# ON A CERTAIN SYSTEM OF ORTHOGONAL STEP FUNCTIONS (1)

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## Introduction

As closed systems of orthonormal step functions, the Haar's system, and the Walsh's system are well known, [1], [2], [3]. In this paper, we shall see a rather general system of orthonormal step functions, including the above two systems.

In § 1, we define a system of orthogonal step functions, as well as other definitions, and shall see the completeness in $C$ and in $L^2$.

In § 2, an estimation of Fourier coefficients (with respect to the above system) of a continuous function is studied.

In § 3, the Lebesgue functions of this system are estimated, and as its corollary, some convergence properties of the Fourier expansion of a function are studied.

In § 4, some summability properties of the Fourier expansion of continuous functions are studied, and finally we shall show some examples of the above system, from which it is seen that the Fourier expansion of a continuous function is not necessarily $C$-summable at some point.

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## § 1. Construction of a system of orthonormal step functions.

In this papar, the treated space is a bounded closed interval of real numbers. Without loss of generality, we may assume that the interval is the closed interval $[0, 1]$.

DEFINITION 1. *A division system* $\boldsymbol{D}$ *is a infinite sequence of divisions* $D_n$ *of the interval* $[0, 1]$, *with the following properties.*

1. *If* $n>m$, *then* $D_n$ *is a subdivision of* $D_m$.
2. *The length of the longest piece in the division* $D_n$ *decreases to zero as* $n$ *increases.*

For convenience, we regard that the notation $D_n$ denotes the set of all pieces which appear by all divisions up to the $n$-th division, as well as the $n$-th division itself. There will be no confusion. Each piece in $D_n$ is assumed to be a half closed interval including the left end point except the most right piece which is closed.

The length of the longest interval in $D_n$ is denoted by $|D_n|$, and the length of the shortest interval in $D_n$ is denoted by $(D_n)$. The number of intervals in

$D_n$ is denoted by $E_n$. The interval in $D_n$ is denoted by $\alpha^{(n)}$ or simply by $\alpha$, and when we have to show that the interval contains a point $x$, it is denoted by $\alpha^{(n)}(x)$ or $\alpha(x)$. The length of $\alpha^{(n)}(x)$ is denoted by $|\alpha^{(n)}(x)|$. The notation $\alpha_i^{(n)}$ may also be used to denote an interval in $D_n$, where the suffix shows the number of an interval in $D_n$, which is properly numbered, for instance, numbered from left.

The set of all step functions which have constant values in each interval in $D_n$ is naturally regarded as an $E_n$-dimensional vector space. We denote the set of all those step functions by $\widetilde{D}_n$.

**2.** Now we shall construct a system of orthogonal step functions. Let $\varphi_1(x)$ be any normal function (e.g. the $L^2$-norm of $\varphi_1(x)$ is equal to 1) in $\widetilde{D}_1$. When for any $i$ such that $i < k \leqq E_1$, $\varphi_i(x)$ is defined in $\widetilde{D}_1$ we choose a function $\varphi_k$ (arbitrary, if there are many) in $\widetilde{D}_1$, in such a way that $\varphi_k$ is normal and is orthogonal to every $\varphi_i$.

After these processes, we shall get a set of $E_1$ functions $\varphi_1, \varphi_2, \cdots \varphi_{E_1}$, which form a complete coordinate system in the vector space $\widetilde{D}_1$. These functions $\varphi_k$ are also regarded as the functions in $\widetilde{D}_2$. We choose a normal function as $\varphi_{E_{n+1}}$ from $\widetilde{D}_2$, which is orthogonal to every $\varphi_k$.

Continuing this process, if $\varphi_i(x)$ is defined, for any $i<k$, $E_n - 1 \leqq k < E_n$, then we choose a normal functions $\varphi_k(x)$ from $\widetilde{D}_n$, which is orthogonal to every preceding function $\varphi_i(x)$, and after every $\varphi_k(x), k \leqq E_n$, is defined, $\varphi_{E_{n+1}}$ is chosen from $\widetilde{D}_{n+1}$ as before.

After such inductive choice of $\varphi_k(x)$, we shall get an infinite sequence of functions

$$\varphi_1,\ \varphi_2, \cdots \varphi_{E_n},\ \varphi_{E_{n+1}}, \cdots$$

We denote this sequence by $\Psi$.

Every $\varphi_k(x)$ is normal, and is orthogonal to another. If $E_{n-1} < k \leqq E_n$, then $\varphi_k \in \widetilde{D}_n$, and the set of all $\varphi_k(x)$, whose index number is equal to or less than $E_n$, makes a complete coordinate system of $E_n$-dimensional vector space $\widetilde{D}_n$.

