A NOTE ON THE PRINCIPAL GENUS THEOREM

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1. Let K be a normal extension of an algebraic number field k, with the Galois group G. We denote the different, the conductor, and the "Geschlechtermodul" of K/k by $\mathfrak{D}, \mathfrak{f}$, and $\mathfrak{F}$, respectively[1]. The author has recently proved the following theorem, which is a generalization of the principal genus theorem in the cyclic case (cf.[2]).

In an arbitrary abelian extension K/k, a necessary and sufficient condition for a system of ideals $\{\mathfrak{A}_\sigma, \sigma \in G\}$ in K to satisfy the condition

$$\mathfrak{A}_\tau^\sigma \mathfrak{A}_\sigma \mathfrak{A}_{\sigma\tau}^{-1} = (a_{\sigma,\tau}) \tag{1}$$

$$a_{\sigma,\tau} \in k, \ a_{\sigma,\tau} \equiv 1 \ mod. \ \mathfrak{m}\mathfrak{f} \tag{2}$$

is that there exists an ideal $\mathfrak{B}$ and numbers A_σ in K such that the following conditions are satisfied:

$$\mathfrak{A}_\sigma = \mathfrak{B}^{1-\sigma}(A_\sigma) \tag{3}$$

$$A_\sigma \equiv 1 \bmod \mathfrak{m}\mathfrak{F} \tag{4}$$

$$A_\tau^\sigma A_\sigma A_{\sigma\tau}^{-1} \in k, \ and \equiv 1 \bmod \mathfrak{m}\mathfrak{f}, \tag{5}$$

where $\mathfrak{m}$ is an arbitrary integral module in k.

The proof of this theorem is based on the cyclic case, and is obtained by means of the inductive method. It was so complicated and a more simple proof is desirable, especially by means of the cohomological method. Recently, Mr. H. Kuniyoshi and S. Takahashi have proved the following generalized theorem, the proof of which is based upon the cohomology theory (cf.[1]).

In an arbitrary normal extension K/k, let $\{\mathfrak{A}_\sigma, \sigma \in G\}$ be a system of ideals of K which satisfies the following conditions

$$\mathfrak{A}_\tau^\sigma \mathfrak{A}_{\sigma\tau}^{-1} \mathfrak{A}_\sigma = (A_{\sigma,\tau}) \tag{6}$$

$$A_{\sigma,\tau} \equiv 1 \bmod \mathfrak{f}\mathfrak{D} \tag{7}$$

$$\{A_{\sigma,\tau}\} \ is \ a \ factor \ set \ of \ K/k. \tag{8}$$

Then, there exists an ideal $\mathfrak{C}$ such that

$$\mathfrak{A}_\sigma = \mathfrak{C}^{1-\sigma}(C_\sigma) \tag{9}$$

where $C_\sigma \equiv 1 \bmod \mathfrak{f}$.

The proof of this theorem is based upon the cohomological method, and is

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1) Concerning the definition of $\mathfrak{f}, \mathfrak{D}$, c.f.[1]

simpler than ours, but this formulation is not a generalization of ours,[2] and the author could not immediately reduce our theorem to the above Kuniyoshi and Takahashi's theorem. Nevertheless, it can be shown that the following theorem, which is rougher than our previous formulation, is reduced to the Kuniyoshi and Takahashi's.

THEOREM. *In an arbitrary abelian extension let* $\{\mathfrak{A}_\sigma, \sigma \in G\}$ *be a system of ideals in* K *which satisfies the conditions* (1) *and* (2), *then there exists an ideal* $\mathfrak{B}$ *and numbers* A_σ *in* K *such that the conditions* (3) *and* (4) *are satisfied.*

It seems to be difficult to show that the numbers A_σ are selected so as to satisfy the condition (5) by this method, and we can not state the formulation as a necessary and sufficient condition for (1) and (2).

In this note, we shall show the reduction of the last theorem to the Kuniyoshi and Takahashi's.

2. Let U be an abelian group, and G be a finite abelian group which operates on U. Let us decompose the abelian group G into n cyclic subgroups G_i $(i=1, \cdots, n)$ of order m_i, a generator of which will be denoted by σ_i. An element of the group ring $[G]$ of G defines an endomorphism of U, and we denote the elements $1-\sigma_i$, $1+\sigma_i \cdots + \sigma_i^{m_i-1}$ by Δ_i and M_i respectively, and also denote by V the subgroup of elements in U invariant under the whole endomorphism from G. We now state a lemma which follows from Schreier's theory concerning the group extension.

