ON SOME EXTENDED PRINCIPAL AXIS THEOREMS FOR COMPLETELY CONTINUOUS OPERATORS

M. NAKAMURA, Z. TAKEDA AND T. TURUMARU

(Received September 11, 1953)

1. Introduction For a square matrix A of order n, a classical theorem of Autonne states the existence of unitary U and V for which UAV is diagonal consisting of proper value of $(AA^*)^{1/2}$. Autonne's theorem was generalized by Eckart-Young [1], Williamson [4] and Wiegmann [5], who observed the simultaneous diagonalizability of two or more matrices as analogues of the principal axis problems of hermitian and normal matrices.

The purpose of the present note is to generalize their above mentioned theorems for completely continuous operators on a Hilbert space.[1]

2. Definitions. Let H be a Hilbert space. The separability of the space is not necessary for us, however, we assume it for the notational convenience. After R. Schatten [3], we shall use the symbol $\varphi \times \psi$ for an operator such as

(1) $$\xi(\varphi \times \psi) = <\xi, \psi> \varphi.$$

It is one-dimensional and so completely continuous. Therefore, their linear combination

(2) $$a = \sum_{i=1}^{\infty} \alpha_i \varphi_i \times \psi_i$$

defines a linear completely continuous operator if $\{\alpha_n\}$ is a non-increasing sequence of complex numbers converging to 0, which are in turn the square roots of proper values of aa^*, $\{\varphi_i\}$ and $\{\psi_i\}$ are orthonormal sets. Conversely, if a is completely continuous, then it is expressible in (2). This is established in R. Schatten [3]. We shall call it as *Schatten's canonical form* for a. If $a \geq 0$ (i.e., a is non-negative definite), then (2) becomes the spectral decomposition of a by $\varphi_i = \psi_i$. Let $\{\phi_i\}$ be a complete orthonormal set in H. An operator a on H will be called *diagonal* with respect to $\{\phi_i\}$ provided that

(3) $$a = \sum_{i=1}^{\infty} \alpha_i \phi_i \times \phi_i$$

for a sequence $\{\alpha_i\}$ of complex numbers. If a is diagonal then it is obvious a is normal. For a linear operator b on H, the *extended principal axis problem* will be called being true if there exists unitary operators u and v such that ubv is *diagonal* with respect to $\{\phi_i\}$. For a set of operators b_i, the problem will be called being true provided that ub_iv is diagonal for all i.

3. Theorems The theorems of Autonne, Eckart-Young, Williamson and

---

1) The material of the note grew under seminar discussions between Misonou, Sakai, Suzuki, Watari and the authors. Especially, Watari pointed out a mistake in earlier stage proof of Theorem 3. The authors express here their hearty thanks.

Wiegmann will be generalized as follows:

THEOREM 1. *The extended principal axis problem is true for a completely continuous operator.*

THEOREM 2. *The extended principal axis problem for two completely continuous operators a and b is true if and only if ab^* and b^*a are normal.*

For a special case of Theorem 2, we shall show

THEOREM 3. *For two completely continuous operators a and b, the extendedly transposed operators uav and ubv are diagonal with real coefficient if and only if ab^* and b^*a are hermitian.*

For three or more operators, Theorems 2 and 3 become

THEOREM 4. *The extended principal axis problem of a set of completely continuous operators a_i is true if and only if $a_ia_j^*$ and $a_ja_i^*$ are normal for each pair of i and j, and $a_ia_j^*a_k = a_ka_j^*a_i$ for each (i,j,k).*

THEOREM 5. *For a set of completely continuous operators a_i, there exist unitary operators u and v such that ua_iv is hermitian and diagonal if and only if $a_ia_i^*$ and $a_ia_j^*$ are hermitian for each pair (i,j).*

4. Proof of Theorem 1. Let a be expressed in Schatten's canonical form (2), and let $\{\phi_i\}$ be a complete orthonormal set. After completing $\{\varphi_i\}$ and $\{\psi_i\}$, there exists unitary operators u and v such that $\phi_i v^* = \varphi_i$ and $\phi_i u = \psi_i$ respectively. Hence,

$$uav = u\ (\sum_i \alpha_i\varphi_i \times \psi_i)v = \sum_i \alpha_i\varphi_i v \times \psi_i u^* = \sum_i \alpha_i\phi_i \times \phi_i,$$

which is desired.