**3. Examples.** Let $D_1$ be the set which consists of only the undivided whole interval $[0,1]$, and $D_n$ be the $(n-1)$-th diadic division, namely, the division by the dividing points $p/2^{n-1}$, where $p$'s are positive integers less than $2^{n-1}$.

If we construct $\Psi$ on this division system with a restrictive condition that the absolute value of every function is constantly 1, we shall get the Walsh's system of step functions, disregarding some unessential change of the order of functions, and the change of signs of functions.

We arrange all proper irreducible diadic fractions in the ascending order of the magnitudes of their denominators, and in the case when their denominators are equal, in the ascending order of the magnitudes of their numerator. That is, put

$$A_i=(2\cdot n_i-1)/2^{m_i} \qquad i=1, 2, 3,\cdots,$$

where $2^{m_i}$ is the least power of 2, which is equal to or greater than $i$, and $n_i=i-2^{(m_i-1)}+1$.

Then for any integer $i$, there corresponds a proper irreducible diadic fraction $A_i=(2\cdot n_i-1)/2^{m_i}$, with the following condition;

If $i<j$ then $m_i<m_j$

or $m_i=m_j$ and $n_i<n_j$.

Let $D_1$ be the same division as the previous example, and $D_n$ be the division by the dividing points $A_1, A_2,\cdots, A_{n-1}$. The Haar's system is a system $\Psi$ constructed on this division system. About this system, it is remarkable that every $E_n$ is equal to $n$,

**4.** Let $f$ be an integrable function (not necessarily in $L^2$). The *n-th mean function* $\tilde{f}^{(n)}$ of $f$ is defined as

$$\tilde{f}^{(n)}(x) = \frac{1}{|\alpha^{(n)}(x)|}\int_{a^{(n)}(x)} f(t)dt \tag{1}$$

that is, the value of $f(x)$ is the mean value of $f(x)$ in the interval $\alpha^{(n)}(x)$.

Let

$$a_k = \int_0^1 \tilde{f}^{(n)}(t)\varphi_k(t)dt, \qquad k\leqq E_n$$

then

$$a_k = \int_{p=1}^{E_n}\int_{a_p^{(n)}} \tilde{f}^{(n)}(t)\varphi_k(t)dt.$$

But then, since in each interval $\alpha_p^{(n)}$, the integrand is constant,

$$a_k = \sum_{p=1}^{E_n} \tilde{f}_p^{(n)}.\varphi_{k,p}\,|\alpha_p^{(n)}|,$$

where $\tilde{f}_p^{(n)}$, and $\varphi_{n,p}$ deonte the constant value of $\tilde{f}^{(n)}$, and $\varphi_k$, respectively in the interval $\alpha_p^{(n)}$.

Now by the equation (1),

$$\begin{aligned} a_k &= \sum_{p=1}^{E_n}\varphi_{k,p}\,|\alpha_p^{(n)}|\cdot\frac{1}{|\alpha_p^{(n)}|}\int_{ap^{(n)}} f(t)dt \\ &= \sum_{p=1}^{E_n}\int_{\alpha_p^{(n)}} \varphi_{k_p} f(t)dt \\ &= \int_0^1 \varphi_k(t)\,f(t)dt, \end{aligned}$$

which shows that the $k$-th Fourier coefficient[1] of $f(x)$ is equal to that of

1) By the terms, Fourier coefficients, a Fourier expansion, etc. we mean those with respect to the sysem $\Psi$. In the case when we say about the usual Fourier series, we particularly call it the trigonometric Fouries series.

$\widetilde{f}^{(n)}(x)$, where $k \leqq E_n$, that is, if $k \leqq E_n$

$$a_k = \int_0^1 \widetilde{f}^{(n)}(t)\varphi_k(t)dt = \int_0^1 f(t)\varphi_k(t)dt.$$

Now, since the set of the functions $\varphi_1, \varphi_2, \cdots, \varphi_{E_n}$ makes a complete coordinate system of the vector space $\widetilde{D}_n$, and since $\widetilde{f}^{(n)} \in \widetilde{D}_n, \widetilde{f}^{(n)}$ can be represented as a linear sum of the functions $\varphi_1, \varphi_2, \cdots, \varphi_{E_n}$, that is

$$\widetilde{f}^{(n)}(x) = \sum_{k=1}^{E_n} b_k \varphi_k(x).$$

But, multiplying $\varphi_k$ to the both terms and integrating from 0 to 1, we get

$$b_k = \int_0^1 \widetilde{f}^{(n)}(t)\varphi_k(t)dt = \int_0^1 f(t)\varphi_k(t)dt.$$

Hence $b_k$ is the $k$-th Fourier coefficient of $f(x)$, and the $E_n$-th partial sum of Fourier expansion of $f(x)$ is equal to the $n$-th mean value function.

