ON THE DIRECT-PRODUCT OF OPERATOR ALGEBRAS II

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1. Introduction. R. Schatten-J. von Neumann [4] introduced the idea of direct-product of Banach spaces, and the author modified this considerations to C^*-algebras in the previous paper [7], and defined the direct-product of C^*-algebras.

Let A_1 and A_2 be any C^*-algebras with unit, and following R. Schatten-J. von Neumann. construct $A_1 \times A_2$ as the set of all expressions $\sum x_i \times y_i$ with the equivalence relation $\cong$, as A_1, A_2 to be Banach spaces; and finally define the multiplication, involution and norm of expressions as follows:

$$\text{product:} \qquad \Big(\sum_{i=1}^{n} x_i \times y_i\Big)\Big(\sum_{j=1}^{m} s_j \times t_j\Big) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i s_j \times y_i t_j,$$

$$\text{involution:} \qquad \Big(\sum_{i=1}^{n} x_i \times y_i\Big)^* = \sum_{i=1}^{n} x_i^* \times y_i^*,$$

$$\text{norm:} \qquad \alpha\Big(\sum_{i=1}^{n} x_i \times y_i\Big) = \sup\Big[\Phi\Big(\Big(\sum_{i=1}^{n} x_i \times y_i\Big)\Big(\sum_{i=1}^{n} x_i \times y_i\Big)^*\Big)^{1/2} : \Phi \in \mathfrak{S}\Big],$$

where $\mathfrak{S}$ denotes the set of positive type functional Φ's such that

$$\Phi\Big(\sum_{j=1}^{m} s_j \times t_j\Big) = \frac{\varphi \times \psi\Big(\Big(\sum_{i=1}^{n} x_i \times y_i\Big)\Big(\sum_{j=1}^{m} s_j \times t_j\Big)\Big(\sum_{i=1}^{n} x_i \times y_i\Big)^*\Big)}{\varphi \times \psi\Big(\Big(\sum_{i=1}^{n} x_i \times y_i\Big)\Big(\sum_{i=1}^{n} x_i \times y_i\Big)^*\Big)}$$

φ and ψ are pure states on A_1 and A_2 respectively, and $\sum_{i=1} x_i \times y_i$ is an arbitrary element of $A_1 \times A_2$; then α becomes a cross-norm on $A_1 \times A_2$ and $A_1 \times A_2$, is a non-complete C^*-algebra [7].

Now, let A_1 and A_2 be C^*-algebras on the Hilbert spaces H_1 and H_2 respectively. Then by [3, 4], $\sum_{i=1}^{n} x_i \times y_i$ can be considered as bounded operator on $H = H_1 \times_\sigma H_2$. In this paper, we consider the relation between C^*-algebra generated by $\sum_{i=1}^{n} x_i \times y_i$ as operator on H with operator bound as norm, and direct-product $A_1 \times_\alpha A_2$ (§2); and we give more detailed discussions in the case where A_i are C^*-algebras of completely continuous operators on H_i (§3); and finally in §4 we prove some algebraic properties of $A_1 \times_\alpha A_2$.

2. Relation between the direct-product as operators and as algebras.[1] Suppose that H_1 and H_2 are Hilbert spaces. Then, F. J. Murray-J. von Neumann's construction [3] of direct-product gives us $H = H_1 \times_\sigma H_2$ as Hilbert space. If x and y are operators on H_1 and H_2 respectively, then

$$\left(\sum_{i=1}^{n} \xi_i \times \eta_i\right)(x \times y) = \sum_{i=1}^{n} \xi_i x \times \eta_i y$$

gives a linear operator $x \times y$ on H, and the operator bound of $x \times y$ on H satisfies the cross-property of Schatten [4]: $|||x \times y||| = |||x||| \cdot |||y|||$ [2].

If A_1 and A_2 are C^*-algebras on H_1 and H_2 respectively, then the set

$$\{x \times y : x \in A_1,\ y \in A_2\}$$

generates a C^*-algebra A.

