TAUBERIAN THEOREMS FOR RIEMANN SUMMABILITY

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(Received February 25, 1953)

1. Introduction. O. Szász [7] has studied Tauberian theorems for summability (R_1). In his case, the given series are convergent or Abel-summable and his Tauberian conditions are satisfactory. Recently S. Izumi-N. Matsuyama [1] have treated the case where the given series are Cesàro summable. But their conditions are somewhat stringent. In § 2, the author gives better conditions. On the other hand, concerning summability $(R, 1)$, O. Szász [4] [5] has studied the analogous type to his own theorems for summability $(R, 1)$ and L. Schmetterer [2] has studied the analogous type to Izumi-Matsuyama's theorem for summabiity (R_1). Since the latter investigation is unsatisfactory, the author gives a better theorem in § 3. These problems are closely connected to the uniform convergence of trigonometrical series. The problem of uniform convergence has been treated by O. Szász [6] and S. Izumi-N. Matsuyama [1]. A related theorem to their investigation is given in § 4.

2. Summability (R_1). In the series $\sum_{\nu=1}^{\infty} a_\nu$, put $S_n = \sum_{\nu=1}^{n} a_\nu$. Then if

$$\sum_{\nu=1}^{\infty} \frac{s_\nu}{\nu} \sin \nu t$$

converges for every t in $0 < t < \delta < 2\pi$, and

$$\lim_{t \to +0} \sum_{\nu=1}^{\infty} \frac{s_\nu}{\nu} \sin \nu t = s,$$

we call that the series is summable (R_1) to sum s. Then we get the following theorem.

THEOREM 1. *In the series* $\sum_{\nu=1}^{\infty} a_\nu$, *if*

$$(1) \qquad \sum_{\nu=1}^{n} s_\nu = o(n^\alpha), \qquad 0 < \alpha < 1$$

and

$$(2) \qquad \sum_{\nu=n}^{\infty} \left| \frac{a_\nu}{\nu} \right| = O(n^{-\alpha}),$$

then the series is summable (R_1) *to sum zero.*

PROOF. If we put

$$\sum_{\nu=n}^{\infty}\left|\frac{a_\nu}{\nu}\right| = r_n,$$

then

$$|a_n| = n(r_n - r_{n+1})$$

and we have

$$\begin{aligned}\sum_{\nu=1}^{n}|a_\nu| &= \sum_{\nu=1}^{n}\nu(r_\nu - r_{\nu+1})\\ &= r_1 + \sum_{\nu=2}^{n} r_\nu - nr_{n+1}\\ &= O\Big(\sum_{\nu=1}^{n}\nu^{-\alpha}\Big) + nO(n^{-\alpha}) = O(n^{1-\alpha}),\end{aligned}$$

by (2). That is

$$s_n = O(n^{1-\alpha}). \tag{3}$$

By this result we get

$$\sum_{\nu=n}^{\infty}\frac{|s_\nu|}{\nu^2} = \sum_{\nu=n}^{\infty}O\left(\frac{\nu^{1-\alpha}}{\nu^2}\right) = \sum_{\nu=n}^{\infty}O(\nu^{-1-\alpha}) = O(n^{-\alpha})$$

and we have

$$\begin{aligned}\sum_{\nu=n}^{\infty}\left|\frac{s_\nu}{\nu} - \frac{s_{\nu+1}}{\nu+1}\right| &= \sum_{\nu=n}^{\infty}\left|\frac{s_\nu - s_{\nu+1}}{\nu} + \left(\frac{1}{\nu} - \frac{1}{\nu+1}\right)s_{\nu+1}\right|\\ &\leqq \sum_{\nu=n}^{\infty}\left|\frac{a_{\nu+1}}{\nu}\right| + \sum_{\nu=n}^{\infty}\frac{|s_{\nu+1}|}{\nu^2} = O(n^{-\alpha}).\end{aligned} \tag{4}$$

