NOTE ON CONSERVATIVE ALGEBRAIC FUNCTION FIELDS

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J. Tate's formula of genus reduction in his article "Genus reduction in purely inseparable extension of algebraic function fields", Proc. Amer. Math. Soc. (1952), gives a solution to the problem to characterize conservative algebraic function fields, stated in E. Artin's "Algebraic numbers and algebraic functions I", New York (1951), which we quote as A. N. F. in the following, but we discuss in the present article the problem directly on the base of the Chapter XV of A. N. F., especially on the Theorem 20 there.

Though our results follows also from above Tate's formula without any difficulties, it seems to the writer that our treatment based on a $\mathfrak{p}$-adic number theoretical lemma (Lemma 4 in the following) has some interest.

THEOREM 1. *Let k be an algebraic function field of transcendental degree 1 with coefficient field k_0 of characteristic p ($\neq 0$). Suppose that there exists an element x of k, not belonging to k_0, such that the rank n of k over $k_0(x)$ is not divided by p. Then k is conservative, if and only if every prime ideals of $k_0[x]$ generated by polynomials of x^p with coefficients in k_0 does not ramify, that is, any irreducible polynomials of x with coefficients in k_0 dividing the discriminant of the principal order of k over $k_0[x]$ are not polynomials of x^p with coefficients in k_0.*

PROOF. To prove the Theorem 1, it is sufficient to show that the Theorem holds when k_0 is separably algebraically closed (i. e. when every separably algebraic elements over k_0 are involved in k_0 itself). Because, when k_0 is not so, we take the separably algebraic closure $\bar{k}_0^s$ of k_0, the field consisting of every separably algebraic elements over k_0, and we extend the coefficient field k_0 of k to $\bar{k}_0^s$, denote k_0k by k'; then clearly the genus of k' is equal to that of k, and k' is conservative, if and only if k is conservative; on the other hand, as it holds clearly that

$$[K: \bar{k}_0^s(x)] = [k: k_0(x)],$$

the same x in k satisfies the assumption of the Theorem for k' with coefficient field $\bar{k}_0^s$; and as irreducible polynomials of x in $k_0[x]$ which are polynomials of x^p with coefficients in k_0 resolve into products of different linear polynomials of x^p with coefficients in $\bar{k}_0^s$, there exists a prime ideal in $\bar{k}_0^s[x]$ satisfying the conditions of the Theorem for k', if and only if there exists a prime ideal of $k_0[x]$ satisfying that for k.

So we suppose from now on that k_0 is separably algebraically closed. Now we state three trivial lemmas without proof.

LEMMA 1. *Let S be an arbitrary algebraic field, S^* its algebraic closure, S_1 a separably algebraic finite extension field of S in S^*, and S_2 a purely inseparable algebraic, not necessarily finite, extension field of S in S^*. Then holds*

$$[S_1S_2 : S_2] = [S_1 : S]$$

LEMMA 2. *Every finite algebraic extension field of k_0 is also separably algebraically closed.*

LEMMA 3[1]. *Let K_0 be an arbitrary inseparable finite algebraic extension field of k_0. We denote the principal order of k over $k_0[x]$ by $\mathfrak{o}_x$, and that of $K_0 k$ over $K_0[x]$ by $\mathfrak{O}_x$. Let $\mathfrak{p}$ be an arbitrary prime ideal of $\mathfrak{o}_x$, then the ideal of $\mathfrak{O}_x$ generated by $\mathfrak{p}$ is a power of a prime ideal of $\mathfrak{O}_x$.*

The following Lemma 4 is fundamental to our proof of the Theorem.

LEMMA 4. *Let S' be a field with a discrete non-archimedean valuation $|\ |_P$, S be a subfield of S', and S_1 and S_2 be finite extension fields of S involved in S'. Let S_{12} denote the field generated by S_1 and S_2 in S'; Σ, Σ_1, Σ_2, and Σ_{12} respectively the rings of integers of S, S_1, S_2, and S_{12} with reference to $|\ |_P$; $\mathfrak{p}$, $\mathfrak{p}_1$, $\mathfrak{p}_2$, and $\mathfrak{p}_{12}$ respectively their prime ideals; $\widetilde{\Sigma}$, $\widetilde{\Sigma}_1$, $\widetilde{\Sigma}_2$, and $\widetilde{\Sigma}_{12}$ respectively the residue class fields $\Sigma/\mathfrak{p}$, $\Sigma_1/\mathfrak{p}_1$, $\Sigma_2/\mathfrak{p}_2$, and $\Sigma_{12}/\mathfrak{p}_{12}$ [we identify the natural images of $\widetilde{\Sigma}$, $\widetilde{\Sigma}_1$, and $\widetilde{\Sigma}_2$ in $\widetilde{\Sigma}_{12}$ respectively with $\widetilde{\Sigma}$, $\widetilde{\Sigma}_1$, and $\widetilde{\Sigma}_2$ themselves, to obtain*