Especially when $f(x)$ is continuous, $\sup_{0 \leqq x \leqq 1} |f(x) - \widetilde{f}(x)^{(n)}|$ becomes to zero uniformly as $n$ increases. Hence

THEOREM 1. *The $E_n$-th partial sum of the Fourier expansion of a continuous function $f(x)$ uniformly converges to $f(x)$ as $n$ increases.*

Hence the whole Fourier expansion of a continuous function $f(x)$ converges to $f(x)$ by $L^2$-norm. Since the set of continuous functions is dense in $L^2$,

THEOREM 2. *The system $\Psi$ is complete in $L^2$.*

By the way, it is a direct corollary that the Fourier expansion of a continuous function $f(x)$ by the Haar's system uniformly converges to $f(x)$.

## § 2. Fourier coefficients of a continuous function.

**1.** First we notice the following: ——

If $k > E_n$, then $\varphi_k$ is orthogonal to every $\varphi_h$; $h \leqq E_n$. But since the set of functions $\varphi_1, \varphi_2, \cdots, \varphi_{E_n}$ makes a complete coordinate system of the vector space $\widetilde{D}_n, \varphi_k$ is orthogonal to every function in $\widetilde{D}_n$. Especially, the characteristic function $\chi_a^{(n)}$ of an interval in $D_n$ is contained in $\widetilde{D}_n$. Hence $\varphi_k$ is orthogonal to it, and hence

$$\int_{a^{(n)}} \varphi_k(t)dt = \int_0^1 \varphi_k(t)\chi_a^{(n)}(t)dt = 0. \tag{2}$$

2. From this fact, we can get some estimation of the absolute value of Fourier coefficients of a continuous function.

Let $f(x)$ be a continuous function with the modulus

$$\omega(\delta, f) = \sup_{0 \leqq x \leqq 1, |h| < \delta} |f(x+h) - f(x)|. \tag{3}$$

Assume that $E_n \leqq k < E_{n+1}$, and put

$$a_k = \int_0^1 f(t)\varphi_k(t)dt$$
$$C_p = 1/2\times(\sup_{a_p} f(x) + \inf_{a_p} f(x)),$$

then

$$|a_k| = \left|\int_0^1 f(t)\varphi_k(t)dt\right| = \left|\sum_{p=1}^{E_n}\int_{a_p^{(n)}} f(t)\varphi_k(t)dt\right|,$$

which, by the equation (2),

$$= \left|\sum_{p=1}^{E_n}\int_{a_p^{(n)}} (f(t)-C_p)\varphi_k(t)dt\right| \leqq \frac{\omega(|D_n|,f)}{2}\sum_{p=1}^{E_n}\int_{a_p^{(n)}} |\varphi_k(t)|\,dt$$
$$= \frac{\omega(|D_n|,f)}{2}\sum_{p=1}^{E_n}|\varphi_{k,p}|\;|\alpha_p^{(n)}|$$
$$\leqq \frac{\omega(|D_n|,f)}{2}\sqrt{\sum_{p=1}^{E_n}|\varphi_{k,p}|^2\cdot|\alpha_p^{(n)}|\cdot\sum_{p=1}^{E_n}|\alpha_q^{(n)}|}$$
$$= \frac{\omega(|D_n|,f)}{2}.$$

Hence

THEOREM 3. *If $k\geqq E_n$, then*

$$\left|\int_0^1 f(t)\varphi_k(t)dt\right| \leqq \frac{\omega(|D_n|,f)}{2}.$$

## § 3. Lebesgue functions and some convergence properties.

**1.** In this chapter, first we shall see the following estimation.

THEOREM 4. *If $E_{n-1}+m\leqq E_n$ then,*

$$L_{E_{n-1}+m}(x) = \int_0^1\left|\sum_{k=1}^{E_{n-1}+m}\varphi_k(x)\varphi_k(t)\right|dt \leqq \frac{1+\sqrt{\alpha^{(n)^{-1}}(x)}}{2}, \tag{4}$$

$$\int_0^1\left|\sum_{k=1}^{m}\varphi_{E_{n-1}+k}(x)\varphi_{E_{n-1}+k}(t)\right|dt$$
$$\leqq\frac{1}{2}\left[2-2\alpha^{(n)}(x)\cdot\alpha^{(n-1)^{-1}}(x)+\sqrt{\alpha^{(n)^{-1}}(x)-\alpha^{(n-1)^{-1}}(x)}\right]. \tag{5}$$

The proof of these two estimations will be established by quitely similar methods, and so, we shall prove the second inequality only.

