CESÀRO SUMMABILITY OF FOURIER SERIES

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1. Introduction. Let $\varphi(t)$ be an even periodic function with Fourier series

(1.1) $$\varphi(t) \sim \sum_{n=0}^{\infty} a_n \cos nt, \qquad a_0 = 0.$$

The α-th integral of $\varphi(t)$ is defined by

(1.2) $$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \varphi(u)\,(t-u)^{\alpha-1} du \quad (\alpha > 0)$$

and the β-th Cesàro sum of (1.1) is defined by s_n^β $(\beta > -1)$.
Especially we put $s_n^0 = s_n$.

Some years ago we have conjectured that if

(1.3) $$\Phi_\beta(t) = o(t^\gamma) \qquad (t \to 0)$$

for $\gamma > \beta > 0$, then

(1.4) $$s_n = o(n^\alpha) \qquad (n \to \infty)$$

for $\alpha = \beta/(\gamma - \beta + 1)$, and proved that this is valid for $0 < \alpha \leqq 1$. See Izumi-Sunouchi [3], Sunouchi [5] and Wang [6]. One of the object of this note is to master this problem thoroughly.

On the other hand Prof. Izumi [2] has proved that if

(1.5) $$s_n^\beta = O(n^\gamma) \qquad (n \to \infty)$$

for $\beta > \gamma > 0$, then

(1.6) $$\Phi_\alpha(t) = o(t^\alpha) \qquad (t \to 0)$$

for $\alpha = (\beta + 1)/(\beta - \gamma - 1)$. If we add to (1.5) a Tauberian condition

(1.7) $$a_n = O(n^{-(1-\delta)}) \qquad (n \to \infty)$$

for $0 < \delta < 1$, then we may expect

$$\Phi_\alpha(t) = o(t^\alpha) \qquad (t \to 0)$$

for $\alpha = \delta(\beta + 1)/(\beta - \gamma + \delta)$. (cf. Sunouchi [5]) The case β = integer was considered by Loo [4]. The case $\beta = 1$ and $-1 < \gamma < 0$ was proved by Chandrasekharan and Szász [1] and S. Izumi [3] proved general case under the restriction $\beta \leqq 1$ or $\delta \leqq 2(\beta-\gamma)/(\beta-1)$. In this note we shall prove general case under a weaker Tauberian condition

(1.8) $$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-(1-\delta)}).$$

(1.5) and so called one-side condition imply (1.8).

The method of proof is a slight modificatoin of Izumi's method. Especially we use Bessel summability instead of Cesàro summability. These two methods of summability are equivalent, and Bessel summability behaves more adequately at the neighborhood of infinity than Cesàro summability.

2. Cesàro summability of Fourier series. Let $J_\mu(t)$ denote the Bessel function of order μ, and put

(2.1) $$\alpha_\mu(t) = J_\mu(t)/t^\mu$$

(2.2) $$V_{1+\mu}(t) = \alpha_{\mu+\frac{1}{2}}(t),$$

then $$\overset{(k)}{V}_{1+\mu}(t) = O(1) \quad \text{as } t \to 0 \text{ and}$$

(2.3) $$\overset{(k)}{V}_{1+\mu}(t) = O(t^{-(\mu+1)}) \qquad \text{as } t \to \infty, \text{ for } k = 0, 1, 2, \cdots.$$

If we denote by σ_ω^a the α-th Bessel mean of the Fourier series (1.1), then

(2.4) $$\sigma_\omega^a = K\omega \int_0^\infty \varphi(t) V_{1+a}(\omega t)dt.$$

THEOREM 1. *If* $0 < \beta < \gamma$ *and*

(2.5) $$\Phi_\beta(t) = o(t^\gamma),$$

then the Fourier series of $\varphi(t)$ *is summable* $(C, \beta/(\gamma-\beta-1))$ *to zero at* $t = 0$.

PROOF. Put $\alpha = \beta/(\gamma-\beta+1) < \beta$ and $\rho = \alpha/(1+\alpha) < 1$. Neglecting the constant factor the equivalent Bessel mean is

$$\sigma_\omega^a = \int_0^\infty \omega\varphi(t) V_{1+a}(\omega t)dt$$

(2.6) $$= \left(\int_0^{C\omega^{-\rho}} + \int_{C\omega^{-\rho}}^\infty\right) \omega\varphi(t) V_{1+a}(\omega t)dt$$

$$= I + J,$$

say, where C is a fixed large constant. Concerning J,

$$J = O\left(\int_{C\omega^{-\rho}}^\infty \omega(\omega t)^{-(1+a)} |\varphi(t)| \, dt\right)$$

(2.7) $$= O\left(\omega^{-a} \int_{C\omega^{-\rho}}^\infty t^{-(1+a)} |\varphi(t)| \, dt\right)$$

$$= O\left\{\omega^{-a} C^{-(1+a)} \left(\omega^{\rho(1+a)} + \sum_{m=1}^\infty m^{-(1+a)}\right) \int_0^{2\pi} |\varphi(t)| \, dt\right\}$$

$$= O\{C^{-(1+a)} \omega^{-a+\rho(1+a)} + O(\omega^{-1})(\omega^{-a})\} = O(C^{-(1+a)}) \leqq \varepsilon,$$

for large C since $\rho = \alpha/(1+\alpha)$.

