DIMENSION OF COMPACT GROUPS AND THEIR REPRESENTATIONS

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In Pontrjagin's theory of duality for compact abelian groups, the following theorem is known:[1]

Let G be a compact abelian group, G^* the dual group. Then the topological dimension of G, in the sense of Lebesgue, is equal to the rank of discrete abelian group G^*.

It was Prof. T. Tannaka who has called my attention to the lack of corresponding theorem in non-commutative case.

I intend to give, in this note, a theorem of this kind in the following form:

THEOREM A. *Let G be an arbitrary compact group, $G^\wedge$ the aggregate of continuous finite dimensional representstions of G, $C[G^\wedge]$ the algebra over the complex numbers C generated by the coefficients of representations in $G^\wedge$, i.e., the "representative ring" of G in the sense of C. Chevalley*[2]. *Then the topological dimension of G, in the sense of Lebesgue, is equal to the transcendental degree of $C[G^\wedge]$ over C.*

Another form of corresponding theorem, which may be ture, is the following:

THEOREM B. *Let $\overline{G}$ be the space consisting of conjugate classes of a compact group G, G^* the characters of representations in $G^\wedge$, $C[G^*]$ the algebra over C generated by G^*. Then the topological dimension of $\overline{G}$ is equal to the transcendental degree of $C[G^*]$ over C.*

In spite of its natural formulation, I cannot prove this theorem at present and merely justified it for connected compact Lie groups.

1. Notations. We shall use the following notations for an arbitrary compact group G:

$n(G)$: the topological dimension of G in the sense of Lebesgue.

$n(G^\wedge) = \langle C[G^\wedge] : C\rangle$ the transcendental degree of "representative ring" $C[G^\wedge]$ over the complex numder field C.

$r(G)$: the topological dimension of the space $\overline{G}$ consists of conjugate classes of G. In case the group G is a connected compact Lie

1) L. PONTRJAGIN, Topological groups (1939), p.148 Example 49.

2) C. CHEVALLEY, Theory of Lie groups I (1946). p. 188.

group, $r(G)$ is the rank of G in the sense of H. Hopf i.e., dimension of a maximal abelian subgroup in G by the well known "principal axis theorem".

$r(G^*) = <C[G^*] : C>$: the transcendental degree of "characterring" $C[G^*]$ over C.

2. Auxiliary theorems.

THEOREM 1[3] *$n(G)=0$ if and only if $n(G^\wedge)=0$.*

PROOF. Assume $n(G)=0$, then any $D \in G^\wedge$ maps G onto a 0-dimensional Lie group, i.e., a finite group. By a suitable coordinate transformation every coefficient $d_{ij}(x)$ of D becomes algebraic, therefore $n(G^\wedge)=0$. On the other hand, if $n(G^\wedge)=0$, then every coefficient $d_{ij}(x)$ of $D \in G^\wedge$ is algebraic ; in particular its character $\Sigma_i d_{ii}(x)$ is algebraic. Hence, by a theorem of Weil[3], $D(G)$ is a finite group. Since G has sufficiently many representations, this means that $n(G)=0$. q. e. d.

THEOREM 2[3]. *G is connected if and only if every element in $C[G^\wedge]$ is constant or transcendental.*

PROOF. Assume G be not connected and put G_0 for the connected component containing the identity 1. Then G/G_0 is a 0-dimensional group and $C[(G/G_0)^\wedge] \subseteq C[G^\wedge]$. By preceding theorem there exists a non-constant algebraic element in $C[(G/G_0)^\wedge]$ and a priori in $C[G^\wedge]$.

Convesrely, if $C[G^\wedge]$ contains a non-constant algebraic element $f(x)$; then $f(x)$ is a finite valued continuous function on G. Therefore G cannot be connected. q. e. d.

3. Proof of Theorem A. The proof is accomplished by a series of elementary lemmas.

LEMMA 1. $n(G) \leq n(G^\wedge)$.

PROOF. Assume first G be a compact Lie group, then G has a faithful representation $D(x) \in G^\wedge$[4]. Since G has a neighborhood of the identity homeomorphic to the euclidean n-space R^n $(n = n(G))$, it follows that among the coefficients $d_{ij}(x)$ of $C(x)$ there exist n topologically, hence algebraically independent elements. Therefore $n(G^\wedge) \geq n(G)$.

Next G be arbitrary, there exists, for any finite number $n^* \leq n(G)$, a sufficiently small invariant subgroup $\mathfrak{U}$ such that $G/\mathfrak{U}$ is a Lie group and $n(G/\mathfrak{U}) \geq n^*$[5].

