HOMOLOGY GROUPS IN CLASS FIELD THEORY

SHUICHI TAKAHASHI

(Riceived December 15, 1952)

Recently, J. Tate[1] has given an interesting theorem that the higher dimensional cohomology groups $H^r(G, A)$ occurring in class field theory, i. e. A: the multiplicative group of nonzero elements in a $\mathfrak{p}$-adic field or the idèle class group in an algebraic number field, G: the galois group, are canonically isomorphic to the integral cohomology groups $H^{r-2}(G, Z)$ for every $r > 2$. It was stated, without details, that one can introduce negative dimensional cohomology groups and the isomorphisms:

$$H^{r-2}(G, Z) \cong H^r(G, A)$$

are valied for every dimensions. Moreover, the isomorphism

$$H^{-2}(G, Z) \cong H^0(G, A)$$

from $H^{-2}(G, Z) =$ commutator factor group of G, to $H^0(G, A) =$ idèle norm residue class group, is the reciprocity law mapping.

I shall show in this note that if we put

$$H^{-r}(G, A) = H_{r-1}(G, A) \qquad (r = 1, 2, \ldots.)$$

where $H_{r-1}(G, A)$ is the $(r-1)$-dimensional homology group, then all statements of Tate hold. Moreover, the isomorphism

$$H^{-3}(G, Z) \cong H^{-1}(G, A)$$

is the isomorphism theorem of H. Kuniyoshi[2] in the theory of T. Tannaka[3] concerning the "*Hauptgeschlechtssatz im Minimalen*".

1. Let G be a finite group, A a G-module, we now define boundary and coboundary operators ∂, δ for the module of q-chains[4] $C_q(G, A)$ of G with value in A:

$$\partial f(x_1 \ldots, x_{q-1}) = \sum_{x \in G} x^{-1} f(x, x_1, \ldots. x_{q-1})$$

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1) J. TATE, Higher dimensional cohomology groups of class field theory. Ann. of Math., 56(1952), 294-297.

2) H. KUNIYOSHI, On a certain group concerning the p-adic number field, Tohoku Math. Journ., 1(1950), 186-193, Theorem 2.

3) T. TANNAKA, Some remarks concerning p-adic number field, Journ. of Math. Soc. of Japan, 3(1951), 252-257, Theorem 2.

4) q-chain is a function of q-variable in G to A; therefore identical with q-cochain. For infinite group G, one must restrict the function to the class that are $\neq 0$ only for some finite systems $(x_1, \cdots, x_q)$ of elements in G.

$$+\sum_{i=1}^{q-1}(-1)^i\sum_{x\in G}f(x_1,\cdots,x_ix^{-1},x,\cdots,x_{q-1})$$

$$+(-1)^q\sum_{x\in G}f(x_1,\cdots,x_{q-1},x),$$

$$\delta f(x_1,\cdots,x_{q+1})=x_1f(x_2,\cdots,x_{q+1})+\sum_{i=1}^{q}(-1)^if(x_1,\cdots,x_ix_{i+1},\cdots,x_{q+1})+(-1)^{q+1}f(x_1,\cdots,x_q).$$

$H_q(G,A)$ and $H^q(G,A)$ denote, respectively, the q-homology and q-cohomology group for every $q>0$; and for $q=0$ put

$$H_0(G,A)=A_N/\Delta A,\quad H^0(G,A)=A_\Delta/NA^{5)}$$

where we used the following conventions:

$$A_N=\{a\,|\,Na=0,\ N=\sum_{x\in G}x\},\qquad NA=\{Na\,|\,a\in A\},$$

$$A_\Delta=\{a\,|\,\Delta_x a=0,\ \Delta_x=1-x,x\in G\},\ \Delta A=\{\Delta_x a\,|\,x\in G,a\in A\}.$$

Let B be another G-module which is paired[6] to a third G-module E:

$$(A,B)\ni(a,b)\rightarrow a\cdot b\in E.$$

Define cap-and cup-product $\cap$, $\cup$ by

$$f\cap g(x_1,\cdots,x_p)=\sum_{x_{p+1},\cdots,x_{p+q}\in G}x_1\cdots x_pf(x_{p+1},\cdots,x_{p+q})\cdot g(x_1,\cdots,x_{p+q})\in C_p(G,E)$$
$$(f\in C^q(G,A),\ g\in C_{p+q}(G,B)),$$

$$f\cup g(x_1,\cdots,x_{p+q})=f(x_1,\cdots,x_p)\cdot x_1\cdots x_pg(x_{p+1}\cdots,x_{p+q})\in C_{p+q}(G,E)$$
$$(f\in C^p(G,A),\ g\in C^q(G,B)).$$

By direct computations one can prove:

$$\partial(f\cap g)=f\cap\partial g+(-1)^p\delta f\cap g\qquad(f\in C^q(G,A),\ g\in C_{p+q}(G,B))$$
$$\delta(f\cup g)=\delta f\cup g+(-1)^pf\cup\delta g\qquad(f\in C^p(G,A),\ g\in C^q(G,B)).$$

2. Let now A be a G-module which satisfies the axiom 1 of J. Tate, and $\alpha\in H^2(G,A)$ be a canonical class whose restriction to any subgroup $U\subset G$ generates the cyclic group $H^2(U,A)$ of order equal to that of U. Tate's isomorphisms are given explicitly by

$$H^{r-2}(G,Z)\ni\zeta\rightarrow\alpha\cup\zeta\in H^r(G,A)\qquad(r\geqq 2)$$

5) This definition of H^0 is due to J. Tate, loc. cit[1].

