ON AN APPROXIMATION PROBLEM IN THE THEORY OF PROBABILITY

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1. Mr K. Ito has proposed the problem: When the series

$$\sum_{k=0}^{\infty} f(k)^2 \frac{\lambda^k}{k!} \tag{1.1}$$

converges, does there exist a polynomial $P(x)$ such that

$$\sum_{n=0}^{\infty} (P(n) - f(n))^2 \frac{\lambda^n}{n!} < \varepsilon, \tag{1.2}$$

ε being any preassigned positive number? Answering this poblem, we prove the following theorems.

THEOREM 1. *If there is a constant w such that the series*

$$\sum_{k=0}^{\infty} |f(k)|/w^k$$

converges; then there is a polynomial $P(x)$ such that

$$\sum_{n=0}^{\infty} |P(n) - f(n)| \frac{\lambda^n}{n!} < \varepsilon$$

ε being a preassigned positive number.

THEOREM 2. *If there is a positive number w such that the series*

$$\sum_{k=0}^{\infty} f(k)^2/w^k$$

converges, then there is a polynomial $P(x)$ such that (1.2) *holds.*

2. We suppose that

$$\sum_{k=0}^{\infty} |f(k)| \frac{\lambda^k}{k!} < \infty \tag{2.1}$$

and that $f(0) = 0$ without any loss of generality. Let us put[1]

$$B_u(x) = e^{-ux} \sum_{k=0}^{\infty} f(k) \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{(kx)^m}{m!} \tag{2.2}$$

whose convergence will be justified easily, and may be seen also from the following estimation. Consider the series

1) This is the discrete analogue of the Bernstein's polynomial, deduced from the Szász form.

$$I = \sum_{n=0}^{\infty} |B_u(n) - f(n)| \frac{\lambda^n}{n!}$$

Then

$$I \leqq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-un} \sum_{k=0}^{\infty} |f(k) - f(n)| \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{(un)^m}{m!} + o(1)$$

where $o(1)$ is the term tending to zero as $u \to \infty$. The right side sum is

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-un} \left(\sum_{k=0}^{n-1} + \sum_{k=n+1}^{\infty} \right) |f(k) - f(n)| \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{(un)^m}{m!} = I_1 + I_2,$$

say. Now

$$I_1 = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \leqq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-un} \sum_{k=0}^{n-1} |f(k)| \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{(un)^m}{m!}$$

$$+ \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-un} \sum_{k=0}^{n-1} |f(n)| \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{(un)^m}{m!} = I_{11} + I_{12},$$

say. Changing the order of summation in I_{11},

$$I_{11} = \sum_{k=0}^{\infty} |f(k)| \sum_{n=k+1}^{\infty} \frac{\lambda^n}{n!} e^{-un} \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{(un)^m}{m!}$$

$$= \sum_{k=0}^{\infty} |f(k)| \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{u^m}{m!} \sum_{n=k+1}^{\infty} \left(\frac{\lambda}{e^u}\right)^n \frac{n^m}{n!}.$$

In the inner summation, the ratio of the consecutive terms is

$$\left(\frac{\lambda}{e^u}\right)^{n+1} \frac{(n+1)^m}{(n+1)!} \Big/ \left(\frac{\lambda}{e^u}\right)^n \frac{n^m}{n!} = \left(1 + \frac{1}{n}\right)^m \frac{\lambda}{e^u(n+1)}.$$

Since $n \geqq k+1$, $m \leqq (k+1/2)u$, and then

$$\left(1+\frac{1}{n}\right)^m \leqq \left\{\left(1+\frac{1}{n}\right)^n\right\}^{m/n} \leqq e^{(k+1/2)u/(k+1)}$$

hence

$$\left(1+\frac{1}{n}\right)^m \frac{\lambda}{e^u(n+1)} \leqq \frac{\lambda}{n e^{u/2(k+1)}},$$

$$\sum_{n=k+1}^{\infty} \left(\frac{\lambda}{e^u}\right)^n \frac{u^m}{n!} \leqq A \left(\frac{\lambda}{e^u}\right)^{k+1} \frac{(k+1)^m}{(k+1)!}.$$