**The proof of the inequality** (5). For the convenience's sake, put

$$\varphi_{E_{n-1}+k}(t) = a_{kq}, \quad \text{when } t\in\alpha_q^{(n)},$$
$$|\alpha_q^{(n)}| = \alpha_q,\; |\alpha_s^{(n-1)}| = \beta_s,\quad E_n = l,\; E_{n-1} = l', \tag{6}$$

and let

$$x\in\alpha_p\subset\beta_t. \tag{7}$$

Then

$$\sum_{q=1}^{l}\alpha_q\left|\sum_{k=1}^{m}a_{kq}\;a_{kp}\right|$$

is to be estimated under constant $\alpha_q$'s and variable $a_{kq}$'s.

First we fix constants $\varepsilon_q = \pm 1$, where $\varepsilon_p = 1$, and determine the maximum value of the expression

$$\sum_{q=1}^{l} \alpha_q \varepsilon_q \sum_{k=1}^{m} a_{kq} a_{kq} \tag{8}$$

then, by suitable choice of the signs of $\varepsilon_q$'s, we find an $\varepsilon_q$-free estimation.

There are restrictive conditions on the domain of the variables $a_{kq}$;

$$\sum_{q=1}^{l} \alpha_q a_{kq} a_{hq} = \delta_{kh}, \tag{9}$$

$$\sum_{q \in s} \alpha_q a_{kq} = 0. \tag{10}$$

(We use the expression $q \in s$, instead of $\alpha_q^{(n)} \subset \alpha_s^{(n-1)}$, for simplicity.)

Hence the domain of the variable vector (all, $a_{12} \cdots, a_{kq} \cdots$) is compact, and the Lagrange's method to determine maximum value is applicable.

Now we are to find the maximum value of the expression,

$$F = \sum_{q=1}^{l} \alpha_q \varepsilon_q \sum_{k=1}^{m} a_{kp} a_{kq} + \lambda_{kh} \left( \sum_{q=1}^{l} \alpha_q a_{kq} a_{hq} - \delta_{kh} \right) + \eta_{ks} \left( \sum_{q \in s} \alpha_q a_{kq} \right) \tag{11}$$

under variable $a_{kq}$ and variable parameters $\lambda_{kh}$ und $\eta_{kq}$.

Differentiate (11) by $a_{kq}$, then

$$\begin{aligned} F_{a_{kq}} &= \delta_{pq} \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kq} + \alpha_q \varepsilon_q a_{kp} \\ &+ \sum_{h=1}^{m} \lambda_{kh} \alpha_q a_{hq} + \lambda_{kk} \alpha_q a_{kq} + \eta_{ks} \alpha_q = 0. \end{aligned} \tag{12}$$

Summing up with respect to $q \in s$, we have by (10),

$$\delta_{st} \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kr} + a_{kp} \sum_{q \in s} \alpha_q \varepsilon_q + \beta_s \eta_{ks} = 0,$$

from which

$$\eta_{ks} = - \beta_s^{-1} \left( a_{kp} \sum_{q \in s} \alpha_q \varepsilon_q + \delta_{st} \sum_{r=1}^{l} \alpha^r \varepsilon^r a_{kr} \right). \tag{13}$$

Now, multiplying $a_{jq}$ to (12), and summing up about whole $q$, we have

$$\begin{aligned} & a_{jp} \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kr} + \sum_{q=1}^{l} \alpha_q \varepsilon_q a_{kq} a_{jq} \\ &+ \sum_{h=1}^{m} \lambda_{kh} \sum_{q=1}^{l} \alpha_q a_{jq} a_{hq} + \lambda_{kk} \sum_{q=1}^{l} \alpha_q a_{jq} a_{kq} \\ &+ \sum_{s=1}^{l'} \eta_{ks} \sum_{q \in s}^{l} \alpha_q a_{jq} = 0. \end{aligned} \tag{14}$$

Putting $j = k$

$$\lambda_{kk} = - a_{kp} \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kr}. \tag{15}$$

Putting $j = h$

$$\lambda_{kh} = - \left( a_{kp} \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kr} + a_{kp} \sum_{q=1}^{l} \alpha_q \varepsilon_q a_{kq} \right). \tag{16}$$