Now there is an integer $k>1$ such that $k-1<\beta \leqq k$. We suppose that $k-1 < \beta < k$, for the case $\beta = k$ can be easily deduced by the following argument. As we have already seen,

(2.8) $$\sigma_\omega^a = \int_0^{C\omega^{-\rho}} \omega\varphi(t) V_{1+a}(\omega t)dt + o(1).$$

By k-times applications of integration by parts, the last integral I becomes

$$I = \sum_{h=1}^k (-1)^h \left[\omega^h \Phi_h(t) V_{1+a}^{(h-1)}(\omega t)\right]_0^{C\omega^{-\rho}} + (-1)^k \omega^{k+1} \int_0^{C\omega^{-\rho}} \Phi_k(t) V_{1+a}^{(k)}(\omega t)dt$$

(2.9) $$= \sum_{h=1}^k (-1)^{h-1} I_h + (-1)^k I_{k+1}, \text{ say.}$$

Since $\Phi_1(t) = o(1)$ and $\Phi_\beta(t) = o(t^\gamma)$, applying M. Riesz's convexity theorem we have

$$\Phi_1(t) = o(1),\ \Phi_2(t) = o(t^{\gamma/(\beta-1)}),\cdots,\ \Phi_h(t) = o(t^{(h-1)\gamma/(\beta-1)}),\cdots$$
$$\cdots,\Phi_{k-1}(t) = o(t^{(k-2)\gamma/(\beta-1)}) \text{ and } \Phi_k(t) = o(t^{k+\gamma-\beta}).$$

Therefore we have

$$\begin{aligned} I_1 &= \Big[\omega\Phi_1(t)V_{1+\alpha}(\omega t)\Big]_0^{C\omega^{-\rho}} \\ &= O(\omega\omega^{-(1+\alpha)}C^{-(1+\alpha)}\omega^{\rho(1+\alpha)}) = O(C^{-(1+\alpha)}\omega^{-\alpha+(1+\alpha)\rho}) \qquad (2.10) \\ &= O(C^{-(1+\alpha)}) \leqq \varepsilon, \end{aligned}$$

and, for $h = 2,3,\cdot\cdot,k-1$,

$$\begin{aligned} I_h &= \Big[\omega^k\Phi_h(t)V_{1+\alpha}^{(h-1)}(\omega t)\Big]_0^{C\omega-\rho} \\ &= O(\omega^h C^{(h-1)\gamma/(\beta-1)}\ \omega^{-\rho(h-1)\gamma/(\beta-1)}\ \omega^{-(1+\alpha)}\ C^{-(1+\alpha)}\omega^{\rho(1+\alpha)}) \end{aligned}$$

by (2.3), Since $\rho = \alpha/(1+\alpha)$ the exponent of ω of the last formula is

$$h - \rho(h-1)\gamma/(\beta-1) - (1+\alpha) + \rho(1+\alpha)$$

$$= h-1-\rho(h-1)\gamma/(\beta-1) = \frac{h-1}{\beta-1}\Big\{(\beta-1)-\rho\gamma\Big\}$$

$$= \frac{h-1}{\beta-1}\Big\{(\beta-1) - \frac{\alpha}{1+\alpha}\gamma\Big\} = \frac{(h-1)}{(\beta-1)(1+\alpha)}\Big\{(1+\alpha)(\beta-1)-\alpha\gamma\Big\}$$

$$= \frac{(h-1)}{(\beta-1)(1+\alpha)}\Big\{\beta-1-\alpha(1+\gamma-\beta)\Big\} = \frac{h-1}{1+\alpha}\Big\{1 - \frac{\alpha(1+\gamma-\beta)}{\beta-1}\Big\} < 0,$$

for $\alpha = \beta/(1+\gamma-\beta)$, and these terms appear for $\beta > 1$. Thus we have

(2.11) $I_h = o(1)$, as $\omega\to\infty$, for $h = 2,3,\cdots,k-1$.

Concerning I_k,

$$\begin{aligned} I_k &= \Big[\omega^k\Phi_k(t)\ V_{1+\alpha}^{(k-1)}(\omega t)\Big]_0^{C\omega^{-\rho}} \\ &= O(\omega^k\omega^{-\rho(k+\gamma-\beta)}\omega^{-(1+\alpha)}\omega^{\rho(1+\alpha)}). \end{aligned}$$

The exponent of ω is

$$k - \rho(k+\gamma-\rho) - (1+\alpha) + \rho(1+\alpha)$$

$$= k-1-\rho(k+\gamma-\beta) = k-1 - \frac{\alpha}{1+\alpha}(k+\gamma-\beta)$$

$$= \frac{1}{1+\alpha}\Big\{(1+\alpha)(k-1) - \alpha(k+\gamma-\beta)\Big\}$$

$$= \frac{1}{1+\alpha}\Big\{k-1-\alpha(1+\gamma-\beta)\Big\} = \frac{k-1-\beta}{1+\alpha} < 0.$$

Therefore

(2.12) $I_k = o(1)$, as $\omega\to\infty$.