On the other hand, we can consider the direct-product $A_1 \times_\alpha A_2$ as in [7], in this case, we define the norm $\alpha(\cdot)$ as follows:

$$\alpha\left(\sum_{i=1}^{n} x_i \times y_i\right) = \sup\left[\Phi\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)^*\right)^{1/2} : \Phi \in \mathfrak{S}'\right],$$

where $\mathfrak{S}'$ denotes the set of positive type functional Φ's on $A_1 \times A_2$ such that $\Phi = \sum_{i,j=1}^{n} \varphi_{ij} \times \psi_{ij}$, $\Phi(1 \times 1) = 1$, and $\varphi_{ij}(x)$, $\psi_{ij}(y)$ have the form

$$\varphi_{ij}(x) = < \xi_i x,\ \xi_j >, \text{ and } \psi_{ij}(y) = < \eta_i y,\ \eta_j >$$

respectively where $\xi_i \in H_1, \eta_j \in H_2$ and $\|\sum_{i=1}^{n} \xi_i \times \eta_j\| = 1$. Then the following theorem holds:

THEOREM 1. *A is isometrically isomorphic to the direct-product $A_1 \times_\alpha A_2$.*

LEMMA 1. *If the expression $\sum_{i=1}^{n} x_i \times y_i$ is equivalent to 0×0 as element of direct product $A_1 \times_\alpha A_2'$ of algebras, then it is zero operator on H.*

PROOF. By the definition of the norm in $A_1 \times_\alpha A_2$,

$$\begin{aligned} 0 &= \sum_{i,j=1}^{n} < \xi x_i x_j^*,\ \xi > < \eta y_i y_j^*,\ \eta > \\ &= \sum_{i,j=1}^{n} < \xi x_i,\ \xi x_j > < \eta y_i,\ \eta y_j > \\ &= < \sum_{i=1}^{n} \xi x_i \times \eta y_i,\ \sum_{j=1}^{n} \xi x_j \times \eta y_j > \end{aligned}$$

for any elements $\xi \in H_1$, $\eta \in H_2$, $\|\xi\| = \|\eta\| = 1$. Then, $\sum_{i=1}^{n} \xi x_i \times \eta y_i = 0$ for any $\xi \in H_1$, $\eta \in H_2$, so $\sum_{i=1}^{n} x_i \times y_i = 0$ as operator on H.

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1) The author expresses his hearty thanks for many discussions of Prof. M. Nakamura; he has pointed out the incompleteness of author's original proof of present and next sections.

2) $|||\cdot|||$ denotes the operator bound.

LEMMA 2. *The converse statement of Lemma* 1 *holds.*

PROOF. Let $\sum_{i=1}^{n} x_i \times y_i = 0$ as operator on H. First we remark that we can assume, without loss of generality, the linear independency of $\{x_i\}$. Indeed, if $x_1 = \sum_{i=2}^{n} a_i x_i$ holds, then for any $\xi \times \eta \in H$,

$$(\xi \times \eta)\sum_{i=1}^{n} x_i \times y_i = (\xi \times \eta)\left(\left(\sum_{i=2}^{n} a_i x_i\right)\times y_1 + \sum_{i=2}^{n} x_i \times y_i\right)$$

$$\cong \sum_{i=2}^{n} a_i \xi x_i \times \eta y_1 + \sum_{i=2}^{n} \xi x_i \times \eta y_i$$

$$\cong \sum_{i=2}^{n} \xi x_i \times \eta (a_i y_1) + \sum_{i=2}^{n} \xi x_i \times \eta y_i$$

$$\cong (\xi \times \eta)\left(\sum_{i=2}^{n} x_i \times (a_i y_1 + y_i)\right),$$

so $\sum_{i=1}^{n} x_i \times y_i \cong \sum_{i=2}^{n} x_i \times (a_i y_1 + y_i)$.