$$\widetilde{\Sigma}_{12} \supseteq \widetilde{\Sigma}_i \supseteq \widetilde{\Sigma} \qquad (i = 1, 2)].$$

And let e_1 and e_2 denote respectively the ramification degrees of S_1 and S_2 over S. Suppose that

$$(1) \qquad e_1 \gneqq 1, \ e_2 \gneqq 1,$$

$$(2) \qquad (e_1, e_2) = 1,$$

and that $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$ are linearly disjoint over $\widetilde{\Sigma}$ to each other. Then holds

$$(3) \qquad \Sigma_{12} \supsetneqq \Sigma_1 \Sigma_2$$

where $\Sigma_1 \Sigma_2$ denotes the subring of Σ_{12} generated by Σ_1 and Σ_2.

PROOF. We take a primitive element π of $\mathfrak{p}$ in S and determine the orders of elements of S' with reference to $|\ |_P$ such that that of π is equal to 1. From (2) there exists in Σ_{12} an element A of order $1/e_1e_2$ (we denote it by $A \sim \pi^{1/e_1e_2}$). We show that every element in S_{12} with order $1/e_1e_2$ does not belong to $\Sigma_1 \Sigma_2$. The denial of this fact leads to a contradiction as follows. Suppose that there exists an element A of $\Sigma_1 \Sigma_2$ with order $1/e_1e_2$. A can be written as

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1). We need not the supposition that k_0 is separably algebraically closed in the Lemma 3.

$$A = \sum_{i=1}^{t} a_i b_i$$

with

(4) $a_i \in \Sigma_1, \ b_i \in \Sigma_2 \qquad (i = 1, 2, \ldots . t).$

Then holds clearly

(5) $$\sum_{i=1}^{t} \tilde{a}_i \tilde{b}_i = 0$$

where we denote by $\tilde{a}_i$ and $\tilde{b}_i$ the elements of $\tilde{\Sigma}_{12}$ naturally determined respectively by a_i and b_i. As from the supposition holds

(6) $$\min. (1/e_1, \ 1/e_2) \gneqq 1/e_1 e_2$$

follows that all of $a_i b_i$ for $i = 1, 2, \ldots . t$ are not divided by $\mathfrak{P}_{12}$. We sum up among $a_i b_i$ $(i = 1, 2, \ldots . t)$ all of such ones which are not divided by $\mathfrak{P}_{12}$, and denote the sum by

$$A' = \sum_{i=1}^{t}{}' a_i b_i,$$

then holds clearly

(7) $$\sum_{t=1}^{i}{}' a_i b_i \sim \pi^{\frac{1}{e_1 e_2}},$$

as our valuation is non-archimedean. Then, changing the suffixes suitably, if necessary, we obtain a natural number $t' \leqq t$ such that

(8) $$\tilde{a}_i \tilde{b}_i \neq 0 \qquad \text{for } i = 1, 2, \ldots . t'$$
$$\tilde{a}_i \tilde{b}_i = 0 \qquad \text{for } i = t' + 1, \ t' + 2, \ldots . t.$$

Now

(9) $$\sum_{i=1}^{t}{}' a_i b_i = \sum_{i=1}^{t'} a_i b_i,$$

and so clearly

(10) $$\sum_{i=1}^{t'} \tilde{a}_i \tilde{b}_i = 0, \quad \tilde{b}_i \neq 0 \qquad \text{for } i = 1, 2, \ldots . t'.$$

As, from the supposition, $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ are linearly disjoint over $\tilde{\Sigma}$ to each other, there exists $c_i \in \Sigma$ $(i = 1, 2, \ldots . t')$ such that for not all of them hold

$$\tilde{c}_i = 0$$

and

(11) $$\sum_{i=1}^{t'} \tilde{a}_i \tilde{c}_i = 0.$$

So we can suppose that

$$\tilde{c}_1 \neq 0.$$

Then there exists clearly $c_i' \in \Sigma$ for $i = 2, \cdots t'$ such that

(12) $$a_1 \equiv \sum_{i=2}^{t'} a_i c_i' \quad \bmod \mathfrak{P}_1.$$

Then from (6) holds clearly

(13) $$\sum_{i=2}^{t'} a_i b_i + b_1 \sum_{i=2}^{t'} a_i c_i' \sim \pi^{\frac{1}{e_1 e_2}}.$$