On the other hand, by Abel's transformation

$$\sum_{\nu=n}^{m}\frac{s_\nu}{\nu}\sin\nu t = \sum_{\nu=n}^{m-1}\left(\frac{s_\nu}{\nu} - \frac{s_{\nu+1}}{\nu+1}\right)T_\nu(t) + \frac{s_m}{m}T_m(t) - \frac{s_n}{n}T_{n-1}(t), \tag{5}$$

where

$$T_n(t) = \frac{\cos\frac{t}{2} - \cos\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}},$$

hence in the interval $0 < \varepsilon \leqq t \leqq 2\pi - \varepsilon$,

$$\left|\sum_{\nu=n}^{m}\frac{s_\nu}{\nu}\sin\nu t\right| \leqq \varepsilon^{-1}\pi\sum_{\nu=n}^{\infty}\left|\frac{s_\nu}{\nu} - \frac{s_{\nu+1}}{\nu+1}\right| + 2\varepsilon^{-1}\pi\left(\frac{|s_m|}{m} + \frac{|s_n|}{n}\right).$$

Thus the series $\sum(s_\nu/\nu)\sin\nu t$ is uniformly convergent in $0 < \varepsilon \leqq t \leqq 2\pi - \varepsilon$.

We write

$$\begin{aligned}\sum_{\nu=1}^{\infty}\frac{s_\nu}{\nu}\sin\nu t &= \left(\sum_{\nu=1}^{n} + \sum_{\nu=n+1}^{\infty}\right)\frac{s_\nu}{\nu}\sin\nu t\\ &= u_1(t) + u_2(t),\end{aligned}$$

say. where $n = [t^{-\frac{1}{\alpha}} \varepsilon^{-\frac{1}{\alpha}}]$. Now, estimating analogously to (5),

$$|u_2(t)| < \pi t^{-1}\left(\sum_{\nu=n+1}^{\infty} \left| \frac{s_\nu}{\nu} - \frac{s_{\nu+1}}{\nu+1} \right| + \left| \frac{s_{n+1}}{n+1} \right| \right)$$

$$= t^{-1}O(n^{-\alpha}) = O(t^{-1}t\,\varepsilon) = \varepsilon \cdot O(1).$$

As to $u_1(t)$, we have

$$u_1(t) = \sum_{\nu=1}^{n} \frac{s_\nu}{\nu} \sin \nu t = \sum_{\nu=1}^{n-1} S_\nu \Delta_\nu(t) + S_n \frac{\sin nt}{n} ,$$

where

$$S_n = \sum_{\nu=1}^{n} s_\nu, \quad \Delta_n(t) = \frac{\sin nt}{n} - \frac{\sin (n+1)t}{n+1}.$$

Since

$$\Delta_n(t) = \frac{\sin nt - \sin (n+1)t}{n} + \sin (n+1)t\left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \frac{-2}{n} \cos \frac{(2n+1)t}{2} \sin \frac{t}{2} + \frac{\sin (n+1)t}{n(n+1)} = O\left(\frac{t}{n} \right),$$

we have

$$|u_1(t)| \leqq \sum_{\nu=1}^{n-1} |S_\nu| \cdot |\Delta_\nu(t)| + |S_n|/n$$

$$= \sum_{\nu=1}^{n} o(\nu^\alpha) O\left(\frac{t}{\nu} \right) + o(n^\alpha)\left(\frac{1}{n} \right)$$

$$= t\sum_{\nu=1}^{n-1} o(\nu^{-1+\alpha}) + o(1)$$

$$= t \cdot o(n^\alpha) + o(1) = o(t\ t^{-1}\varepsilon^{-1}) + o(1) = o(1).$$

Hence if ε is arbitrally small, we have

$$\lim_{t\to+0} \sum_{\nu=1}^{\infty} \frac{s_\nu}{\nu} \sin \nu t = 0.$$

Thus we get the theorem.