Concerning I_{k+1}, we split up three parts,

$$I_{k+1} = \omega^{k+1} \int_0^{C\omega^{-\rho}} \Phi_k(t)\ V_{1+a}^{(k)}(\omega t)\ dt$$

$$(2.13) \qquad = \int_0^{C\omega^{-\rho}} \omega^{k+1} V_{1+a}^{(k)}(\omega t)\ dt \int_0^t \Phi_\beta(u)(t-u)^{k-\beta-1}\ du$$

$$= \int_0^{C\omega^{-\rho}} du \int_u^{u+\omega^{-1}} dt + \int_0^{C\omega^{-\rho}-\omega^{-1}} du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} dt - \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} du \int_{C\omega^{-\rho}}^{u+\omega^{-1}} dt$$

$$= K_1 + K_2 - K_3,$$

say. Let K_1 split in two parts

$$(2.14) \qquad K_1 = \int_0^{\omega^{-1}} du \int_u^{u+\omega^{-1}} dt + \int_{\omega^{-1}}^{C\omega^{-\rho}} du \int_u^{u+\omega^{-1}} dt$$

$$= L_1 + L_2.$$

Since $V_{1+a}^{(k)}(t) = O(1)$ for $0 \leqq t \leqq 1$,

$$L_1 = \omega^{k+1} \int_0^{\omega^{-1}} \Phi_\beta(u) du \int_u^{u+\omega^{-1}} V_{1+a}^{(k)}(\omega t)(t-u)^{k-\beta-1} dt$$

$$= O\{\omega^{k+1} \int_0^{\omega^{-1}} \Phi_\beta(u)\ du \int_u^{u+\omega^{-1}} (t-u)^{k-\beta-1} dt\}$$

$$= o\{\omega^{k+1} \int_0^{\omega^{-1}} u^{\gamma} \Big[(t-u)^{k-\beta}\Big]_u^{u+\omega^{-1}} du\}$$

$$= o(\omega^{k+1} \int_0^{\omega^{-1}} u^{\gamma} \omega^{-(k-\beta)}\ du)$$

$$(2.15) \qquad = o(\omega^{\beta+1}\ [u^{\gamma+1}]_0^{\omega^{-1}}) = o(\omega^{\beta-\gamma}) = o(1), \quad \text{for } \gamma > \beta.$$

$$L_2 = \omega^{k+1} \int_{\omega^{-1}}^{C\omega^{-\rho}} \Phi_\beta(u)\ du \int_u^{u+\omega^{-1}} V_{1+\gamma}^{(k)}(\omega t)(t-u)^{k-\beta-1} dt$$

$$= o\{\omega^{k+1} \int_{\omega^{-1}}^{C\omega^{-\rho}} u^{\gamma} du \int_u^{u+\omega^{-1}} (\omega t)^{-(1+\alpha)}\ (t-u)^{k-\beta-1} dt\}$$

$$= o\{\omega^{k-\alpha} \int_{\omega^{-1}}^{C\omega^{-\rho}} u^{\gamma}\ u^{-(1+\alpha)} du \int_u^{u+\omega^{-1}} (t-u)^{k-\beta-1} dt\}$$

$$= o\{\omega^{k-\alpha} \int_{\omega^{-1}}^{C\omega^{-\rho}} u^{\gamma-(1+\alpha)} du \Big[(t-u)^{k-\beta}\Big]_u^{u+\omega^{-1}}\}$$

$$= o\{\omega^{k-\alpha}\ \omega^{-(k-\beta)} \Big[u^{\gamma-\alpha}\Big]_{\omega^{-1}}^{C\omega^{-\rho}}\}$$

$$= o(\omega^{\beta-\alpha}\ \omega^{-\rho(\gamma-\alpha)}), \quad \text{for } \gamma - \alpha > 0.$$

Since

$$\beta-\alpha-\rho(\gamma-\alpha) = \beta-\alpha - \frac{\alpha}{1+\alpha}(\gamma-\alpha)$$

$$= \frac{1}{1+\alpha}\{\beta - \alpha(1-\beta+\gamma)\} = 0,$$

we have

(2.16) $$L_2 = o(1) \quad \text{as } \omega \to \infty.$$

Concerning K_2, if we use integration by parts in the inner integral, then

$$K_2 = \omega^{k+1}\int_0^{C\omega^{-\rho}-\omega^{-1}} \Phi_\beta(u)\, du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} V_{1+\alpha}^{(k)}(\omega t)\,(t-u)^{k-\beta-1}\, dt$$

(2.17) $$= \omega^{k+1}\int_0^{\omega^{-\rho}-\omega^{-1}} \Phi_\beta(u)\, du \left\{ \left[\omega^{-1} V_{1+\alpha}^{(k-1)}(\omega t)(t-u)^{k-\beta-1}\right]_{u+\omega^{-1}}^{c\omega^{-\rho}} - (k-\beta-1)\int_{u+\omega^{-1}}^{C\omega^{-\rho}} \omega^{-1} V_{1+\alpha}^{(k-1)}(\omega t)(t-u)^{k-\beta-2}\, dt\right\}$$

$$= M_1 - (k-\beta-1)M_2,$$

say. Then

(2.18) $$M_1 = \omega^{k+1}\int_0^{C\omega^{-\rho}-\omega^{-1}} \Phi_\beta(u)\, du \,\{\omega^{-1}\omega^{-(1+\alpha)(1-\rho)}(C\omega^{-\rho}-u)^{k-\beta-1} - \omega^{-1}\omega^{-(1+\alpha)}(u+\omega^{-1})^{-(1+\alpha)}\omega^{-(k-\beta-1)}\}$$