Now, assume that $\sum_{i=1}^{n} x_i \times y_i = 0$ as operator on H, and $\{x_i\}$ are linearly independent, then for any $\xi,\ \xi' \in H_1$ and $\eta,\ \eta' \in H_2$

$$0 = <(\xi \times \eta)\sum_{i=1}^{n} x_i \times y_i,\ \xi' \times \eta'>$$

$$= <\sum_{i=1}^{n} \xi x_i \times \eta y_i,\ \xi' \times \eta'>$$

$$= \sum_{i=1}^{n} <\xi x_i,\ \xi'><\eta y_i,\ \eta'>$$

$$= \sum_{i=1}^{n} <<\eta y_i,\ \eta'>\xi x_i,\ \xi'>$$

$$= <\sum_{i=1}^{n} <\eta y_i,\ \eta'>\xi x_i,\ \xi'>.$$

Since ξ' is any element of H_1, $\Sigma<\eta y_i,\ \eta'>\xi x_i = \xi(\Sigma<\eta y_i,\ \eta>' x_i) = 0$, and furthermore, by the arbitrariness of $\xi \in H_1$, $\Sigma<\eta y_i,\ \eta'>x_i = 0$ as operator on H_1. While $\{x_i\}$ are linearly independent, so $<\eta y_i,\ \eta'> = 0$ for $i = 1, 2, \cdots, n$. Again by the arbitrariness of η, and η', $y_i = 0$, $i = 1, 2, \cdots, n$ as operators on H_2, so finally $\sum_{i=1}^{n} x_i \times y_i \cong 0 \times 0$ as element of direct-product.

LEMMA 3. $$\alpha\left(\sum_{i=1}^{n} x_i \times y_i\right) = \left\|\left|\sum_{i=1}^{n} x_i \times y_i\right\|\right|.$$

PROOF.

$$\left\|\left|\sum_{i=1}^{n} x_i \times y_i\right\|\right|^2 = \sup\left[\left\|\left(\sum_{j=1}^{n'} \xi_j \times \eta_j\right)\left(\sum_{i=1}^{n} x_i \times y_i\right)\right\|^2 : \left\|\sum_{j=1}^{n'} \xi_j \times \eta_j\right\| = 1\right]$$

$$= \sup\left[\left\|\sum_{i,j} \xi_j x_i \times \eta_j y_i\right\|^2 : \left\|\sum_{j} \xi_j \times \eta_j\right\| = 1\right]$$

$$= \sup\left[\sum_{i,j,k,m} < \xi_j x_i,\ \xi_k x_m > < \eta_j y_i,\ \eta_k y_m > : \left\|\sum_{j} \xi_j \times \eta_j\right\| = 1\right]$$

$$= \sup\left[\sum_{i,j,k,m} \varphi_{jk}(x_i x_m^*)\, \psi_{jk}(y_i y_m^*) : \left\|\sum_{j} \xi_j \times \eta_j\right\| = 1\right],$$

where $\varphi_{jk}(x) = < \xi_j x,\ \xi_k >$ and $\psi_{jk}(y) = < \eta_j y,\ \eta_k >$

$$= \sup\left[\Phi\left(\left(\sum_{i=1}^{n} x_i \times y_i\right)\left(\sum_{j=1}^{n} x_j^* \times y_j^*\right)\right) : \Phi \in \mathfrak{S}'\right]$$

$$= \alpha\left(\sum_{i=1}^{n} x_i \times y_i\right)^2.$$

PROOF OF THEOREM 1. By Lemmas 1 and 2, A is algebraically isomorphic to $A_1 \times_\alpha A_2$, and by Lemma 3, this isomorphism is also isometric. This completes the proof.

3. Direct-product of completely continuous operators. Our principal aim of this section is to show

THEOREM 2. *If A_1 and A_2 are C^*-algebras of completely continuous operators on H_1 and H_2 respectively, then $A = A_1 \times_\alpha A_2$ is also a C^*-algebra of completely continuous operators on $H = H_1 \times_\sigma H_2$.*

To prove the statement, we shall begin by proving

LEMMA 4. *For any $\xi_1,\ \xi_2 \in H_1$ and $\eta_1,\ \eta_2 \in H_2$,*

$$(\xi_1 \times \eta_1) \times (\xi_2 \times \eta_2) = (\xi_1 \times \xi_2) \times (\eta_1 \times \eta_2)$$

where $\zeta(\zeta_1 \times \zeta_2) = < \zeta,\ \zeta_2 > \zeta_1$ for $\zeta, \zeta_1, \zeta_2 \in H$.