Put (13), (15), and (16) into (12). Then we have

$$\begin{aligned}\delta_{pq} \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kr} + \alpha_p \varepsilon_p a_{kp} &= \alpha_q \sum_{h=1}^{m} a_{hp} a_{hq} \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kr} \\ &+ \alpha_q \ a_{kq} \sum_{h=1}^{m} a_{hp} \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{hr} \\ &+ \alpha_q \ \beta_s^{-1} \ (a_{kp} \sum_{r \in s} \alpha_r \varepsilon_r + \delta_{st} \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kr}).\end{aligned} \tag{17}$$

Now let us set

$$\sum_{q=1}^{l} \alpha_q \varepsilon_q \sum_{k=1}^{m} a_{kq} a_{kq} = x, \tag{18}$$

which is to be estimated, and put

$$\sum_{k=1}^{m} a^2_{kq} = A. \tag{19}$$

In (17), putting $p=q$, multiplying $a_{qk}$ and, summing up about $k$, then, by the expression (18) and (19), we get

$$X + \alpha_p A = 2\alpha_p \ AX + \alpha_p \beta_t^{-1} A \sum_{\in t} \alpha_r \varepsilon_r + \alpha_p \beta_t^{-1} X,$$

which is,

$$(\alpha_p^{-1} - \beta_h^{-1}) \ X = A \ (2X + \beta_t^{-1} \sum_{r \in t} \alpha_r \varepsilon_r - 1). \tag{20}$$

On the other hand, multiply $\varepsilon_q a_{kp}$ to (17) and sum up about $q$ and $k$. Then

$$\begin{aligned}X + A = X^2 + A \cdot \sum_{h=1}^{m} \Big(\sum_{q=1}^{l} \alpha_q \varepsilon_q a_{hq}\Big)^2 + A \sum_{s=1}^{l'} \beta_s^{-1} \Big(\sum_{q \in s} \alpha_q \varepsilon_q\Big)^2 \\ + \beta_t^{-1} \sum_{q \in t} \alpha_q \varepsilon_q \cdot X.\end{aligned} \tag{21}$$

To calculate $A \sum_{h=1}^{m} \Big(\sum_{q=1}^{l} \alpha_q \varepsilon_q a_{hq}\Big)^2$, we put $p = q$ in (17) and put in order

then

$$\sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kr} \ (\alpha_p^{-1} - \beta_t^{-1} - A) = a_{kp} (X + \beta_t^{-1} \sum_{r \in t} \alpha_r \varepsilon_r - 1)$$

which is, by (20),

$$= a_{kp} X \ ((\alpha_p^{-1} - \beta_t^{-1}) \ A^{-1} - 1).$$

Hence

$$A \sum_{r=1}^{l} \alpha_r \varepsilon_r a_{kr} = a_{kp} X.$$

Squaring both sides, and summing up about $k$, we have

$$A \cdot \sum_{k=1}^{m} \Big(\sum_{q=1}^{l} \alpha_q \varepsilon_q a_{kq}\Big)^2 = X^2. \tag{22}$$

(We may assume that $A \neq 0$, since otherwise $X = 0$).

Put (22) into (21). Then

$$X + A = 2X^2 + A\cdot \sum_{s=1}^{l'} \beta_s^{-1} \Big(\sum_{q\in s} \alpha_q \varepsilon_q\Big)^2 + \beta_t^{-1} \sum_{q\in t} \alpha_q \varepsilon_q \; X. \tag{23}$$

That is

$$A^2\Big(1 - \sum_{s=1}^{l'} \beta_s^{-1} \Big(\sum_{q\in s} \alpha_q \varepsilon_q\Big)^2\Big) = X\Big(2X + \beta_t^{-1} \sum_{q\in t} \alpha_q \varepsilon_q - 1\Big)A$$

which is, by (20), equal to

$$(\alpha_p^{-1} - \beta_t^{-1})\, X^2. \tag{24}$$

Eliminating $A$ from (20) and (24), we get

$$(\alpha_p^{-1} - \beta_\tau^{-1}) - \sum_{s=1}^{l'} \beta_s^{-1} \Big(\sum_{q\in s} \alpha_q \varepsilon_q\Big)^2 = \Big(2X - 1 + \beta_\tau^{-1} \sum_{q\in t} \alpha_q \varepsilon_q\Big)^2.$$

Hence

$$1X = 1/2 \Big[ - \beta_\tau^{-1} \sum_{q\in \tau} \alpha_q \varepsilon_q + \sqrt{\alpha_p^{-1} - \beta_\tau^{-1}}\, \sqrt{1 - \sum_{s=1}^{l'} \beta_s^{-1} \Big(\sum_{q\in s} \alpha_q \varepsilon_q\Big)^2} \Big],$$

or, since $\sum_{q\in t} \alpha_q \varepsilon_q \geqq -(\beta_t - 2\alpha_p)$, we have

$$X \leqq 1/2\, [2 - 2\beta_t^{-1} \alpha_p + \sqrt{\alpha_p^{-1} - \beta_t^{-1}}],$$

which is the proposition of the inequality (5). Q. E. D.