LEMMA. *Let* $a_1, \cdots, a_n$ *be* n *elements in* V. *For any two elements* $\sigma = \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $\tau = \sigma_1^{\beta_1} \cdots \sigma_n^{\beta_n}$ *in* G, *define an element* $b_{\sigma,\tau}$ *in* V *by*

$$b_{\sigma,\tau} = a_1^{\left[\frac{\alpha_1+\beta_1}{m_1}\right]} \cdots a_n^{\left[\frac{\alpha_n+\beta_n}{m_n}\right]} \tag{10}$$

where the sign [ ] *means the integral part of a rational number. Then these elemeuts* $b_{\sigma,\tau}$ *constitute a factor set of* G *in* U, *and satisfy the condition*

$$b_{\sigma,\tau} = b_{\tau,\sigma}. \tag{11}$$

Now, as in the previous note (the reduction of the Theorem 1″ in [2]), we may reduce our theorem to the following:

If a system $\{\mathfrak{A}_i, i = 1, \cdots, n\}$ of ideals in K satisfies the following conditions

$$\mathfrak{A}_i^{M_i} = (a_i), \quad a_i \equiv 1 \bmod \mathfrak{m}\mathfrak{f}\mathfrak{d}, \quad a_i \in K \qquad (i = 1, \cdots, n) \tag{12}$$

$$\mathfrak{A}_j^{\Delta_i} = \mathfrak{A}_i^{\Delta_j} \qquad (i, j = 1, \cdots, n) \tag{13}$$

2) These two theorems differ in the following points;
1. concerning the modulus.
2. we does not assume that the numbers $a_{\sigma,\tau}$ in (1) constitute a factor set, and on the other hand we have to assume (8) in the latter.
3. concerning the condition (5).

for an integral module $\mathfrak{d}$ in k such that $\mathfrak{D} | \mathfrak{d}$, then there exists an ideal $\mathfrak{B}$ and numbers A_i in K such that

(14) $\qquad \mathfrak{A}_i = \mathfrak{B}^{\Delta_i} (A_i), \; A_i \equiv 1 \bmod \mathfrak{m}\mathfrak{f} \qquad (i = 1, \cdots, n)$

The reason of the reducibility is simpler than the previous note according to the lack of the condition (5), and we do not refer to its detail.

Now let us apply the above lemma to this case. As the groups U and V we consider the groups K^* and k^* respectively. We define a factor set $b_{\sigma,\tau}$ by (10) from n numbers $a_1, \cdots, a_n$ in (12) and also define a system of ideals $\{\mathfrak{B}_\sigma, \sigma \in G\}$ from n ideals $\mathfrak{A}_i$ in (12) by the following manner. That is, after arranging the cyclic groups G_i in fixed order, we define an ideal $\mathfrak{B}_\sigma$ for each element $\sigma = \sigma_{i_1}^{\alpha_1} \cdots\cdots \sigma_{i_r}^{\alpha_r}$ α^r $(0 < \alpha_i < m_i)$ by

$$\mathfrak{B}_1 = 1 \quad , \quad \mathfrak{B}_{\sigma_i} = \mathfrak{A}_i$$
$$\mathfrak{B}_\sigma = \mathfrak{B}_{\sigma 1}^{\sigma_{i1}} \mathfrak{B}_{\sigma_{i_1}} \quad (\text{where } \sigma = \sigma_{i_1}\sigma'),$$

Then it is shown from (10), (13) by a simple calculation that these ideals $\mathfrak{B}_\sigma$ and $b_{\sigma,\tau}$ satisfy the conditions (6), (8), and the condition (9) shows for $\sigma = \sigma_i$ our condition (14.)

Thus we can reduce our Theorem to the Kuniyoshi and Takahashi's.

REFERENCES

[1] H.Kuniyoshi and S.Takahashi ; On the principal genus theorem, in this same volume.

[2] F.Terada ; On the principal genus theorem concerning the abelian extensions. Tôhoku Mathematical Journal, vol. 4(1952).

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