PROOF. Let $\xi \in H_1$ and $\eta \in H_2$, we have, for $\zeta = \xi \times \eta$,

$$\begin{aligned}
&(\xi \times \eta)((\xi_1 \times \eta_1) \times (\xi_2 \times \eta_2)) \\
&= < \xi \times \eta,\ \xi_2 \times \eta_2 > \xi_1 \times \eta_1 \\
&= < \xi,\ \xi_2 > < \eta,\ \eta_2 > \xi_1 \times \eta_1 \\
&= \xi(\xi_1 \times \xi_2) \times \eta(\eta_1 \times \eta_2)
\end{aligned}$$

$$= (\xi \times \eta)[(\xi_1 \times \xi_2) \times (\eta_1 \times \eta_2)].$$

To prove the theorem, it is sufficient to show that each $x \times y$ is completely continuous, since $A_1 \times_\alpha A_2$ is generated by such $x \times y$'s. If $x = \sum_{i=1}^{\infty} \xi_i \times \xi_i'$ and $y = \sum_{j=1}^{\infty} \eta_j \times \eta_j'$ are the canonical form of J. Dixmier [1], then it is also sufficient to show for

$$x = \sum_{i=1}^{n} \xi_i \times \xi_i' \quad \text{and} \quad y = \sum_{j=1}^{m} \eta_j \times \eta_j',$$

since such $x \times y$ are dense in $A_1 \times_\alpha A_2$. Therefore by Lemma 4,

$$x \times y = \left(\sum_{i=1}^{n} \xi_i \times \xi_i'\right) \times \left(\sum_{j=1}^{m} \eta_j \times \eta_j'\right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (\xi_i \times \eta_j) \times (\xi_i' \times \eta')$$

shows that $x \times y$ is a completely continuous operator on $H_1 \times_\sigma H_2$. This proves Theorem 2.

THEOREM 3. *If A_i of the previous theorem are the algebra $C(H_i)$ of all completely continuous operators on H_i, then A is the algebra $C(H)$ of all completely continuous operators on H: i.e.,*

$$C(H_1) \times_\alpha C(H_2) = C(H_1 \times_\sigma H_2).$$

PROOF. By the previous theorem, it is sufficient to show that A contains all one-dimensional projections of H. This is proved when $\left(\sum_{i=1}^{\infty} \xi_i \times \eta_i\right) \times \left(\sum_{i=1}^{\infty} \xi_i \times \eta_i\right)$ is contained in A, if $\sum_{i=1}^{\infty} \xi_i \times \eta_i$ exists and is of norm unity. Since, for each n,

$$\frac{\sum_{i=1}^{n} \xi_i \times \eta_i}{\left\| \sum_{i=1}^{n} \xi_i \times \eta_i \right\|} \times \frac{\sum_{i=1}^{n} \xi_i \times \eta_i}{\left\| \sum_{i=1}^{n} \xi_i \times \eta_i \right\|}$$

exists in A and converges uniformly to $(\Sigma \xi_i \times \eta_i) \times (\Sigma \xi_i \times \eta_i)$, the latter is contained in A.

Following Corollary is an immediate consequence of our Theorem 3 and I. Kaplansky [2].

COROLLARY. *If two C^*-algebras of completely continuous operators are simple, then their direct-product is simple too.*

REMARK. This corollary is not yet decided when the condition of "completely continuous operators" is replaced by "operators."