COROLLARY 1. *In the series* $\sum_{\nu=1}^{\infty} a_\nu$, *if*

(1) $$\sum_{\nu=1}^{n} s_\nu = o(n^\alpha), \qquad (0 < \alpha < 1)$$

(2′) $$\sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) = O(n^{1-\alpha}),$$

then the series is summable (R_1) *to sum zero.*

PROOF. Applying Szász's argument [3] to (1) and (2′), we get (2). For the sake of completeness, we shall repeat his argument. If we put

$$v_n = \sum_{\nu=1}^{n} \nu a_\nu, \quad (v_0 = s_0 = a_0), \quad 1 \leqq k \leqq 1 + (n+1)$$

then we have

$$v_n = \sum_{\nu=0}^{n} (s_n - s_\nu)$$

and

$$v_{n+k} - v_n = (n+1)(s_{n+k} - s_n) + \sum_{\nu=n+1}^{n+k} (s_{n+k} - s_\nu)$$

$$\geqq - p \cdot n^{2-\alpha} - n^{1-\alpha} p \cdot k \geqq - p n^{1-\alpha}(n+k),$$

by (2′), where p is a bound of (2′). Now put

$$n_\nu = [n 2^{-\nu}] \qquad (\nu = 0, 1, 2, \ldots),$$

so that

$$v_n = \sum_{\nu=0}^{\infty} (v_{n_\nu} - v_{n_{\nu+1}})$$

then

$$v_n \geqq -p n^{1-\alpha} \sum_{\nu=0}^{\infty} n_\nu \geqq - p n^{1-\alpha}\, n \sum_{\nu=0}^{\infty} 2^{-\mu} = 2p\ n^{2-\alpha}.$$

Under the assumption (1)

$$\sigma_n = \left(\sum_{\nu=1}^{n} s_\nu\right)/n = o(n^{-1+\alpha})$$

and

$$s_n = \frac{v_n}{n+1} + \sigma_n > -2 p n^{1-\alpha} + o(n^{-1+\alpha}) > -3 p n^{1-\alpha}$$

for large n. On the other hand we have

$$s_n = \sigma_{n+1} + (\sigma_{2n+1} - \sigma_n) - \frac{1}{n+1} \sum_{\nu=1}^{n+1} (s_{n+\nu} - s_n)$$

whence

$$s_n < o(n^{-1+\alpha}) + o(n^{-1+\alpha}) - p n^{1-\alpha} < 2 p n^{1-\alpha}$$

for large n. By combining these two inequalities for s_n, we get

$$s_n = O(n^{1-\alpha}).$$

On the other hand, we have

$$\sum_{\nu=n}^{2n} |a_\nu| = \sum_{n}^{2n} (|a_\nu| - a_\nu + s_{2n} - s_{n-1})$$

$$= O(n^{1-\alpha}) + O(n^{1-\alpha}) = O(n^{1-\alpha}).$$

Consequently

$$\sum_{\nu=n}^{2n} \nu^{-1}|a_\nu| \leqq n^{-1}\sum_{\nu=n}^{2n} |a_\nu| = O(n^{-\alpha}),$$

$$\sum_{\nu=2^k}^{2^{k+1}-1} \nu^{-1}|a_\nu| = O(2^{-k\alpha})$$

and

$$\sum_{\nu=1}^{2^l} \nu^{-1}|a_\nu| = O\left(\sum_{k=0}^{l} 2^{-k\alpha}\right) = O(1).$$

Hence we have

$$\sum_{\nu=n}^{\infty} \nu^{-1}|a_\nu| = \sum_{k=0}^{\infty}\sum_{n\cdot 2k}^{n\cdot 2^{k+1}-1} \nu^{-1}|a_\nu| = O\left(n^{-\alpha}\sum_{k=0}^{\infty} 2^{-k\alpha}\right) = O(n^{-\alpha})$$

which is the desired inequality (2).

3. Summability (R, **1**). In the series $\sum_{\nu=1}^{\infty} a_\nu$, if

$$\sum_{\nu=1}^{\infty} a_\nu \frac{\sin \nu t}{\nu t}$$

converges for every t in $0 < t < 2\pi$, and

$$\lim_{t\to 0}\sum_{\nu=1}^{\infty} a_\nu \frac{\sin \nu t}{\nu t} = s,$$

then we say that the series is summable $(R, 1)$ to sum s. For the summability $(R, 1)$, we get the analogous theorem.