5. Proof of Theorem 3. Since the necessity follows easily, it will be proved only the sufficiency. For this purpose, it can be assumed without loss of generality that a is already diagonal, and positive hermitian by Theorem 1, since the conditions are unchanged after the transposition: If $a' = uav$ and $b' = ubv$ then

$$a'b'^* = uav(ubv)^* = uab^*u^* = uba^*u^* = ubv(uav)^* = b'a'^*$$

and similarly $a'^*b' = b'^*a'$. Under this assumption, the condition becomes:

$$ab = b^*a, \quad ab^* = ba. \tag{4}$$

Let φ be a proper vector of a belonging to a proper value λ. Then (4) implies $\varphi b = \varphi b^*a/\lambda$ and $\varphi b^* = \varphi ba/\lambda$. This shows, a vector φ from the range E of a is transformed into E again by both b and b^*, since E is the linear closure of all characteristic vectors, whence E reduces b, that is, $eb = be$ and $eb^* = b^*e$ where e is the projection belonging to E.

At first, we shall suppose $E = H$, and

$$a = \sum_{i=1}^{\infty} \alpha_i\varphi_i \times \varphi_i = \sum_{i=1}^{\infty} \lambda_i\ e_i \tag{5}$$

where $\{\varphi_i\}$ is a complete orthonormal set of E, $\{\alpha_i\}$ and $\{\lambda_i\}$ are (non-negative) proper values of a, and e_i is the projection of E_i which is belonging to λ_i, (i.e., E_i is spanned by finite number of vectors $\varphi_{i_1}, \cdots \varphi_{i_k}$ which are belonging to the same proper value $\alpha_{i1} = \alpha_{i2} = \cdots = \alpha_{i_k} = \lambda_i$). By (4) we have

$$(6) \qquad a^2b = ab^*a = ba^2,\ a^2b^* = aba = b^*a^2$$

which implies, for each j and k,

$$(7) \quad \alpha_j^2 < \varphi_j b, \varphi_k > = < \varphi_j ba^2, \varphi_k > = < \varphi_j b, \varphi_k a^2 > = \alpha_k^2 < \varphi_j b, \varphi_k >.$$

This shows either $\alpha_j = \alpha_k$ or $< \varphi_j b, \varphi_k> = 0$, if and only if either φ_j and φ_k belong to the same E_n, or $E_n b$ is orthogonal to E_m, $n \neq m$. Thus, $e_n b e_n = e_n b$ for all n. Similarly, $e_n b^* e_n = e_n b^*$.

Combining these equalities, we have

$$(8) \qquad e_n b = e_n e_n b = b e_n,\ e_n b^* = e_n b^* e_n = b^* e_n,$$

that is, E_n reduces b. Multiplying e_n from left and right at (4), it is easy to deduce $e_n b e_n = e_n b^* e_n$. Consequently, the restriction $e_n b e_n$ of b on E_n is hermitean and commutes with $e_n a e_n$, the restriction of a on E_n, since $e_n a e_n = \lambda_n e_n$ and scalar on E_n. Since $1 = e = \sum_i e_i$, $ab = ba$ and $b = b^*$ by the above. Therefore, applying the principal axis theorem for commuting hermitian operators, there exists u such that uau^* and ubu^* is diagonal with respect to the given $\{\varphi_n\}$.

Next, consider the case $E \neq H$. By the above considerations, we have

$$(9) \qquad a(1-e) = (1-e)a = 0,\ (1-e)b = b(1-e).$$

Putting $a_1 = eae$, $b_1 = eae$, $b_2 = (1-e)b(1-e)$, there are matrix representation of a and b on $H = E + E^\perp$:

$$(10) \qquad a = \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

By Theorem 1, there are unitary operatos u_2 and v_2 such that $u_2 b_2 v_2$ is diagonal.