4. Some algebraic properties of $A_1 \times_\alpha A_2$.

THEOREM 4. *If A_i are C^*-algebra with unit, and if $A_1 \times_\alpha A_2$ is simple, in the sense of non-existence of proper two-sided closed ideals, then each A_i is simple.*

PROOF. We prove this theorem by an indirect argument. If A_1 were not simple, then there exists a proper closed ideal I_1. Then $I_1 \times_\alpha A_2$ is a closed proper ideal in $A_1 \times_\alpha A_2$, so by assumption, $I_1 \times_\alpha A_2 = A_1 \times_\alpha A_2$. Then for every positive real number ε, there exists $\sum_{i=1}^{n} x_i \times y_i \in I_1 \times A_2$ such that

$$\alpha\left(\sum_{i=1}^{n} x_i \times y_i - 1 \times 1\right) < \varepsilon.$$

Then, by the definition of $\alpha(\cdot)$ [7],

$$\left|\sum_{i=1}^{n} \varphi(x_i)\,\psi(y_i) - 1\right| < \varepsilon,$$

for any pure states φ and ψ on A_1 and A_2 respectively.

On the other hand, by I. E. Segal's result [6], there exists a pure state φ_0 of A_1 which vanishes on I_1. For a such state φ_0,

$$\varphi_0(x_i) = 0, \ i = 1, 2., \cdots n.$$

Then the above inequality implies the contradiction : $|1| < \varepsilon$.

THEOREM 5. *If A_i are C^*-algebras with unit, and if $A_1 \times_\alpha A_2$ is factorial (that is, the center is a multiple of unit), then each A_i is so.*

PROOF. If x is in the center of A_1, $x \times 1$ is in the center of $A_1 \times_\alpha A_2$, so by assumption $x \times 1 = a(1 \times 1) = (a1) \times 1$ for some scalar a. Thus $x = a1$, this is desired.

THEOREM 6. *If $A_1 = I + J$ (direct sum), then*

$$A_1 \times_\alpha A_2 = I \times_\alpha A_2 + J \times_\alpha A_2 \ (\textit{direct sum}).$$

PROOF. By assumption, if $x \in A_1$, there exists a unique expression

$$x = x' + x'', \ x' \in I, \ x'' \in J.$$

Then for any $y \in A_2$,

$$x \times y = x' \times y + x'' \times y, \ \ x' \times y \in I \times_\alpha A_2, \ \ x'' \times y \in J \times_\alpha A_2.$$

Thus $(I + J) \times_\alpha A_2 \subseteqq I \times_\alpha A_2 + J \times_\alpha A_2$ (right-hand side is not necessarily direct). On the other hand, since $I + J \supseteqq I, \ J$,

$$(I + J) \times_\alpha A_2 \supseteqq I \times_\alpha A_2, \ J \times_\alpha A_2,$$

so

$$(I + J) \times_\alpha A_2 \supseteqq I \times A_2 + J \times_\alpha A_2.$$

Thus, to prove the theorem it is sufficient to show

$$(I \times_\alpha A_2) \cap (J \times_\alpha A_2) = 0.$$

Now if $x' \in I$, $x'' \in J$, then for any y', $y'' \in A_2$,

$$(x' \times y')(x'' \times y'') = (x'x'') \times (y'y'') = 0 \times (y'y'') = 0.$$

Since $I \times_\alpha A_2$ and $J \times_\alpha A_2$ are generated from

$$\{x' \times y' \colon x' \in I, y' \in A_2\} \text{ and } \{x'' \times y'' : x'' \in J,\ y'' \in A_2\}$$

respectively, $(I \times_\alpha A_2)(J \times_\alpha A_2) = 0$, by the above relation.

On the other hand, $I \times_\alpha A_2$ and $J \times_\alpha A_2$ are closed two-sided ideals in C^*-algebra $A_1 \times_\alpha A_2$, so of course self-adjoint, then if $u \in I \times_\alpha A_2 \cap J \times_\alpha A_2$, $uu^* \in (I \times_\alpha A_2)(J \times_\alpha A_2) = 0$, that is $uu^* = 0$, $u = 0$. This completes the proof.

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