THEOREM 2. *In the series* $\sum_{\nu=1}^{\infty} a_\nu$, *if*

$$\sum_{\nu=1}^{n} s_\nu = o(n^\alpha), \qquad 0 < \alpha < 1 \tag{6}$$

and

$$\sum_{\nu=n}^{\infty} \left|\frac{a_\nu}{\nu}\right| = O(n^{-\alpha}), \tag{7}$$

then the series is summable $(R, 1)$ *to sum zero.*

PROOF. The proof is analogous to § 2. Since

$$\sum_{\nu=n}^{\infty} \left|\frac{a_\nu}{\nu} - \frac{a_{\nu+1}}{\nu+1}\right| \leqq 2\sum_{\nu=n}^{\infty}\left|\frac{a_\nu}{\nu}\right| = O(n^{-\alpha}),$$

in the interval $0 < \varepsilon \leqq t \leqq 2\pi - \varepsilon$, we have

$$\sum_{\nu=n}^{m} \frac{a_\nu}{\nu} \sin \nu t = \sum_{\nu=n}^{m-1} \left| \frac{a_\nu}{\nu} - \frac{a_{\nu+1}}{\nu+1} \right| T_\nu(t) + \frac{a_m}{m} T_m(t) - \frac{a_n}{n} T_{n-1}(t),$$

hence

$$\left| \sum_{\nu=n}^{m} \frac{a_\nu}{\nu} \sin \nu t \right| \leqq \varepsilon^{-1} \pi \sum_{\nu=n}^{\infty} \left| \frac{a_\nu}{\nu} - \frac{a_{\nu+1}}{\nu+1} \right| + 2\, \varepsilon^{-1} \pi \left(\frac{|a_m|}{m} + \frac{|a_n|}{n} \right)$$

and the series

$$\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu} \sin \nu t$$

converges uniformly in this interval. We write

$$\sum_{\nu=1}^{\infty} \frac{a_\nu}{\nu} \frac{\sin \nu t}{t} = \sum_{\nu=1}^{n} + \sum_{\nu=n+1}^{\infty} = u_1(t) + u_2(t),$$

say, where $n = [t^{-\frac{1}{\alpha}} \varepsilon^{-\frac{1}{\alpha}}]$. Then

$$|u_2(t)| < \pi t^{-1} \left(\sum_{\nu=n+1}^{\infty} \left| \frac{a_\nu}{\nu} - \frac{a_{\nu+1}}{\nu+1} \right| + \left| \frac{a_n}{n+1} \right| \right)$$
$$= t^{-1} O(n^{-\alpha} \varepsilon^{-1}) \leqq \varepsilon.$$

Applying Abel's transformation twice to $u_1(t)$, we get

$$u_1(t) = \sum_{\nu=1}^{n-1} a_\nu \frac{\sin \nu t}{\nu t} = \sum_{\nu=1}^{n} S_\nu \Delta_\nu^2(t) + S_{n-1} \Delta_n(t)$$
$$+ s_n \frac{\sin nt}{nt},$$

where

$$\Delta_n(t) = \frac{\sin nt}{n} - \frac{\sin (n+1)t}{n+1}, \quad \Delta_n^2(t) = \Delta(\Delta_n(t)).$$

Since we have easily

$$\Delta_n(t) = O\left(\frac{t}{n}\right), \qquad \Delta_n^2(t) = O\left(\frac{t}{n}\right),$$

$$|u_1(t)| = \sum_{\nu=1}^{n-1} o(\nu^\alpha)\, O\left(\frac{t}{\nu}\right) + o(n^\alpha)\, O\left(\frac{t}{n}\right) + o(n^{1-\alpha})\, O\left(\frac{1}{nt}\right)$$
$$= o(n^\alpha t) + o(n^{-1+\alpha} t) + o(n^{-\alpha} t^{-1}) = o(1).$$

Thus we have the desired results.