3) These theorems 1, 2 are founded independently by Y. KAWADA. His results are published in Japanese periodical "Shijo-Sugaku-Danwakai". WEIL's theorem quoted in the proof is in C.R. Paris 198, 1739-42; 199, 180-2(1934).

4) e.g., CHEVALLEY, l.c.[2] p.211.

5) e.g., PONTRJAGIN, l.c.[1] p.211 F). Separability assumption is not essential in this proof.

Obviously $n(G^\wedge) \geq n((G/\mathfrak{U})^\wedge) \geq n(G/\mathfrak{U}) \geq n^*$. This means $n(G^\wedge) \geq n(G)$. q. e. d

LEMMA 2. *If G is connected $C[G^\wedge]$ has no zero-divisors.*

PROOF. Let $f_1, f_2 \in C[G^\wedge]$ be $f_1(x) f_2(x) = 0$ everywhere on G. We must show that at least one of f_1, f_2 is zero everywhere. Since the problem concerns two elements $f_1, f_2 \in C[G^\wedge]$ it is sufficient to assume that G is a Lie group. Now f_1, f_2 are analytic functions on G, hence, by a property of analytic functions, at least one of f_1, f_2 is zero in a sufficiently small neighborhood of the identity. Since G is connected this holds everywhere. q. e. d.

LEMMA 3. *For the proof of $n(G) \geq n(G^\wedge)$ it is sufficient to assume that G is connected.*

PROOF. Let G_0 be the connected component containing 1. At first it holds obviously $n(G) \geq n(G_0)$. We show that $n(G_0^\wedge) \geq n(G^\wedge)$. For this we put $n(G_0^\wedge) = n$ and assume n is finite. Take $n+1$ arbitrary elements $f_1, \ldots, f_{n+1} \in C[G^\wedge]$ and a sufficiently small invariant subgroup $\mathfrak{U}$ such that $H = G/\mathfrak{U}$ is a Lie group and $f_1, \ldots, f_{n+1}$ are functions on H. If H_0 is the component in H, $H_0 = G_0\mathfrak{U}/\mathfrak{U} \cong G_0/G_0 \cap \mathfrak{U}$ and $C[H_0^\wedge] \subseteq C[G_0^\wedge]$ hence $n(H_0^\wedge) \leq n$.

If $H = \sum_{i=1}^{h} s_i H_0$ is a coset decomposition of H by H_0, the set of elements in $C[H^\wedge]$ which vanish on $s_i H_0$ constitutes an ideal $\mathfrak{P}_i$ in $C[H^\wedge]$ such that $C[H_0^\wedge]/\mathfrak{P}_i \cong C[H_0^\wedge]$ has no zero-divisors by Lemma 2 and its transcendental degree $n(H_0^\wedge) \leq n$. Hence there exist $h = [H:H_0]$ polynomials P_i such that

$$P_i(f_1, \cdots, f_{n+1}) \in \mathfrak{P}_i \quad (i = 1, 2, \cdots, h).$$

Since $\bigcap_{i=1}^{h} \mathfrak{P}_i = 0$, $\prod_{i=1}^{h} P_i(f_1, \cdots, f_{n+1}) = 0$. This means that $f_1, \cdots, f_{n+1}$ are algebraically dependent i.e., $n(G^\wedge) \leq n$. q. e. d.

LEMMA 4. *If G is connected $n(G) \geq n(G^\wedge)$.*

PROOF. Let $n \leq n(G^\wedge)$ be a finite number, we want to show that $n(G) \geq n$.

We take $D_0 \in G^\wedge$ such that, among the coefficients $d_{ij}(x)$ of D_0, theree xist n algebraically independent elements in $C[G^\wedge]$. Put $\mathfrak{U} = \{x \mid D_0(x) = 1\}$. Then $H = G/\mathfrak{U}$ is a Lie group with D_0 as a faithful representation. Hence by Kampen's theorem[6] coefficients of $D(x)$, $\overline{D(x)}$ generate the algebra $C[G^\wedge]$. Let $M(H)$[7] be the associated algebraic group of H, then by definition the point $(d_{ij}(x), \overline{d_{kl}(x)})$ in complex $2r$-space C^{2r}, where $r = \deg D$, is a generic point of $M(H)$ over a suitable field k. Therefore $M(H)$ is the set of specializations of the point $(d_{ij}(x), \overline{d_{kl}(x)})$ over k and

$$\textit{complex dimension of } M(H) = \langle C[H^\wedge] : C \rangle \geq n.$$

6) e.g., CHEVALLEY, l.c.[2], p.193-4.