$$= N_1 + N_2,$$

(2.19) $$N_1 = o\left(\omega^{k+(1+\alpha)(\rho-1)}\int_0^{C\omega^{-\rho}} u^\gamma (C\omega^{-\rho}-u)^{k-\beta-1}\, du\right)$$

$$= o\left(\omega^{k+(1+\alpha)(\rho-1)}\int_0^{C\omega^{-\rho}} u^\gamma (C\omega^{-\rho}-u)^{k-\beta-1} du\right)$$

$$= o\left(\omega^{k+(1+\alpha)(\rho-1)}\left[u^{\gamma+k-\beta}\right]_0^{C\omega^{-\rho}}\right)$$

$$= o\left(\omega^{k+(1+\alpha)(\rho-1)-\rho(\gamma+k-\beta)}\right)$$

Since the exponent of ω is

$$k + (1+\alpha)\left(\frac{\alpha}{1+\alpha} - 1\right) - \frac{\alpha}{1+\alpha}(\gamma + k - \beta)$$

$$= \frac{1}{1+\alpha}\{k(1+\alpha) - (1+\alpha) - \alpha(\gamma+k-\beta)\}$$

$$= \frac{1}{1+\alpha}\{k - 1 - \alpha(1+\gamma-\beta)\} = \frac{1}{1+\alpha}(k-1-\beta) < 0,$$

(2.20) $$N_1 = o(1), \quad \text{as } \omega \to \infty.$$

(2.12) $$N_2 = o\left(\omega^{k-(1+\alpha)-(k-\beta-1)}\int_0^{C\omega^{-\rho}-\omega^{-1}} u^\gamma (u+\omega^{-1})^{-(1+\alpha)}\, du\right)$$

$$= o\left(\omega^{\beta-\alpha}\int_0^{C\omega^{-\rho}} u^{\gamma-(1+\alpha)}\, du\right)$$

$$= o\left(\omega^{\beta-\rho}\omega^{-(\gamma-\alpha)}\right) = o(1).$$

Similar estimations give

$$
\begin{aligned}
M_2 &= \omega^k \int_0^{C\omega^{-\rho}-\omega^{-1}} \Phi_\beta(u)\,du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} V_{1+a}^{(k-1)}(\omega t)(t-u)^{k-\beta-2}\,dt \\
&= o\{\omega^k \int_0^{C\omega^{-\rho}-\omega^{-1}} u^\gamma\,du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} \omega^{-(1+a)}\,t^{-(1+a)}\,(t-u)^{k-\beta-2}\,dt\} \\
&= o\{\omega^{k-1-a} \int_0^{C\omega^{-\rho}-\omega^{-1}} u^\gamma\,u^{-(1+a)}\,du \int_{u+\omega^{-1}}^{C\omega^{-\rho}} (t-u)^{k-\beta-2}\,dt\} \\
&= o\{\omega^{k-1-a} \int_0^{C\omega^{-\rho}} u^{\gamma-(1+a)}\,du \Big[(t-u)^{k-\beta-1}\Big]_{u+\omega^{-1}}^{C\omega^{-\rho}}\} \qquad (2.22)\\
&= o\{\omega^{k-1-a} \int_0^{C\omega^{-\rho}} u^{\gamma-(1+a)}\,\omega^{-(k-\beta-1)}\,du\} \\
&= o\{\omega^{k-1-a-(k-\beta-1)} \Big[u^{\gamma-a}\Big]_0^{C\omega^{-\rho}}\} \\
&= o(\omega^{\beta-a-\rho(\gamma-a)}) \\
&= o(1), \quad \text{as } \omega\to\infty.
\end{aligned}
$$

We have easily

$$
\begin{aligned}
K_3 &= \omega^{k+1} \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} \Phi_\beta(u)\,du \int_{C\omega^{-\rho}}^{u+\omega^{-1}} V_{1+a}^{(k)}(\omega t)(t-u)^{k-\beta-1}dt \\
&= \omega^{k+1} \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} \Phi_\beta(u)\,du \int_{C\omega^{-\rho}}^{u+\omega^{-1}} \omega^{-(1+a)}\,t^{-(1+a)}(t-u)^{k-\beta-1}dt \\
&= \omega^{k-a} \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} \Phi_\beta(u)\,du\ \omega^{\rho(1+a)} \int_{C\omega^{-\rho}}^{u+\omega^{-1}} (t-u)^{k-\beta-1}dt \\
&= \omega^{k-a-\rho(1+a)} \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} \Phi_\beta(u)\,du \Big[(t-u)^{k-\beta}\Big]_{C\omega^{-\rho}}^{u+\omega^{-1}} \qquad (2.23)\\
&= o\{\omega^k\omega^{-(k-\beta)} \int_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}} u^\gamma du\} \\
&= o\{\omega^\beta \Big[u^{\gamma+1}\Big]_{C\omega^{-\rho}-\omega^{-1}}^{C\omega^{-\rho}}\} \\
&= o(\omega^\beta\omega^{-\rho(\gamma+1)}) = o(\omega^{\beta-\rho(\gamma+1)}) = o(1),
\end{aligned}
$$

for

$$
\beta-\rho(\gamma+1) = \beta - \frac{\alpha}{1+\alpha}(\gamma+1) = \frac{1}{1+\alpha}(\beta+\alpha\beta-\alpha\gamma-\alpha)
$$

$$
= \frac{1}{1+\alpha}\{\beta-\alpha\,(1+\gamma-\beta)\} = 0.
$$

Summing up (2.7), (2.10), (2.11), (2.12), (2.15), (2.16), (2.20), (2.21), (2.22) and (2.23) we have

$$\sigma_\omega^\alpha = o(1)$$

which is the required.