Let u_1 be the obtained unitary operator on E as in the preceding paragraph.

Put

$$(11) \qquad u = \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \quad , \quad v = \begin{pmatrix} u_1^* & 0 \\ 0 & v_2 \end{pmatrix}.$$

Then

$$(12) \qquad uav = a' \text{ and } ubv = b'$$

are hermitian and diagonal with respect to $\{\varphi_i\}$ as desired.

5. Proof of Theorem 2. The main feature of the proof is as same as that of Theorem 3. Only a few minor changes are necessary. As in the before, we shall only prove the sufficiency. Again, it is enough to assume that a is positive hermitian.

Let E be the range of a as before. The proof of Theorem 2 that E reduce b must be changed. However, in this case the condition, which is same as (6) by a theorem of I. Kaplansky [2; Theorem 1], implies the range F of a^2 reduces b. Since a is positive, $E = F$ in our case,and so E reduces b.

The reduction of (8) from (6) is unchanged. Using (8) the normality[2)] of

2) Under a discussion, Misonou tells us a simple proof of this thesis basing on his decomposition theorem for a hyperreducible W^*-algebra.

ebe follows from that of ab:

(13) $$abb^*a = b^*a^2b = b^*ba^2,$$

multiplying e_n from both sides, and by $e = \sum_n e_n$. The remainder of the proof needs only minor verbal changes.

6. Proof of Theorems 4 **and** 5. To prove these theorems, it will be sufficient by Theorems 1, 2 and 3 that, the restrictions of a_i and a_k on the range E (with the projection e) of a_i which is assumed to be hermitian positive commutes each other. For an example, we shall show this in Theorem 4.

Use the condition $a_ia_j^*a_k = a_ka_j^*a_i$ for $j = 1$. As in §4, it is not hard to see e_l commutes with each a_i and a_k, where e_l is the spectral projection of a_i, whence $e_la_ie_la_ke_l = e_la_ke_la_ie_l$. Using $e = \sum_l e_l$, we have

$$ea_ie\cdot ea_ke = ea_ke\cdot ea_ie$$

as desired.

7. Remarks. It is to be noticed, the constructions of u and v shows that Theorems 2 and 3 are true for a completely continuous a and an arbitrary b if the proper vectors of a corresponding non-zero proper value of $(aa^*)^{1/2}$ spans H, i. e., 0 is not a proper value of a, since the complete continuity of b is used in $H^\perp$.

Also, it will be remarked that Theorems 1, 2 and 3 are still true if a and b are not completely continuous and to have Schatten's canonical form in which the coefficients α_i and β_i need not converging to zero in general.

Finally, it will be noted, a remark of Wiegmann [6] for the normality of the product of two normal matrices can be carried out for completely continuous operators as follows. If a and b are completely continuous normal operators, then ab is normal (whence ba is normal too by Wiegmann-Kaplansky's theorem) if and only if there exist unitary operators u and v such that uav and ub^*v are diagonal. This easily follows from Theorem 3 and a Theorem of Kaplansky [2; Theorem 1].

REFERENCES

[1] ECKART AND G. YOUNG, A principal axis transformation for non-hermitean matrices, Bull. Amer. Math. Soc., 45(1939), pp. 118-121.

[2] I. KAPLANSKY, Products of normal operators, Duke Math. Jorun., 20 (1953) pp. 257-260.

[3] R. SCHATTEN, A theory of Closs-space, Princeton, (1950).

[4] J. WILLIAMSON, Note on a principal axis transformation for non-hermitean matrices, Bull. Amer. Math. Soc., 45(1939), pp. 922-929.

[5] N. A. WIEGMANN, Some analogs of the generalized principal transformation, ibid., 54 (1941), pp. 905-908.

[6] N. A. WIEGMANN, A note on infinite normal matrices, Duke Math. Journ., 16(1949), pp. 535-538.

ÔSAKA NORMAL COLLEGE, ÔSAKA; TÔHOKU UNIVERSITY, SENDAI
and TÔHOKU UNIVERSITY, SENDAI.