7) For the definition and properties of associated algebraic group used in the following see CHEVALLEY, l.c.[2] pp. 194-202,

On the other hand, if $h = n(H)$ is the dimension of H, then $M(H)$ is homeomorphic to $H \times R^h$, therefore $2h \geq 2n$, i.e., $h \geq n$.

More precisely, any $A \in H_1 = M(H) \cap \mathfrak{U}(r)$ (unitary restriction) is associated with $a \in H$ by

$$A(f) = f(a) \qquad f \in C[H^\wedge]$$

(duality theorem). $C[H^\wedge] \subseteq C[G^\wedge]$ and A is a representation of $C[H^\wedge]$. We shall show that A can be extended continuously on $C[G^\wedge]$; continuity means that $A \to 1$ implies convergency of extension $\widetilde{A}$:

$$\widetilde{A}(f) \to f(1) \qquad f \in C[G^\wedge].$$

Since H_1 has a neighborhood of the identity homeomorphic to R^h, this continuous one to one image in $G^{\wedge\wedge} = G$ has dimension h. Hence $n(G) \geq h \geq n$.

Consider couples (F_1, A_1) consisting of a sub-algebra F_1 generated by a set of representations $\{D_1, \overline{D}_1, D_2, \overline{D}_2, \cdots\}$ in $G^\wedge$ and continuous extensions A_1 on F_1 of every $A \in H_1$.

$$(F_1, A_1) \leq (F_2, A_2)$$

means $F_1 \subseteq F_2$ and each A_1 coincides on F_1 with unique A_2. Then all couples (F_1, A_1) satisfy condition of Zorn's lemma and there exists a maximal couple (F_∞, A_∞). We must show $F_\infty = C[G^\wedge]$. Otherwise there would exists $D \in G^\wedge$ such that at least one coefficient of D or $\overline{D}$ does not belong to F_∞. Take one of such coefficient $d_{ij}(x) = f$ and define

1) $A'_\infty(f) = f(1)$ if f is transcendental over F_∞.
2) If f is algebraic over F_∞, take an irreducible equation satisfied by f (since $C[G^\wedge]$ is without zero-divisors by Lemma 2):

$$f^m g_m + f^{m-1} g_{m-1} + \cdots + g_0 = 0 \quad (g_i \in F_\infty).$$

By assumption $A \to 1$ implies $A_\infty(g_i) \to g_i(1)$, there exists a root of equation

$$X^n A_\infty(g_m) + X^{n-1} A_\infty(g_{m-1}) + \cdots + A_\infty(g_0) = 0$$

such that $A \to 1$ implies $\alpha \to f(1)$. We define then

$$A'_\infty(f) = \alpha.$$

Thus we can extend A_∞ to the algebra $C[F_\infty, D, \overline{D}]$ as an algebra-representation with continuity preserved.

Now consider a direct product

$$\mathfrak{G} = \underbrace{GL(r(\overline{D}_1)) \times GL(r(D_1)) \times \cdots}_{\text{on } F} \times GL(r(D)) \times GL(r(\overline{D}))$$

where $GL(r(D_i)) = GL(r(D_i), C)$ means complex general linear group of degree $r(D_i) = \deg D_i$. In this product algebra-representations of $C[F_\infty, D, \overline{D}]$ constitute a generalized algebraic group $\mathfrak{M}$ in the sense that its elements are defined by an infinity of algebraic equations. $M \in \mathfrak{M}$ implies $M^* = {}^t M^{-1} \in \mathfrak{M}$ and the subset satisfying conjugate condition is precisely $\mathfrak{M} \cap [\mathfrak{U}(r(D_1)) \times \cdots]$. In particular above A'_∞ determines

$$M = A_\infty(D_1) \times A_\infty(\bar{D}_1) \times \cdots \times A'_\infty(D) \times A'_\infty(\bar{D})$$

which is an element in $\mathfrak{M}$ such that on F_∞ its components are unitary. Now decompose M into a unitary matrix M_1 and a positive definite hermitian matrix $M_2 : M = M_1 \cdot M_2$ continuously. It is easy to verify that $M_1 \in \mathfrak{M}$ again. Define A_∞ on D by

$$M_1 = A_\infty(D_1) \times A_\infty(\bar{D}_1) \times \cdots \times A_\infty(D) \times A_\infty(\bar{D}),$$

then A_∞ is an algebra-representation of $C[F_\infty, D, \bar{D}]$ preserving conjugate condition. This would imply $(C[F_\infty, D, \bar{D}], A_\infty) \gneqq (F_\infty, A_\infty)$ contrary to the hypothesis. q.e.d.