3. Converse problem.

THEOREM 2. *If*

(3.1) $$s_n^\beta = o(n^\gamma),\ (n\to\infty)$$

for $\beta > \gamma > -1,\ 1+\gamma > \delta$, *and*

(3.2) $$\sum_{\nu=n}^{\infty} |a_\nu|/\nu = O(n^{-(1-\delta)}),\quad (n\to\infty)$$

for $0<\delta<1$, *then*

(3.3) $$\Phi_\alpha(t) = o(t^\alpha),\quad (t\to 0)$$

for $\alpha = \delta(\beta+1)/(\beta-\gamma+\delta)$.

We need the following lemma.

LEMMA 1. *If* $2 \geqq \alpha > 0$ *and* $\beta \geqq 0$, *then*

(3.4) $$\int_0^t u^\beta \cos nu\,(t-u)^{\alpha-1}\,du = O(t^\beta/n^\alpha)$$

PROOF. *If* $\beta = 0$

$$\int_0^t \cos nu\,(t-u)^{\alpha-1}du = O(n^{-\alpha}),$$

which is proved easily as Young's function. For $\beta>0$, using the second mean value theorem,

$$\begin{aligned}
&\int_0^t u^\beta \cos nu(t-u)^{\alpha-1}\,du \\
&= t^\beta \int_h^t \cos nu(t-u)^{\alpha-1}du \qquad (0<h<t) \\
&= t^\beta \left\{ \int_0^t \cos nu(t-u)^{\alpha-1}\,du - \int_0^h \cos nu(t-u)^{\alpha-1}\,du \right\} \\
&\leqq t^\beta \left\{ \left| \int_0^t \cos nu(t-u)^{\alpha-1}\,du \right| + \max_{0\leqq\tau\leqq t} \left| \int_0^\tau \cos nu(\tau-u)^{\alpha-1}du \right| \right\} \\
&= O(t^\beta/n^\alpha).
\end{aligned}$$

Proof of the theorem for $0 \leqq \alpha \leqq 2$. We begin with the case $-1<\beta<0$.

(3.5) $$\begin{aligned}
\Gamma(\alpha)\,\Phi_\alpha(t) &= \sum_{n=0}^{\infty} a_n \int_0^t \cos nu(t-u)^{\alpha-1}\,du \\
&= \sum_{n=0}^{M} + \sum_{n=M+1}^{\infty} = I + J,
\end{aligned}$$

say, where $M = [Ct^{-1/(1+\gamma-\delta)}]$ for a fixed large C. Since $1+\gamma>\delta$, M is determined exactly. By the well known formula

(3.6) $$a_n = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{\beta+1}{n-\nu} s_\nu^\beta,$$

we have

$$I = \sum_{n=0}^{M} a_n \int_0^t \cos nu(t-u)^{\alpha-1} du$$

$$= \sum_{\nu=0}^{M} s_\nu^\beta \int_0^t \left\{ \sum_{\nu=0}^{M} (-1)^{n-\nu} \binom{\beta+1}{n-\nu} \cos nu \right\} (t-u)^{\alpha-1} du$$

$$= \sum_{\nu=0}^{M} s_\nu^\beta \int_0^t \Big[2^{\beta+1} \left(\sin\frac{u}{2}\right)^{\beta+1} \cos\left\{ \left(\frac{\beta+1}{2} + \nu\right) u + \frac{(\beta+1)\pi}{2} \right\}$$

$$- \sum_{m=M-\nu+1}^{\infty} (-1)^m \binom{\beta+1}{m} \cos (m+\nu)u \Big] (t-u)^{\alpha-1} du$$

$$= I_1 - I_2,$$

say. From Lemma 1,

$$I_1 = \sum_{\nu=0}^{M} o(\nu^\gamma)\, (t^{\beta+1}/\nu^\alpha) = o(t^{\beta+1} M^{\gamma-\alpha+1}) \cdot o(t^\alpha t^{\beta+1-\alpha} M^{\gamma-\alpha+1}) = o(t^\alpha).$$

(3.8) $$I_2 = \sum_{\nu=0}^{M} s_\nu^\beta \int_0^t \sum_{n=M-\nu+1}^{\infty} (-1)^m \binom{\beta+1}{m} \cos (m+\nu)u(t-u)^{\alpha-1} du$$

$$= \sum_{\nu=0}^{M} o(\nu^\gamma) \sum_{m=M-\nu+1}^{\infty} \frac{1}{m^{\beta+2}(m+\nu)^\alpha}.$$

Since $\beta < 0$,

(3.9) $$I_2 = o\Big(\sum_{\nu=0}^{M} \nu^\gamma \frac{1}{M^\alpha (M-\nu+1)^{\beta+1}}\Big) = o(M^{-\alpha-\beta+\gamma})$$

$$= o\big(t^{\frac{\beta+1-\alpha}{\gamma+1-\alpha}(\alpha+\beta-\gamma)}\big) = o(t^\alpha)$$

for $\alpha < \frac{\beta+1-\alpha}{\gamma+1-\alpha}(\alpha+\beta-\gamma)$, which is reduced to $0 < (\beta-\gamma)(1+\beta)$.