**2.** Now if a continuous function $f(x)$ has the modulus of continuity $\omega(\delta, f)$, that is,

$$\underset{|h|\leqq \delta,\ 0\leqq x\leqq 1}{\text{l. u. b.}} |f(x+h) - f(x)| = \omega(\delta, f)$$

and if $|D_n| = \varepsilon < \delta$, then

$$\int_0^1 f(t) \sum_{K=E_n+1}^{E_n+m} \varphi_k(t) \cdot \varphi_k(x)\, dt$$

$$= \int_0^1 [\tilde{f}^{(n)}(t) + \varepsilon(t)] \sum_{K=E_n+1}^{E_n+m} \varphi_k(t)\, \varphi_k(x)\, dt,$$

where $\varepsilon(t)$ is a function of $x$ such that

$$|\varepsilon(t)| \leqq \omega(\varepsilon, f).$$

Since

$$\int_0^1 \tilde{f}^{(n)}(t) \cdot \varphi_k(t)\, dt = 0 \text{ for } k > E_n,$$

we have

$$\int_0^1 f(t) \sum_{K=E_n+1}^{E_n+m} \varphi_k(t)\, \varphi_k(x)\, dt$$

$$\leqq \omega(\varepsilon, f) \int_0^1 \Big| \sum_{k=E_n+1}^{E_n+m} \varphi_k(t)\, \varphi_k(x) \Big| dt$$

$$\leqq \omega(\varepsilon, f) \cdot 1/2 \Big[2 + \sqrt{\alpha_p^{-1} - \beta_t^{-1}}\Big],$$

where it is assumed that $x \in \alpha_p^{(n)} \subset \alpha_t^{(n-1)}$.

Especially if $\omega(\varepsilon, f) = K\varepsilon^a$, then

$$\int_0^1 f(t) \sum_{k=E_n+1}^{E_n+m} \varphi_k(t)\, \varphi_k(x)\, dt \leqq K/2 \left[\sqrt{\frac{\varepsilon^{2a}}{\alpha_p} - \frac{\varepsilon^{2a}}{\beta_t}} + 2\varepsilon^a\right]. \tag{25}$$

Now, we define an additional condition of the division system $D=\{D_n\}$

**Definition** 2. A division system is *moderate* at $x$, if

1. $|D_n|/|\alpha^{(n)}(x)| < M(x)$
2. $|\alpha^{(n)}(x)|/|\alpha^{(n+1)}(x)| < N(x)$,

where $M(x)$ and $N(x)$ are finite functions of $x$ independent on $n$.

If there exist $x$-free constants $M$ which satisfy 1 and 2, for every $x$ (or for $x$'s in a set), then we say that the division system is *uniformly moderate.*

As easily seen, a division system is moderate at $x$ if and only if

$$|D_n|/|\alpha^{(n+1)}(x)| < M^*(x),$$

and it is uniformly moderate if and only if

$$|D_n|/(D_n) < M^*, \qquad \text{where } (D_n) = \inf_{a\in D_n} |\alpha|.$$

Now since $\varepsilon = |D_n|$, in the inequality (25), if the division system is moderate at $x$,

$$\frac{\varepsilon}{\alpha_p} - \frac{\varepsilon}{\beta_t} < N(x)M(x) + M(x).$$

Hence the inequality (25) shows that if the division system is moderate at $x$ and $\alpha > 1/2$, then the left side of (26) converges to 0 as $n$ increases. Since

$$\tilde{f}^{(n)}(x) = \int_0^1 f(t)\cdot \sum_{k=1}^{E_n} \varphi_k(t)\ \varphi_k(X) dt$$

converges to $f(x)$ uniformly, we get,

THEOREM 5. *If the division system is moderate at $x$, and if $f(x)$ satisfies the Lipschitz condition for an index $\alpha$ such that $1/2<\alpha\leq 1$, then the Fourier expansion of $f(x)$ by $\Psi$ converges to $f(x)$ at $x$. If the division system is also uniformly moderate, the convergence of the expansion is uniform.*

## § 4. Some summability property.

The following theorem is known.

Let $\varphi_1, \varphi_2, \cdots$ be any complete system of orthogonal functions in $L^2(0, 1)$, (not necessarily the system $\Psi$ ). If

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} a_k \varphi_k(x)$$

converges almost everywhere, then $\sum_{k=1}^{\infty} a_k \varphi_k(x)$ is $C$-1 summable almost everywhere.