Thus we have

$$I_{11} \leqq A \sum_{k=0}^{\infty} |f(k)| \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{u^m}{m!} \left(\frac{\lambda}{e^u}\right)^{k+1} \frac{(k+1)^m}{(k+1)!}$$

$$\leqq A \sum_{k=0}^{\infty} \frac{|f(k)|}{(k+1)!} \left(\frac{\lambda}{e^u}\right)^{k+1} \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{(u(k+1))^m}{m!}$$

$$\leqq A\sum_{k=0}^{\infty}\frac{|f(k)|}{(k+1)!}\left(\frac{\lambda}{e^u}\right)^{k+1} k\frac{(u(k+1))^{(k+1/2)u}}{((k+1/2)u)!}$$

$$\leqq A\sum_{k=0}^{\infty}\frac{|f(k)|}{k!}\frac{\lambda^{k+1}}{\sqrt{(k+1/2)u}},$$

which becomes small for large u. Now

$$I_{12}=\sum_{n=0}^{\infty}|f(n)|\frac{\lambda^n}{n!}e^{-un}\sum_{m=0}^{(n-1/2)u}\frac{(un)^m}{m!}.$$

Since for large u

$$e^{-v}\sum_{m=0}^{v-u/2}\frac{v^m}{m!}\qquad (v=un)$$

becomes small, I_{12} is also.

Let us now estimate I_2 by the similar argument.

$$I_2=\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{-un}\sum_{k=n+1}^{\infty}|f(k)|\sum_{m=(k-1/2)u}^{(k+1/2)u}\frac{(un)^m}{m!}$$

$$+\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{-un}\sum_{k=n+1}^{\infty}|f(n)|\sum_{m=k)-1/2)u}^{(k+1/2)u}\frac{(un)^m}{m!}=I_{21}+I_{22},$$

say. We have

$$I_{21}=\sum_{k=0}^{\infty}|f(k)|\sum_{n=0}^{k-1}\frac{\lambda^n}{n!}e^{-un}\sum_{m=(k-1/2)u}^{(k+1/2)u}\frac{(un)^m}{m!}.$$

In the inner sum, terms are monotone decreasing, and then the coefficient of $|f(k)|$ is less than

$$J=u\sum_{n=0}^{k-1}\left(\frac{\lambda}{e^u}\right)^n\frac{1}{n!}\frac{(un)^{(k-1/2)u}}{((k-1/2)u)!}=\frac{u\cdot u^{(k-1/2)u}}{((k-1/2)u)!}\sum_{n=0}^{k-}\left(\frac{\lambda}{e^u}\right)^n\frac{n^{(k-1/2)u}}{n!}.$$

Now

$$\left(\frac{\lambda}{e^u}\right)^n\frac{n^{(k-1/2)u}}{n!}$$

becomes maximum when

$$n=\frac{(k-1/2)u}{u+\log((k-1/2)u)}(1+\varepsilon),$$

where ε tends to zero as $k\to\infty$ or $u\to\infty$. For such n,

$$J\leqq\frac{\sqrt{ku}\,e^{(k-1/2)u}}{(k-1/2)^{(k-1/2)u}}\left(\frac{\lambda}{e^u}\right)^n\frac{n^{(k-1/2)u}}{n^n\sqrt{n}}e^n$$

$$=\sqrt{\frac{ku}{n}}\left(\frac{ne}{(k-1/2)u}\right)^{(k-1/2)u}\left(\frac{e\lambda}{ne^u}\right)^n$$

$$=\sqrt{\frac{ku}{n}}\left(\frac{e(1+\varepsilon)}{u+\log((k-1/2)u)}\right)^{(k-1/2)u}\times$$

$$\times \frac{(e\lambda)^{(1+\varepsilon)(k-1/2)u/(u+\log((k-1/2)u))}}{\left(\dfrac{(k-1/2)ne^u}{u+\log((k-1/2)u)}\right)^{(1+\varepsilon)(k-1/2)u/(u+\log)(k-1/2)u}}$$

$$= \sqrt{\frac{ku}{n}}\left(\frac{1+\eta}{u+\log((k-1/2)u)}\right)^{(k-1/2)u}$$

where $\eta \to 0$ as $u \to \infty$ or $k \to \infty$. Hence, if there is a w such that

$$\sum_{k=0}^{\infty} |f(k)|/w^k < \infty, \tag{2.3}$$

then I_{21} tends to zero as $n \to \infty$.