COROLLARY 2. *In the series* $\sum_{\nu=1}^{\infty} a_\nu$, *if*

(6) $$\sum_{\nu=1}^{n} s_\nu = o(n^\alpha) \qquad (o < \alpha < 1)$$

and

(7′) $$\sum_{\nu=n}^{2n}(|a_\nu| - a_\nu) = O(n^{1-\alpha})$$

then the series is summable $(R, 1)$ *to sum zero.*

The proof is obvious from the proof of Corollary 1. The Corollary 2 is a solution of the problem proposed by Schmetterer [2].

4. Uniform convergence of the trigonometrical series. The problem of uniform convergence of the trigonometrical series is closely related to Riemann summability.

THEOREM 3. *If*

(8) $$\sum_{\nu=1}^{n} \nu a_\nu = o(n^\alpha) \qquad (0 < \alpha < 1)$$

and

(9) $$\sum_{\nu=n}^{\infty} |\Delta a_\nu| = O(n^{-\alpha})$$

then the trigonometrical series

$$\sum_{\nu=1}^{\infty} a_\nu \sin \nu t$$

converges uniformly in the interval $0 \leqq t \leqq \pi$.

Proof. We write

$$\sum_{\nu=1}^{\infty} a_\nu \sin \nu t = \sum_{\nu=1}^{n} a_\nu \sin \nu t + \sum_{\nu=n+1}^{\infty} a_\nu \sin \nu t$$
$$= u_1(t) + u_2(t),$$

where n is determined in a little moment. If we put

$$t_n = \sum_{\nu=1}^{n} \nu a_\nu$$

then

$$u_1(t) = \sum_{\nu=1}^{n} a_\nu \sin \nu t = \sum_{\nu=1}^{n} \nu a_\nu \frac{\sin \nu t}{\nu}$$
$$= \sum_{\nu=1}^{n-1} t_\nu \Delta_\nu(t) + t_n \frac{\sin nt}{n},$$

and

$$\Delta_n(t) = O\left(\frac{t}{n}\right),$$

where O is independent on n. From the assumption (8), we have

$$u_1(t) = O\left(\sum_{\nu=1}^{n-1} o\left(\nu^\alpha \frac{t}{\nu}\right)\right) + o(n^{\alpha-1})$$

$$= o(n^{\alpha}t) + o(n^{\alpha-1})$$

and

$$u_2(t) = \sum_{\nu=n+1}^{\infty} a_\nu \sin \nu t$$

$$= -a_{n+1} T_n(t) + \sum_{\nu=n+1}^{\infty} \Delta a_\nu \cdot T_\nu(t)$$

$$= O\left(\frac{|a_{n+1}|}{t} + \frac{1}{t} \sum_{\nu=n+1}^{\infty} |\Delta a_\nu|\right)$$

$$= O\left(\frac{1}{t} \sum_{\nu=n+}^{\infty} |\Delta a_\nu|\right)$$

$$= O(t^{-1}n^{-\alpha}).$$

Hence

$$\sum_{\nu=1}^{\infty} a_\nu \sin \nu t = o(n^{\alpha}t) + O(t^{-1}n^{-\alpha}) + o(1), \tag{10}$$

where $o(1)$ does not depend on t. Of course we have

$$\sum_{\nu=1}^{N(t)} a_\nu \sin \nu t = o(n^{\alpha}t) + O(t^{-1}n^{-\alpha}) + o(1). \tag{11}$$

To say the uniform convergence of

$$\sum_{\nu=1}^{\infty} a_\nu \sin \nu t,$$

it is sufficient to say that

$$\sum_{\nu=1}^{N} a_\nu \sin \nu t_N$$

converges as t_N tend to $t \in [o, \pi]$. If t_N tend to $t \neq 0$, it is obvious from the similar argument to (5). If t_N tend to zero, the formula (11) is

$$\sum_{\nu=1}^{N} a_\nu \sin \nu t_N = o(n^{\alpha} t_N) + O(n^{-1}t_N^{-1}) + o(1)$$

and taking $n = [t_N^{-1} {}^{\alpha} \varepsilon^{-1/\alpha}]$, we get the desired results.

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