If $\alpha \geqq 1$,

$$J = \sum_{n=M+1}^{\infty} a_n \int_0^t \cos nu \cdot (t-u)^{\alpha-1}\, du$$

(3.10) $$\leqq \sum_{n=M+1}^{\infty} \left| \frac{a_n}{n^\alpha} \right| = \sum_{n=M+1}^{\infty} \left| \frac{a_n}{n} \right| n^{1-\alpha}$$

$$= O(M^{1-\alpha} M^{-1+\delta}) = O(M^{-\alpha+\delta})$$

$$= O(C^{-(\alpha-\delta)} t^\alpha) \leqq \varepsilon t^\alpha,$$

for $\alpha - \delta = \alpha(1+\gamma-\delta) > 0$.

If $\alpha < 1$, we choose ε such as $\alpha > \varepsilon > \delta$. Let us put

$$\sum_{\nu=m}^{\infty} |a_\nu| / \nu = r_n, \quad |a_n| = n(r_n - r_{n-1}),$$

then

$$\sum_{\nu=m}^{n} \frac{|a_\nu|}{\nu^\varepsilon} = \sum_{\nu=m}^{n} \nu^{1-\varepsilon}(r_\nu - r_{\nu-1})$$

$$= o(1) + \sum_{\nu=m}^{n} n^{-\varepsilon-1+\delta} = o(m^{-\varepsilon+\delta}).$$

Thus we have

$$J \leqq \sum_{n=M+1}^{\infty} \frac{|a_n|}{n^\alpha} = \sum_{n=M+1}^{\infty} \frac{|a_n|}{n^\varepsilon} n^{\varepsilon-\alpha} = o(M^{\varepsilon-\alpha} M^{-\varepsilon+\delta}) = o(M^{-\alpha+\delta}) \tag{3.11}$$

$$\leqq \varepsilon t^\alpha.$$

From (3.8), (3.9) and (3.10) or (3.11), we get the required.

Let us now consider $0 < \beta < 1$. if we choose $M = [Ct^{-1/(1+\gamma-\delta)}]$ then

$$I = \sum_{n=0}^{M} a_n \int_0^t \cos nu\,(t-u)^{\alpha-1} du$$

$$= \sum_{n=0}^{M-1} s_n \int_0^t \Delta\cos nu(t-u)^{\alpha-1} du + s_M \int_0^t \cos Mu(t-u)^{\alpha-1} du$$

$$= K + L,$$

say. By the formula

$$s_n = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{\beta}{n-\nu} s_\nu^\beta,$$

$$K = \sum_{\nu=0}^{M} s_\nu^\beta \int_0^t \left\{ \sum_{n=\nu}^{M} (-1)^{n-\nu} \binom{\beta}{n-\nu} \sin\left(n+\frac{1}{2}\right) u \, \sin\frac{u}{2} \right\} (t-u)^{\alpha-1} du$$

$$= \sum_{\nu=0}^{M} s_\nu^\beta \int_0^t \left[2^{\beta+1} \left(\sin\frac{u}{2}\right)^{\beta+1} \sin\left\{ \left(\nu + \frac{\beta+1}{2}\right) u + \frac{(\beta+1)}{2}\pi \right\} \right.$$

$$\left. - \sum_{n=M-\nu}^{\infty} (-1)^m \binom{\beta}{m} \sin\frac{u}{2} \sin(m+\nu)u \right] (t-u)^{\alpha-1} du$$

$$= K_1 - K_2,$$

$$K_1 = \sum_{\nu=0}^{M-1} o(\nu^\gamma)(t^{\beta+1}/\nu^\alpha) = o(t^{\beta+1} M^{\gamma-\alpha+1})\, o(t^\alpha M^{\gamma-\alpha+1} t^{\beta-\alpha+1}) = o(t^\alpha) \tag{3.12}$$

and

$$K_2 = \sum_{\nu=0}^{M-1} s_\nu^\beta \int_0^t \sum_{m=M-\nu}^{\infty} (-1)^m \binom{\beta}{m} \sin\frac{u}{2} \sin(m+\nu)u(t-u)^{\alpha-1} du$$

$$= o\Big(\sum_{\nu=0}^{M}\nu^{\gamma}\sum_{m=M-\nu+1}^{\infty}\frac{t}{m^{\beta+1}(m+\nu)^{\alpha}}\Big) = O\Big(\frac{1}{M^{\alpha}}\sum_{\nu=0}^{M}\nu^{\gamma}t(M-\nu+1)^{-\beta}\Big)$$

$$= o(tM^{\gamma+1-\alpha-\beta}) \ ,$$

for $0<\beta<1$.