Finally

$$I_{22} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-un} \sum_{k=n+1}^{\infty} |f(n)| \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{(un)^m}{m!}$$

$$= \sum_{n=0}^{\infty} |f(n)| \frac{\lambda^n}{n!} e^{-un} \sum_{m=(n+1/2)u}^{\infty} \frac{(un)^m}{m!}.$$

For large u,

$$e^{-un} \sum_{m=(n+1/2)u}^{\infty} \frac{(un)^m}{m!} = o(1)$$

and hence I_{22} is $o(1)$.

Thus we have proved that, for any $\varepsilon > 0$, there is a u such that

$$I = \sum_{n=0}^{\infty} |B_u(n) - f(n)| \frac{\lambda^n}{n!} < \varepsilon,$$

3. We suppose that there is a w such that

$$\sum_{k=0}^{\infty} f(k)^2/w^k < \infty \tag{3.1}$$

and consider the series

$$J = \sum_{n=0}^{\infty} \left[B_u(n) - f(n)\right]^2 \frac{\lambda^n}{n!} \tag{3.2}$$

where $B_u(x)$ is defined by (2.2) whose convergence may easily be verified. By (2.2),

$$J = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} [B_u(n) - f(n)]^2$$

$$= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-2un} \sum_{k=0}^{\infty} [f(k) - f(n)] \sum_{\mu=(k-1/2)u}^{(k+1/2)u} \frac{(un)^\mu}{\mu!}$$

$$\cdot \sum_{\Upsilon=0}^{\infty} [f(\lambda) - f(n)] \sum_{\nu=(\lambda-1/2)u}^{(\lambda+1/2)u} \frac{(un)^\nu}{\nu!} + o(1)$$

$$= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n-1} + \sum_{k=n+1}^{\infty}\Big)\Big(\sum_{\lambda=0}^{n-1} + \sum_{\lambda=n+1}^{\infty}\Big) + o(1)$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\sum_{\lambda=0}^{n-1} + \sum_{n=0}^{\infty}\sum_{k=0}^{n-1}\sum_{\lambda=n+1}^{\infty} + \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\sum_{\lambda=0}^{n-1} + \sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\sum_{\lambda=n+1}^{\infty} + o(1)$$

$$= J_1 + J_2 + J_3 + J_4 + o(1),$$

say. Now

$$J_1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-2un} \sum_{k=0}^{n-1}\sum_{\lambda=0}^{n-1} [f(k) - f(n)][f(\lambda) - f(n)]\ S(k, n, u)\ S(\lambda, n, u)$$

where

$$S(k, n, u) = \sum_{\mu=(k-1/2)u}^{(k+1/2)u} \frac{(un)^{\mu}}{\mu!}.$$

Let us consider a part of J_1:

$$J_{11} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-2un} \sum_{k=0}^{n-1}\sum_{\lambda=0}^{n-1} f(k)f(\lambda)S(k, n, u)S(\lambda, n, u)S(\lambda, n, u)$$

$$\leqq \frac{1}{2}\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-2un} \sum_{k=0}^{n-1}\sum_{\lambda=0}^{n-1} f(k)^2\, S(k, n, u)\, S(\lambda, n, u)$$

$$+ \frac{1}{2}\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-2un} \sum_{k=0}^{n-1}\sum_{\lambda=0}^{n-1} f(\lambda)^2 S(k, n, u)S(\lambda, n, u)$$

$$= \frac{1}{2} J_{111} + \frac{1}{2} J_{112},$$

say.