REMARK. After completion of above proof of Theorem A, I found another proof of Theorem A for separable compact groups by using a result of A. Weil[8] which states that, if G is a compact separable group, $\mathfrak{U}$ an invariant subgroup such that $G/\mathfrak{U}$ is a Lie group, then $n(G) \geq n(G/\mathfrak{U})$. Since $n(G/\mathfrak{U}) \geq n(G)$ for sufficiently small subgroup $\mathfrak{U}$, $n(G) = \lim_{\mathfrak{U}\to 1} n(G/\mathfrak{U})$. On the other hand $n(G^\wedge) = \lim_{\mathfrak{U}\to 1} n((G/\mathfrak{U})^\wedge)$ is obvious. First part of the proof of Lemma 4 gives a proof of $n(G/\mathfrak{U}) = n((G/\mathfrak{U})^\wedge)$, therefore $n(G) = n(G^\wedge)$.

4. Proof of Theorem B for connected compact Lie groups.

Every group considered in this section are assumed to be connected compact Lie group.

LEMMA 5. *If $\tilde{G}$ is a finite sheeted covering group of G, then $r(\tilde{G}) = r(G)$, $r(\tilde{G}^*) = r(G^*)$.*

PROOF. $r(\tilde{G}) = r(G)$ is obvious by Hopf's definition of rank. $r(\tilde{G}^*) \geq r(G^*)$ is a consequence of $\tilde{G}^* \supseteq G^*$. Now let $D(x)$ be an irreducible representation in $\tilde{G}^\wedge$ and $\chi(x)$ be the character of $D(x)$. Put $G = \tilde{G}/N$ with N as a finite central subgroup of $\tilde{G}$. By Schur's lemma,

$$D(z) = \lambda(z) \cdot 1 \quad (z \in N),$$

where $\lambda(z)$ is a root of unity such that $\lambda(z)^n = 1$ if n denotes the order of N. Hence the representation

$$\underbrace{D(x) \times \cdots \times D(x)}_{n}$$

maps N into 1, i.e., this is a representation of $G = \tilde{G}/N$. This means $\chi^n \in G^*$, therefore, every character $\chi \in \tilde{G}^*$ is algebraic over G^*. Hence $(\tilde{G}^*) \leq r(G^*)$. q e.d.

LEMMA 6. *If G is a direct product of G_1 and a central subgroup G_2 of G, then $r(G) = r(G_1) + r(G_2)$, $r(G^*) = r(G_1^*) + r(G_2^*)$.*

8) Bull. Amer. Math. Soc. 55(1949), pp. 272-3.

PROOF. Let T_1 be a maximal torus of G_1, $T_2=G_2$, then $T_1\times T_2$ is a torus in G; hence $r(G_1)+r(G_2)\leq r(G)$. On the other hand if T is a maximal torus in G, then, since G_2 is central, $G_2\subseteq T$. As T/G_2 is a torus in $G_1=G/G_2$, dimension of $T/G_2\leq r(G_1)$. Thus dimension of $T=r(G)\leq r(G_1)+r(G_2)$.

Next, every irreducible representation of G is a Kronecker product of irreducible representations of G_1 and G_2. Therefore every irreducible character χ of G is a product $\chi=\chi_1\chi_2$ of characters of G_1 and G_2, i.e., $r(G^*)\leq r(G_1^*)+r(G_2^*)$. Conversely, if $\chi_1,\cdots,\chi_{\nu_1}$ and $\psi_1,\cdots,\psi_{\nu_2}$ are algebraically independent characters of G_1^* and G_2^* respectively, then $\chi_i\psi_j(i=1,\cdots,v_1,j=1,\cdots,v_2)$ are algebraically independent. For if

$$F(\chi_i\psi_1,\cdots,\chi_{\nu_1}\psi_1,\cdots,\chi_{\nu_1}\psi_{\nu_2})=0$$

is a polynomial in r_1r_2 arguments, it can be written in the form :

$$\sum_{n_1\cdots n_{\nu_2}} F_{n_1\dots n_{\nu_2}}(\chi_1,\cdots,\chi_{\nu_1})\,\psi_1^{n_1}\cdots\psi_{\nu_2}^{n_{\nu_2}}=0$$

where $F_{n_1\dots n_{\nu_2}}(\chi_1,\cdots,\chi_{\nu_1})$ are polynomials in $\chi_1,\cdots,\chi_{\nu_1}$. If we fix $x\in G_1$ then $F_{n_1\cdots n_{\nu_2}}(\chi_1(u),\cdots,\chi_{\nu_1}(x))$ is a complex number $=0$ by hypothesis on ψ's. This implies by hypothesis on χ's. $F_{n_1\cdots n_{\nu_2}}\equiv 0$. Hence the equation $F\equiv 0$, and $r(G^*)\geq r(G_1^*)+r(G_2^*)$. q.e.d.