The following lemma and its proof are slight modifications of the above theorem and its proof mentioned in [1] p.190.

LEMMA *Let $\{\varphi_k\}$ be any complete system of orthogonal functions in $L^2$*

(0, 1). *If, for some sequence of integers* $E_1, E_2, \cdots, E_n, \cdots$ *such that* $E_{n+1}<ME_n$ *for some constant* $M$, *the series*

$$S_{F_n}(x)=\sum_{k=1}^{E_n} a_k\varphi_k(x)$$

*converges to* $f(x)$ *almost everywhere, then the series*

$$S(x)=\sum_{k=1}^{\infty} a_k\varphi_k(x)$$

*is* $C$-1 *summable to* $f(x)$ *almost everywhere.*

PROOF. We take out a sub-sequence $F_1, F_2, \cdots, F_n, \cdots$, from the sequence $E_1, E_2, \cdots, E_n, \cdots$, in such a way that

$$M^n \leqq F_n < M^{n+1}.$$

First we prove that the sequence

$$\sigma_{F_n}(x) = 1/F_n\sum_{k=1}^{F_n} S_k(x),$$

where

$$S_k(x)=\sum_{i=1}^{k} a_i\,\varphi_i(x),$$

converges to $f(x)$ almost everywhere.

Since

$$S_m(x)-\sigma_m(x)=1/m\sum_{k=1}^{m} a_k(k-1)\varphi_k(x),$$

$$\int_0^1[S_{F_n}(x)-\sigma_{F_n}(x)]^2\,dx=1/F_n^2\cdot\sum_{k=1}^{F_n} a_k^2(k-1)^2.$$

$$\leqq\sum_{k=1}^{\infty} a_k^2(k-1)^2\sum_{n=[\log_M k]-1} 1/F_n^2.$$

$$\therefore\sum_{m-1}^{\infty}\int_0^1[S_F(x)-\sigma_F(x)]^2dx \leqq\sum_{k=1}^{\infty} a_k^2(k-1)^2\sum_{n\geq(\log_M k)-2}^{\infty} 1/M^{2n}$$

$$\leqq\sum_{k=1}^{\infty} a_k^2(k-1)^2\frac{M^2}{M^2-1}\cdot\frac{M^2}{k^2} \leqq M^4(M^2-1)^{-1}\sum_{k=1}^{\infty} a_k^2<\infty.$$

Hence the series

$$\sum_{n=1}^{\infty}[S_{F_n}(x)-\sigma_{F_n}(x)]^2$$

converges almost everywhere, and so the sequence

$$S_{F_n}(x)-\sigma_{F_n}(x)$$

converges to zero almost everywhere. Hence

$$\lim_{n\to\infty}\sigma_{F_n}(x) = \lim_{n\to\infty} S_{F_n}(x) = f(x)$$

almost everywhere.

Next we shall show that the sequence

$$R_{N,F_n}=\sigma_N(x)-\sigma_{F_n}(x)$$

converges to zero.

Let $F_n \leqq N < F_{n+1}$, then

$$\begin{aligned}(\sigma_N(x)-\sigma_{F_n}(x))^2 &= \left[\sum_{k=F_n}^{N-1}(\sigma_{k+1}(x)-\sigma_k(x)\right]^2 \\ &\leqq \sum_{k=F_n}^{N-1} k(\sigma_{k+1}(x)-\sigma_k(x))^2 \sum_{k=F_n}^{N-1} 1/k \\ &\leqq \frac{M^{n+1}-M^n+1}{M^n} \sum_{k=F_n}^{N-1} k(\sigma_{k+1}(x)-\sigma_k(x))^2 \\ &\leqq (M+1)\sum_{k=F_n}^{N-1} h(\sigma_{k+1}(x)-\sigma_k(x))^2. \end{aligned} \tag{26}$$

But since the series

$$\sum_{k=1}^{\infty} k(\sigma_{k+1}(x)-\sigma_k(x))^2$$

converges almost everywhere (cf. [1], p. 188), the expression of (26) converges to zero almost eveywhere, and the proof of the Lemma is established.

If the system $\{\varphi_k\}$ is the system $\Psi$, and its division system is uniformly moderate, that is

$$\frac{|D_n|}{(D_{n+1})} < M,$$

then obviously $E_{n+1} < ME_n$, and the partial sum of the expansion of a continuous function $f(x)$,

$$\sum_{k=1}^{F_n} a_k\varphi_k(x), \quad \text{where } a_k = \int_0^1 f(t)\varphi_k(t)\, dt,$$

converges to $f(x)$ uniformly as $n$ increases, we can state

THEOREM 6. *For a system $\Psi$, if $E_{n+1} < M\cdot E_n$, or especially its division system is uniformly moderate, then the Fourier expansion of a continuous function $f(x)$ is C-1 summable to $f(x)$ almost everywhere.*

But the everywhere summability theorem, as in the cases of the trigonometric Fourier expansion, or the Walsh's Fourier expansion, does not hold in general. We shall show an example, in which the everywhere summability theorem false.