6) I. e. bilinear map of (A,B) to E satisfying $x(a,b)=xa\cdot xb$ for any $x\in G$, $a\in A$, $b\in B$.

$$H^{-1}(G, Z) = 0 \rightarrow \qquad H^1(G, A) = 0,$$

and

$$H^{-2}(G, Z) = H_1(G, Z) \ni f(x) \rightarrow \sum_{x \in G} a(N, x) f(x) \in H^0(G, A$$

where $H_1(G, Z) \ni f(x) \rightarrow \prod_{x \in G} x^{f(x)} \in G/G'$ is a canonical isomorphism and $a(N, x) = \sum_{y \in G} a(y, x)$ with $a(y, x) \in \alpha$. Therefore, the isomorphism $G/G' \cong H^{-2}(G, A) \cong H^0(G, A) = A_\Delta/NA$ is given by

$$G/G' \ni x \rightarrow a(N, x) \in A_\Delta/NA,$$

i. e. the reciprocity law mapping as was stated by J. Tate.

For negative dimensions $-r (r > 0)$ we have

$$H^{-r-2}(G, Z) = H_{r+1}(G, Z) \ni \zeta \rightarrow \alpha \cap \zeta \in H_{r-1}(G, A) = H^{-r}(G, A).$$

The proof of this isomorphism theorem can be obtained, word for word, from that of J. Tate.

From the theory of universal coefficients group[7] it follows readily that

$$H_q(G, Z) \cong H^{q+1}(G, Z) \qquad (q = 0, 1, 2, \ldots).$$

Hence we conclude from the above isomorphism theorem that

$$H_r(G, A) \cong H^{r+1}(G, A) \qquad (r = 0, 1, 2, \ldots).$$

3. We shall finally mention the meaning of the isomorphism :

$$H_2(G, Z) \ni \zeta \rightarrow \alpha \cap \zeta \in H_0(G, A) = A_N/\Delta A.$$

For this, we assume that G be abelian of type $(n_1, \cdots, n_m)$, $n_{i+1} | n_i (i = 1, \ldots, m-1)$, with m generators $s_1, \cdots, s_m$. If we write $\Delta_i = 1 - s_i^{-1}$, $N_i = 1 + s_i + \cdots + s_i^{n-1}$, O. Schreier's normalization process[8] can be applied to 2-homology group $H_2(G, B)$ for any G-module B and yields the following statements : $H_2(G, B)$ has a representative system consists of 2-cycles :

$$f_a(s_i^k, s_i) = a_{ii} \qquad 1 \leqq k \leqq n_i - 1, \qquad 1 \leqq i \leqq m,$$
$$f_a(s_i, s_j) = -f_a(s_j, s_i) = a_{ij}, \qquad 1 \leqq j < i \leqq m,$$
$$f_a(x, y) = 0 \qquad \text{for all other cases,}$$

associated to a system (a_{ij}), $i \geqq j$, in B satisfying

$$-\sum_{j<i} \Delta_j a_{ji} + N_i a_{ii} + \sum_{j<i} \Delta_j a_{ij} = 0, \qquad 1 \leqq i \leqq m.$$

$f_a \sim 0$ if and only if there exists a system (a_{ijk}), $i \geqq j \geqq k$, in B such that

7) E. g. S. EILENBERG, Topological methods in abstract algebra, Bull. A. M. S., 55 (1949), 3–37, Formula (13. 1).

8) O. SCHREIER, Über die Erweiterung von Gruppen I, Monatsh. f. Math. u. Phys., 34 (1926), 165–180. Satz III. Cf. also a forthcoming paper by Prof. T. Tannaka.

$$a_{ii} = \sum_{j>i} \Delta_i a_{jii} + \sum_{j\leq i} \Delta_j a_{iij} \qquad 1 \leqq i \leqq m,$$

$$a_{ij} = \sum_{k>i} \Delta_k a_{kij} - N_i a_{iij} - \sum_{i>k>j} \Delta_k a_{ikj} + N_j a_{ijj} + \sum_{k<j} \Delta_k a_{ijk} \qquad 1 \leqq j < i \leqq m.$$

We now apply this results to our group $H_2(G, Z)$ and obtain the following basis

$$f_{ij}(s_i, s_j) = -f_{ij}(s_j, s_i) = 1 \qquad 1 \leqq j < i \leqq m$$

$$f_{ij}(x, y) = 0 \qquad \text{for all other cases,}$$

of order n_j. Therefore

$$H_2(G, Z) \cong \sum_{i=2}^{m} (i-1)\cdot Z/(n_i)$$ [9]

and consequently

$$A_N/\Delta A \cong \sum_{i=2}^{m} (i-1)\cdot Z/(n_i),$$

this is the isomorphism theorem of H. Kuniyoshi[2].

Since

$$a \cap f_{ij} = a(s_i, s_j) - a(s_j, s_i)$$

we see that

$$A_N = \{a(s_i, s_j) - a(s_j, s_i), \Delta A\},$$

this is the T. Tannaka's "*Hauptgeschlechtssatz im Minimalen*"[3].

MATHEMATICAL INSTITUTE, TOHOKU UNIVERSITY, SENDAI.

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9) This is a theorem of H. HOPF, Fundamentalgruppe und zweite Bettische Gruppe, Comm. Math. Helv., 14(1941-2), 257-309, Nr. 13, c), and is a special case of Lyndon's formula; R. C. LYNDON, The cohomology theory of group extensions, Duke Math. Journ,. 15(1948), 271-292, Theorem 6.