\beta \frac{\eta_k'}{k^{1+\delta}} - A_n^\beta r_1\right\} + \frac{1}{A_n^\beta}\sum_{k=[n/2]+1}^{n} A_{n-k}^\beta \frac{\eta_k'}{k^{1+\delta}}$$

$$\equiv P + Q$$

say. Then

$$P = \frac{1}{A_n^\beta}\left\{\sum_{k=1}^{[n/2]} A_{n-k}^\beta (r_k - r_{k+1}) - A_n^\beta r_n\right\}$$

$$= \frac{1}{A_n^\beta}\left\{-\sum_{k=1}^{[n/2]} A_{n-k+1}^{\beta-1} r_k - A_{n-[n/2]}^\beta r_{[n/2]+1}\right\},$$

$$|P| = O\left(\frac{1}{n^\beta}\left\{\sum_{k=1}^{[n/2]} n^{\beta-1} k^{-\delta} + n^\beta n^{-\delta}\right\}\right)$$

$$= O(n^{-\beta}\{n^{\beta-1} n^{1-\delta} + n^{\beta-\delta}\}) = O(n^{-\delta}),$$

and

$$|Q| = O(n^{-\beta} n^{-1-\delta} \sum_{k=1}^{[n/2]-1} A_k^\beta)$$

$$= O(n^{-\beta-1-\delta} n^{\beta+1}) = O(n^{-\delta}).$$

From these estimations we get

$$\sigma_n^\beta(0) - f(0) = O(n^{-\delta}) = O(n^{\gamma-\beta}).$$

The estimation of $\Phi_\alpha(t)$ is the same as in the former case.

Thus the theorem was completey proved.

6. Remarks. The theorems 1 and 3 will be shown by using examples of the type used by one of the authors [2,3]. An example of the Paley type [1] may be also used for the proof of Theorem 3.

References

[1] R. E A. C. Paley, On the Cesàro summability of Fourier series and allied series, Proc. Cambridge Phil. Soc., 26(1929-30) p. 173-203.

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