As

(14) $$b_i + b_1 c_i' \in \Sigma_2 \qquad (i = 2, 3, \cdots t'),$$

denoting them respectively by b_i' for $i = 2, 3, \cdots t'$, we obtain that

(15) $$A_1 = \sum_{i=2}^{t'} a_i b_i' \sim \pi^{\frac{1}{e_1 e_2}}$$

with

$$a_i \in \Sigma_1, \quad b_i \in \Sigma_2.$$

Now we consider A_1 as A, repeat the above process, obtain A_1, A_2, and repeat it to A_2 again, obtain A_2', A_3 and so on. Then we obtain $a \in \Sigma_1$ and $b \in \Sigma_2$ such that

(16) $$ab \sim \pi^{\frac{1}{e_1 e_2}},$$

which contradicts to (6), as easily seen, and we obtain the Lemma.

From the above proof we obtain also

COROLLARY. *If one replace the condition*

$$(e_1, e_2) = 1$$

in the Lemma 4 *with the condition that there exists an element in* Σ_{12} *with order smaller than* min $(1/e_1, 1/e_2)$, *it holds also*

$$\Sigma_{12} \supsetneqq \Sigma_1 \Sigma_2$$

Now we prove the Theorem (for separably algebraically closed k_0).

NECESSITY. Suppose that $(x^{p^r} - \alpha)$ is a prime ideal of $k_0[x]$ and it ramifies in $k/k_0(x)$, where r is a natural number and $\alpha \in k_0$. We take $k_0(\sqrt[p^r]{\alpha})$ and denote it by K_0. Let $\overline{k_0(x)}$, $\overline{k}$, $\overline{K_0(x)}$, and $\overline{K_0 k}$ denote respectively the completion fields of $k_0(x)$, k, $K_0(x)$, and $K_0 k$ with reference to the valuation determined by a prime divisor of $K_0 k$ dividing $x^{p^r} - \alpha$. Then, applying Lemma 1, 2, and 3, we see easily that we can apply Lemma 4 to $\overline{k_0(x)}$, $\overline{k}$, $\overline{K_0(x)}$, and $\overline{K_0 k}$, instead of S, S_1, S_2, and S_{12} respectively. So follows from A. N. F. Theorem 20, Chap. XV, 5

$$g(k) \gneqq g(K_0 k),$$

where we denote by $g(k)$ and $g(K_0 k)$ the genera of k and $K_0 k$, and the necessity is proved.

SUFFICIENCY. Now suppose that every prime ideal of $k_0[x]$ written as

$(x^{p^r}-\alpha)$ with natural number r and $\alpha \in k_0$, does not ramify in $k/k_0(x)$. We take an arbitrary purely inseparable algebraic simple extension field $k_0(\sqrt[p^t]{\beta})$ of k_0, denote it by K_0, and K_0k by K and prove

$$g(k) = g(K),$$

from which results easily the sufficiency.

Now let $\mathfrak{P}$ be an arbitrary prime divisor of the algebraic function field K with coefficient field K_0; $| \ |_{\mathfrak{p}}$ denote the valuation of K determined by $\mathfrak{P}$; $\overline{k_0(x)}$, $\overline{k}$, $\overline{K_0(x)}$, and $\overline{K}$ respectively the completion fields of $k_0(x)$, k, $K_0(x)$ and K; $\widetilde{k_0(x)}$, $\widetilde{k}$, $\widetilde{K_0(x)}$, and $\widetilde{K}$ the residue class fields of $\overline{k_0(x)}$, $\overline{k}$, $\overline{K_0(x)}$, and $\overline{K}$; e_1, e_2, and e' respectively the ramification degrees of $\overline{k}/\overline{k_0(x)}$, $\overline{K_0(x)}/\overline{k_0(x)}$, and $\overline{K}/\overline{K_0(x)}$; f_1, f_2, and f' respectively the ranks $[\widetilde{k}:\widetilde{k_0(x)}]$, $[\widetilde{K_0(x)}:\widetilde{k_0(x)}]$, and $[\widetilde{K}:\widetilde{K'}]$. Then, as