$$J_{111} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-2un} \sum_{k=0}^{n-1} f(k)^2 S(k, n, u) \sum_{\lambda=0}^{n-1} S(\lambda, n, u)$$

$$\leqq \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-un} \sum_{k=0}^{n-1} f(k)^2 S(k, n, u),$$

which may be estimated similarly as I_{11}. J_{112} is also similarly estimated. The part of J_1 with factor $f(k)f(n)$ is estimated similarly as I_{11} and I_{12}. J_2, J_3 and J_4 may be similarly estimated, and hence, for any $\varepsilon > 0$, there is a u such that

$$\sum_{n=0}^{\infty} [B_u(n) - f(n)]^2 \frac{\lambda^n}{n!} < \varepsilon.$$

4. Let us put

$$C_u(x) = e^{-ux} \sum_{k=0}^{M} f(k) \sum_{m=(k-1/2)u}^{(k+1/2)u} \frac{(kx)^m}{m!}$$

Then we can easily see that

$$\sum_{n=0}^{\infty} |B_u(n) - C_u(n)| \frac{\lambda^n}{n!} < \varepsilon$$

$$\sum_{n=0}^{\infty}(B(n)-C(n))^2\frac{\lambda^n}{n!}<\varepsilon$$

for sufficiently large M and u, if the condition (2.3) and (3.1) are satisfied, respectively. Thus we have proved that if (2.3) is satisfied, then there are a constant a and a polynomial $P(x)$ such that

$$\sum_{n=0}^{\infty}|e^{-an}P(n)-f(n)|\frac{\lambda^n}{n!}<\varepsilon$$

and that if (3.1) is satisfied, then

$$\sum_{n=0}^{\infty}(e^{-an}P(n)-f(n))^2\frac{\lambda^n}{n!}<\varepsilon.$$

5. Let us put

$$D_u(x)=\sum_{j=0}^{N}\frac{(-ux)^j}{j!}\sum_{k=0}^{M}f(k)\sum_{m=(k-1/2)u}^{(k+1/2)u}\frac{(kx)^m}{m!}.$$

Then we have

$$\sum_{n=0}^{\infty}|C_u(n)-D_u(n)|\frac{\lambda^n}{n!}$$

$$\leqq\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\sum_{j=N+1}^{\infty}\frac{(-un)^j}{j!}\sum_{k=0}^{M}|f(k)|\sum_{m=(k-1/2)u}^{(k+1/2)u}\frac{(ln)^m}{m!}$$

$$\leqq A\sum_{k=0}^{M}|f(k)|\sum_{m=(k-1/2)u}^{(k+1/2)u}\frac{(k\lambda)^m}{m!}\sum_{j=N+1}^{\infty}\frac{(u\lambda)^j}{j!}.$$

For large u

$$\sum_{m=(k-1/2)u}^{(k+1/2)u}\frac{(k\lambda)^m}{m!}\leqq u\frac{(k\lambda)^{(k-1/2)u}}{((k-1/2)u)^{(k-1/2)u}}$$

$$\leqq u\left(\frac{k}{k-1/2}\right)^{(k-1/2)u}\left(\frac{\lambda}{u}\right)^{(k-1/2)u}\leqq Au/v^k$$

and for large N

$$u\sum_{j=N+1}^{\infty}\frac{(u\lambda)^j}{j!}$$

becomes small. Hence, if there is a w such that

$$(5.1)\qquad \sum_{k=0}^{\infty}|f(k)|/w^k<\infty,$$

then

$$\sum_{n=0}^{\infty}|C_u(n)-f(n)|\frac{\lambda^n}{n!}<\varepsilon$$

for sufficiently large u, M and N. Similarly, if

$$\sum_{k=0}^{\infty} f(k)^2/w^k < \infty \tag{5.2}$$

then

$$\sum_{n=0}^{\infty} |C_u(n) - f(n)|^2 \frac{\lambda^n}{n!} < \varepsilon.$$

Thus we have proved that, if (5.1) is satisfied for a w, then there is a polynomial $P(x)$ such that

$$\sum_{n=0}^{\infty} |P(n) - f(n)| \frac{\lambda^n}{n!} < \varepsilon,$$

and if (5.2) is satisfied, then

$$\sum_{n=0}^{\infty} (P(n) - f(n))^2 \frac{\lambda^n}{n!} < \varepsilon.$$

Thus Theorem 1 and 2 are proved.

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