As is well known, every connected compact Lie group G has a finite sheeted covering group G such that

$$\widetilde{G}=G_1\times G_2$$

where G_1 is a simply connected semi-simple compact Lie group and G_2 a torus[9]. Hence by Lemmas 5, 6, it is sufficient to prove $r(G)=r(G^*)$ for simply connected semi-simple compact Lie groups. In the following let G be such a group.

LEMMA 7. $r(G)\geq r(G^*)$.

PROOF. There exists one to one correspondence between representations of G and those of its Lie algebra $\mathfrak{g}$. Every irreducible representation of $\mathfrak{g}$ is determined by a highest weight Λ, which can be written uniquely by Cartan basis $\Lambda_1,\cdots,\Lambda_r$, $r=r(G)$, as

$$\Lambda=m_1\Lambda_1+\cdots+m_r\Lambda_r \quad (m_i \text{ integers } \geq 0).$$

Conversely to every such weight Λ, there exists unique irreducibel representation of $\mathfrak{g}$ having Λ as highest weight[10]. Let $D_1,\cdots,D_r$ and $\chi_1,\cdots,\chi_\nu$ be the irreducibl representations of $\mathfrak{g}$ and characters of G respectively corresponding to the weights $\Lambda_1,\cdots,\Lambda_r$.

We show that $C[G^*]=C[\chi_1,\cdots,\chi_r]$. Take an irreducible character $\chi\in G^*$ such that its weight is

9) e.g. PONTRJAGIN, 1 c.1) p. 282 THEOREM 87.

10) For the theory of representations of semi-simple Lie algebra see CARTAN: Bull. Soc. Math. de France 41(1913), pp, 53-96, WEYL: Math. Zeitsch., 24(1925)pp. 328-395.

$$\Lambda = m_1\Lambda_1 + \cdots + m_r\Lambda_r \quad (m_i \text{ integers} \geq 0)$$

and that if $\Lambda' < \Lambda$ then the character χ' with highest weight Λ' is contained in $C[\chi_1 \cdots, \chi_r]$. Irreducible representation of G which has Λ as its highest weight is contained in the Kronecker product

$$\underbrace{D_1 \times \cdots \times D_1}_{m_1} \times \cdots \times \underbrace{D_r \times \cdots \times D_r}_{m_r}$$

as the irreducible representation with highest weight (Cartan composite). Hence

$$\chi + \chi' + \chi'' + \cdots = \chi_1^{m_1} \chi_2^{m_2} \cdots \chi_r^{m_r}$$

where $\chi', \chi'', \cdots$ are characters with highest weight $\Lambda', \Lambda'', \cdots, < \Lambda$. By hypothesis on χ, $\chi', \chi'', \cdots \in C[\chi_1, \cdots, \chi_r]$, hence $\chi \in C[\chi_1, \cdots, \chi_r]$ and $C[G^*] \subseteq C[\chi_1, \cdots, \chi_r]$ by an inductive argument. q.e.d.

LEMMA 8. $r(G) \leq r(G^*)$.

PROOF. We show that the characters $\chi_1, \cdots, \chi_r$ corresponding to a Cartan basis $\Lambda_1, \cdots, \Lambda_r$ of highest weights are algebraically independent. Let $F(\chi_1, \cdots, \chi_r) = 0$ be a polynomial. If $\mathfrak{h}$ is a maximal abelian subalgebra of the Lie algebra $\mathfrak{g}$ of G, then

$$\chi_i(x) = \exp \Lambda_i(h) + \exp \Lambda_i'(h) + \cdots \quad \Lambda_i' < \Lambda_i, \text{ etc. },$$

where $x = \exp h \; (h \in \mathfrak{h})$. Inserting into the polynomial $F = \Sigma a_{n_1 \ldots n_r} \chi_1^{n_1} \cdots \chi_r^{n_r}$, we see that highest term exists in the sum

$$\Sigma a_{n_1 \ldots n_r} \exp (n_1\Lambda_1(h) + \cdots + n_r\Lambda_r(h)).$$

Now if $n_1^0\Lambda_1 + \cdots + n_r^0\Lambda_r$ is highest, then $a_{n_1^0 \cdots n_r^0} = 0$. By repeated application of this argument we arrive at $F \equiv 0$, i.e., $r(G^*) \geq r = r(G)$. q.e.d.

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