$$e_1f_1 \mid n, \qquad (n, p) = 1, \tag{17}$$

follows from Lemma 1 and 2 that

$$e_1f_1 = e'f', \qquad f_1 \leqq f'. \tag{18}$$

On the other hand, from $(e_1, e_2) = 1$ it follows clearly

$$e_1 \leqq e'. \tag{19}$$

Thus we obtain

$$e_1 = e', \qquad f_1 = f', \tag{20}$$

and so

$$e_1e_2 = e_{12}, \tag{21}$$

where we denote by e_{12} the ramification degree of $\overline{K}/\overline{k_0(x)}$. Now we distinguish the case when $\mathfrak{P}$ divides $1/x$, from when not. If $\mathfrak{P}$ divides $1/x$, then clearly e_2 is 1, while, if $\mathfrak{P}$ does not divide $1/x$, then follows from the assumption that any prime ideal of the form $(x^{p^r}-\alpha)$ with natural number r and $\alpha \in k_0$, that either of e_1 or e_2 is equal to 1. Thus for each divisor of K holds always either

$$e_1 = e_{12} \text{ or } e_2 = e_{12}. \tag{22}$$

Then there exists clearly an integer in $K_0\mathfrak{o}_p$ divided just by $\mathfrak{P}$, not by $\mathfrak{P}^2$, where we denote by $\mathfrak{o}_p$ the ring of integers of $\overline{k}$. On the other hand, from

$$f_1 = f'$$

follows that we can take representatives of the residue classes of $\widetilde{K}$ within $K_0\mathfrak{o}_p$. So we can approximate each element of $\mathfrak{O}_P$ by elements of $K_0\mathfrak{o}_p$ in the sense of the metric defined by $| \ |_P$, where we denote by $\mathfrak{O}_P$ the ring of integers of $\overline{K}$. As $K_0\mathfrak{O}_p$ is, as easily seen, a closed subset of $\overline{K}$ in the sense of that topology[2], we obtain

$$\mathfrak{O}_P = K_0\mathfrak{o}_p \tag{23}$$

2). Cf. A. N. F. Chap. II.

which satifies

(24) $$g(k) = g(K)$$

from A. N. F. Theorem 20, Chap. XV, 5. From Lemma 3 no prime ideal of $K_0[x]$ written as $(x^{p^r} - \alpha')$ with natural number r and $\alpha' \in K_0$ ramifies in $K/K_0(x)$. So repeating the above considerations as to K, we conclude that K is also genus-conservative for purely inseparable algebraic simple extensions of the coefficient field K_0. Thus k is genus-conservative for purely inseparable algebraic finite extensions of the coefficient field k_0, and the sufficiency is proved. q. e. d.

As to the necessary condition for the conservativity holds moreover

THEOREM 2. *Let k be an arbitrary algebraic function field of one variable with coefficient field k_0. If k is conservative, then for each element x of k not involved in k_0, the prime ideal of k_0 $[x]$ written as $(x^{p^r} - \alpha)$ with natural number r and $\alpha \in k_0$ can not be divided by 2nd power of any prime ideal of the ring of integers of k.*

This can be proved without any essential difficulties in a similar way as in the first part of the above proof of the Theorem 1, applying Corollary of Lemma 4 in place of Lemma 4.

Finally we add the following remark due to Prof. T. Tannaka.

If we presuppose the Tate's formula and a proposition on p. 405 of his paper quoted above, and also the book "Introduction to the theory of algebraic functions of one variable, (1951)" of C. Chevalley, then we have immediately the following generalization of our Theorem 1.

THEOREM. *Let k be a separably generated algebraic function field, and x be a separating variable. Then the theorem 1 remains true.*

We confine ourselves by indicating the facts:

(i) By separable constant extension, the genus is invariant (Chevalley, l. c. p. 99).

(ii) If k is a separably generated algebraic function field, and L an extension of constant field, then the constant field of the constant extension $k(L)$ coincides with L (Chevalley l. c. p. 91).

(iii) If k is a separably generated algebraic function field, then none of the prime divisors of k is ramfied by constant extension (Chevalley, l. c. p. 92).

As the genus change occurs already in a finite constant extension, and so by (i) already by finite purely inseparable constant extension we can restrict ourselves to the case of such constant extension. From (iii) it suffices to investigate the case of prime degree p, where p is the characteristic of k. The fact (ii) is used when we apply the formula of Tate.

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