Since $\alpha-1<(\beta+1-\alpha)(\alpha+\beta-\gamma-1)/(\gamma+1-\alpha)$, which is reduced to

$$0<\beta(\beta-\alpha),$$

we have

$$K_2 = o(t^{\alpha}). \tag{3.15}$$

Since we can easily get

$$s_n = O(n^{\delta}),$$

from (3.2),

$$\begin{aligned} L &= s_M\int_0^t \cos Mu(t-u)^{\alpha-1}\,du \\ &= O(M^{\delta}M^{-\alpha}) = O(M^{-(\alpha-\delta)}) = O(C^{-(\alpha-\delta)}t^{\alpha}) \\ &\leqq \varepsilon\, t^{\alpha}. \end{aligned} \tag{3.16}$$

$|J|\leqq \varepsilon t^{\alpha}$ is proved analogously. The general case $n<\beta<n+1$ $(n=1,2,\cdots)$ may be proved by n-times applications of Abel's lemma. The case β = integer is proved easily.

If $\alpha>2$, we can not get

$$\int_0^t \cos nu(t-u)^{\alpha-1}du = O(n^{-\alpha}).$$

Therefore we take the integral

$$\int_0^t \cos nu(t^2-u^2)^{\alpha-1}du.$$

If we put

$$\varphi_\alpha(t) = \frac{1}{\Gamma(\alpha)\,t^{\alpha}}\int_0^t (t-u)^{\alpha-1}\varphi(u)du, \quad \alpha>0$$

$$\varphi_\alpha^*(t) = \frac{2\Gamma(\alpha+1/2)}{\Gamma(1/2)\Gamma(\alpha)}\frac{1}{t^{2\alpha-1}}\int_0^t (t^2-u^2)^{\alpha-1}\varphi(u)du, \ \alpha>0,$$

then, Chandrasekharan and Szász [1] proved that

$\varphi_\alpha(t)\to l$ is equivalent to $\varphi_\alpha^*(t)\to l$ as $t\to 0$.

$$\begin{aligned} \varphi_\alpha^*(t) &= K_\alpha\frac{1}{t^{2\alpha-1}}\int_0^t\Big(\sum_{n=0}^{\infty}a_n\cos nu\Big)(t^2-u^2)^{\alpha-1}du \\ &= \frac{K_\alpha}{t^{2\alpha-1}}\sum_{n=0}^{\infty}a_n\int_0^t (t^2-u^2)^{\alpha-1}\cos nu\,du \end{aligned} \tag{3.17}$$

and

$$\frac{1}{t^{2\alpha-1}}\int_0^t (t^2-u^2)^{\alpha-1}\cos nu\,du = \alpha_\alpha(nt),$$

where $\alpha_\alpha(t)$ has been defined by (2.1) and (2.2). (cf. Chandrasekharan and Szász [1]) From (2.3),

$$\int_0^t (t^2-u^2)^{\alpha-1} \cos nu\, du = O(n^{-\alpha}t^{\alpha-1}). \tag{3.18}$$

LEMMA 2. *If* $\alpha \geqq 1$ and $\beta \geqq 0$,

$$\int_0^t u^\beta (t^2-u^2)^{\alpha-1} \cos nu = O(t^{\alpha+\beta-1}n^{-\alpha}). \tag{3.19}$$

PROOF. The case $\beta = 0$ is mentioned above. For $\beta > 0$

$$\int_0^t u^\beta \cos nu (t^2-u^2)^{\alpha-1} du$$

$$= t^\beta \int_h^t \cos nu (t^2-u^2)^{\alpha-1} du \qquad (0 < h \leqq t)$$

$$= t^\beta \left\{ \int_0^t \cos nu (t^2-u^2)^{\alpha-1} du - \int_0^h \cos nu (t^2-u^2)^{\alpha-1} du \right\}$$

$$\leqq t^\beta \left\{ \left| \int_0^t \cos nu (t^2-u^2)^{\alpha-1} du \right| + \max_{0 \leqq \tau \leqq t} \left| \int_0^\tau \cos nu (\tau^2-u^2)^{\alpha-1} du \right| \right\}$$

$$= o(t^\beta) \{ n^{-\alpha}t^{\alpha-1} + \max_{0 \leqq \tau \leqq t} (n^{-\alpha}\tau^{\alpha-1}) \}$$

$$= O(t^{\alpha+\beta-1}n^{-\alpha})$$

for $\alpha > 1$.

Proof of the theorem for $\alpha > 1$. Let us put

$$\Phi_\alpha^*(t) = \sum_{n=0}^\infty a_n \int_0^t \cos nu (t^2-u^2)^{\alpha-1} du$$

$$= \sum_{n=0}^M + \sum_{n=M+1}^\infty = I + J,$$

where $M = [Ct^{-1/(1+r-\delta)}]$.

From (3.19), we get

$$J = \sum_{n=M+1}^\infty a_n \int_0^t \cos nu (t^2-u^2)^{\alpha-1} du$$

$$= \sum_{n=M+1}^\infty \left| \frac{a_n}{n} \right| n\, O(n^{-\alpha}t^{\alpha-1})$$

$$= O\left\{ t^{\alpha-1} M^{1-\alpha} \sum_{n=M+1}^\infty \left| \frac{a_n}{n} \right| \right\} \tag{3.20}$$

$$= O(t^{\alpha-1}M^{1-\alpha}M^{-(1-\delta)}) = O(t^{\alpha-1}M^{-\alpha+\delta})$$

$$\leqq C^{-(\alpha-\delta)}\, t^{2\alpha-1} \leqq \varepsilon t^{2\alpha-1},$$

for $\alpha - \delta > 0$.