Let $D$ be the diadic division system as that for the Walsh's system (cf. the example in § 1). Let $\{\chi_k(x)\}$ be Haar's system and $\{\phi_k(x)\}$ be the Walsh's system mentioned in the example in § 1. Now put

$$\begin{aligned}\varphi_1(x) &\equiv 1 \\ \varphi_2(x) &= 1 \text{ for } 0 \leqq x < 1/2 \\ &= -1 \text{ for } 1/2 \leqq x \leqq 1\end{aligned}$$

and if $0 \leqq k < 2^{n-1}$, $n \geqq 2$,

$$\begin{aligned}\varphi^{2^n+k}(x) &= \chi_{2^{n-1+k}}(2x) && \text{for } 0 \leqq x < 1/2 \\ &= \phi_{2^{n-1}+k}(2x-1) && \text{for } 1/2 \leqq x \leqq 1,\end{aligned}$$

and if $2^{n-1} \leqq h < 2^n$

$$\varphi_{2^n+h}(x) = \chi_{2^{n-1}+k}(2x) \qquad \text{for } 0 \leqq x < 1/2$$
$$= \phi_{2^{n-1}+k}(2x-1) \qquad \text{for } 1/2 \leqq x \leqq 1.$$

Then $\varphi_k(x)$ makes the system $\Psi$, as easily seen.

Now let $f(x)$ be a continuous function such that,

$$f(x) = 0 \qquad \text{for } 0 \leqq x < 1/2$$
$$= \sum_{n=2}^{\infty} (c_n(x) \cdot b^n \ |\sin 2^{n+1} \pi x|) \qquad \text{for } 1/2 \leqq x \leqq 1$$

where $b$ is a positive constant less than 1, and $c(x) = \pm 1$ is the following function.

Let $x_0$ be a point between 0 and 1/2. There is a $\varphi_{2^n+k}(x)$; $0 \leqq k < 2^n$ which is not zero at $x_0$. We choose the sign as the following way;

$$c_n(x) = \operatorname{sign}(\varphi_{2^n+k}(x_0)\, \varphi_{2^n+k}(x)).$$

Then, as easily calculated,

$$\int_0^1 f(x)\, \varphi_{2^n+k}(u)\, du = b^n/\pi,$$
$$\int_0^1 f(x)\, \varphi_{2^n+2^{n+1}+k}(x)\, dx = -b^n/\pi$$

and

$$\int_0^1 f(u)\, \varphi_{2^n+h}(x)\, dx = 0,$$

where $h \neq k$, $h \neq 2n-1+k$, $0 \leqq h < 2^{n-1}$.

Hence

$$\int_0^1 f(u)\, \varphi_{2^n+k}(u)\, \varphi_{2^n+k}(x_0)\, dx = (2b)^n/2\pi.$$

And we have

$$\frac{1}{2^n + 2^{n-1} + k - 1} \sum_{s=2^n+k}^{2^n+2^{n+1}+k-1} \int_0^1 f(x)\, \varphi_s(x)\, \varphi_s(x_0)\, dx$$
$$= \frac{1}{2^n + 2^{n-1} + k - 1} \cdot \frac{(2b)^n}{2\pi} \cdot 2^{n-1}$$
$$\geqq 2^{n-5} \cdot b^n.$$

Hence if $b$ is greater than 1/2, then the Cesàro means of the Fourier expansion of $f(x)$ are not bounded. Hence it does not converge.

This example also shows that the localization theorem does not hold for a general system $\Psi$.

By a slight modification of this system $\{\varphi_k\}$, we can get a system $\Psi'$, such that for any $0 \leqq x_0 \leqq 1$, there exists a continuous function whose Fourier expansion by $\Psi'$ is not summable at $x_0$.

## REFERENCES

[1] S. KACZMARZ and H. STEINHAUS, Theorie der Orthngonalreihen, Warszawa-Lwow, 1953.

[2] J. L. WALSH. A closed set of normal orthogonal functions. Amer. Journ. Math., 55, No 5 (1953), 5-24.

[3] N. J. FINE, On the Walsh's Functions. Trans. Amer. Math. Soc., 65 (1949), No.3, 372-379.

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