If $0 < \beta < 1$, Applying Ahel's Lemma,

$$I = \sum_{n=0}^M a_n \int_0^t \cos nu (t^2-u^2)^{\alpha-1} du$$

$$= \sum_{n=0}^{M-1} s_n \int_0^t \Delta \cos nu (t^2-u^2)^{\alpha-1} du$$

$$+ s_M \int_0^t \cos Mu\cdot(t^2-u^2)^{a-1}\,du$$
$$= K + L,$$

say.

(3.21)
$$|L| = O(M^\beta C^{-(a-\delta)} t^{a-1} M^{-a})$$
$$= O(C^{-(a-\delta)} t^{2a-1}) \leqq \varepsilon' t^{2a-1}$$

From the formula

$$s_n = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{\beta}{n-\nu} s_\nu^\beta,$$

$$K = \sum_{n=0}^{M-1} \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{\beta}{n-\nu} s_\nu^\beta \int_0^t \Delta \cos nu (t^2-u^2)^{a-1}\,du$$
$$= \sum_{n=0}^{M-1} s_\nu^\beta \int_0^t \left\{ \sum_{n=0}^{M-1} (-1)^{n-\nu} \binom{\beta}{n-\nu} \Delta \cos nu (t^2-u^2)^{a-1} \right\} du$$
$$= \sum_{\nu=0}^{M-1} s_\nu^\beta \int_0^t \left\{ \sum_{n=\nu}^{M-1} (-1)^{n-\nu} \binom{\beta}{n-\nu} 2\sin\frac{u}{2} \sin\left(n+\frac{1}{2}\right) u (t^2-u^2)^{a-1}\,du.\right.$$

The inner sum is

$$2^{\beta+1}\left(\sin\frac{u}{2}\right)^{\beta+1} \sin\left\{\left(\nu+\frac{\beta+1}{2}\right) u + \frac{(\beta+1)\pi}{2}\right\}$$
$$- \sum_{M=m-\nu+1}^{\infty} 2^{\beta+1} \sin\frac{u}{2} (-1)^m \binom{\beta}{m} \sin\left(m+\nu+\frac{1}{2}\right)u.$$

Let us split K into P and Q, where

$$P = \sum_{\nu=0}^{M-1} s_\nu^\beta \int_0^t 2^{\beta+1}\left(\sin\frac{u}{2}\right)^{\beta+1} \sin\left\{\left(\nu+\frac{\beta+1}{2}\right)u + \frac{(\beta+1)\pi}{2}\right\} (t^2-u^2)^{a-1}\,du$$

(3.22)
$$= \sum_{\nu=0}^{M-1} o(\nu^\gamma)\,(t^{a+\beta}\nu^{-a})$$
$$= o(t^{a+\beta}\sum_{\nu=0}^{M-1} \nu^{\gamma-a}) = o(t^{a+\beta} M^{\gamma-a+1})$$
$$= o(t^{a+\beta}\sum_{\nu=0}^{M-1} \nu^{\gamma-a}) = o(t^{a+\beta} M^{\gamma-a+1})$$
$$= o(t^{2a-1} t^{\beta-a+1} M^{\gamma-a+1}) = o(t^{2a-1}), \qquad \text{for } 1+\gamma > \delta,$$

and

$$Q = \sum_{\nu=0}^{M-1} s_\nu^\beta \int_0^t \left\{ \sum_{m=M-\nu-1}^{\infty} 2^{\beta+1} \sin\frac{u}{2} (-1)^m \binom{\beta}{m} \sin\left(m+\nu+\frac{1}{2}\right) u \right\} (t^2-u^2)^{a-1} du$$

$$= \sum_{\nu=0}^{M-1} o\binom{\gamma}{\nu} \sum_{m=M-\nu+1}^{\infty} O\{m^{-(\beta+1)}\, t^a (m+\nu)^{-a}\}$$

$$= o\{\sum_{\nu=0}^{M-1} \nu^{\gamma}\ t^{\alpha}M^{-\alpha}(M-\nu)^{-\beta}\}$$

$$= o\{t^{\alpha}M^{-\alpha}\sum_{\nu=0}^{M-1} \nu^{\gamma}(M-\nu)^{-\beta}\}$$

$$= o\{t^{\alpha}M-\alpha M^{\gamma}\sum_{\nu=0}^{M-1} (M-\nu)^{-\beta}\} = o(t^{\alpha}M-\alpha M^{\gamma}M^{-\beta+1}\}$$

$$= o(t^{\alpha}M^{\gamma-\alpha-\beta+1}) = o(t^{2\alpha-1}t^{-\alpha+1+(1+\beta-\alpha)(\alpha+\beta-\gamma-1)/(1+\gamma-\alpha)}).$$

Since $\beta(\beta-\gamma) > 0$, we have

$$-\alpha+1+(1+\beta+\alpha)(\alpha+\beta-\gamma-1)/(1+\gamma-\alpha) > 0,$$

and

$$Q = o(t^{2\alpha-1}). \tag{3.23}$$

Summing up (3.20), (3.21), (3.22) and (3.23), we get

$$\Phi^{*}{}_{\alpha}(t) = o(t^{2\alpha-1}),$$

which is the required. If $1 < \beta < 2$, we may apply Abel's lemma two times to sum I. Thus proceeding, we get the theorem for all fractional β. The case integral β, the theorem may be proved more easily.

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