ON THE GENERALIZED INVARIANT DIFFERENTIAL FORM ON MANIFOLDS WITH GENERAL CONNECTION

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0. Introduction. It was shown by E. Cartan [1][1)] that any invariant differential form in an n-dimensional homogeneous space with a finite, continuous transformation group of Lie as the group of structure is a certain exterior form with constant coefficients invariant under the linear group of isotropy of the space and conversely. Recently, his famous two theorems for compact isogeneous spaces, except the above one in his paper, that any closed differential form is equivalent to an invariant differential form and that any invariant differential form equivalent to 0 is invariantly null are generalized for connected manifolds of class C^2 with a compact connected topological group of transformations by C. Chevalley and S. Eilenberg [2]. On the other hand, late H. Iwamoto studied differential forms invariant under the holonomy group in compact, orientable, positive-definite Riemannian spaces [3], which form a contrast in certain sense with invariant differential forms in compact isogeneous spaces especially when both spaces are symmetric. A trial in the following is to generalize the above E. Cartan's theorem first stated for abstract manifolds with general connection which may be nonlinear in the style of C. Chevalley and moreover so as to include the generalization for homogeneous spaces with a topological transtormation group as the group of structure as a special case. Since, exterior differential forms in Banach spaces were treated by M. Kerner [4], we shall take general manifolds with Banach coordinate in our consideration.

It will be seen that spaces with projective connection and spaces with conformal connection must be excluded and that if tangent spaces be nonlinear, the condition that the points of contact are changed to another points of contact under displacements must be imposed.

1. The general homogeneous holonomy group. Let *a manifold X of class $C^k (k \geqq 2)$ with Banach coordinate, general connection and a topological transformation group G as the fundamental group* be defined as follows:

(1.1) X is *a general manifold of class C^k with Banach space C as the local coordinate space*, namely, a connected topological space with neighborhoods U_p at each point p such that to each point p, a class $\mathfrak{F}_p = \{f\}$ of functionals $f: X \to C$, called the functionals of class C^k on X at p, defined in a

1) Numbers in brackets refer to the references at the end of the paper.

neighborhood of p is assigned as follows: (i) any $f_2: X \to C$ which is C^k-dependent on $f_1 \in \mathfrak{F}_p$ in a neighborhood U_p of p, by which we mean that $f_2 = F(f_1)$ where $F(u)$ is a functional $C \to C$ of class C^k at any point $u = f_2(q)$ $(q \in U_p)$ in the sense of the topological differential in C, belongs to $\mathfrak{F}_p$; (ii) there exist $x_p \in \mathfrak{F}_p$ and a neighborhood $U_p(x_p)$ and a number a with the following properties: (α) x_p is defined in $U_p(x_p)$ and if $q \in U_p(x_p)$, then $x_p \in \mathfrak{F}_q$ and every $f \in \mathfrak{F}_q$ is C^k-dependent on x_p in a neighborhood of q; (β) the mapping $q \in U_p(x_p) \to x_p(q) \in C$ gives a homeomorphism of $U_p(x_p)$ with $\{u \in C: \|u - x_p(p)\| < a\} \subset C$, and

(1.2) to each point p of X, a general manifold Σ_p of class C^k, called the *associated manifold* at p, with a transformation group G and the same local coordinate space C as the one of X corresponds and supposed a unique definite point $p^* \in \Sigma_p$ called the *centre* of Σ_p to be determined, and

(1.3) there exists a class $\mathfrak{C} = \{\Gamma\}$ of transformations $\Gamma: \Sigma_p \to \Sigma_q$ of class C^k, called the *displacements*, among associated manifolds, where by transformation $\Gamma: \Sigma_p \to \Sigma_q$ of class C^k we mean the same as in [2], satisfied the following rules called *connection*:

(α) for any two displacements $\Gamma, \Gamma' \in \mathfrak{C}: \Sigma_p \to \Sigma_q$ $(p, q \in X)$, $\Gamma^{-1}\Gamma'$ be a transformation of G, (β) for two displacements $\Gamma_1: \Sigma_p \to \Sigma_q$ and $\Gamma_2: \Sigma_q \to \Sigma_r$ $(p, q, r \in X)$, $\Gamma_2\Gamma_1 \in \mathfrak{C}: \Sigma_p \to \Sigma_r$, (γ) for any $\Gamma \in \mathfrak{C}$, its inverse transformation Γ^{-1} is also a displacement. (δ) for any two points p and $q \in X$, there exists a displacement $\Gamma \in \mathfrak{C}: \Sigma_p \to \Sigma_q$.

Above x_p and $U_p(x_p)$ will be called the *local coordinate system* on X at p and the *local coordinate neighborhood* of p with respect to x_p respectively. The set of coordinate transformations does not form a group but a pseudo-group. Let us now suppose that $\bar{f} \in \mathfrak{F}_p$ is expressed in a neighbourhood of p as $\bar{f} = \bar{f}^*(f)$ $(f \in \mathfrak{F}_p)$ where $\bar{f}^*(u)$ is a functional $C \to C$ of class C^k in a neighbourhood of a point $f(p)$ and the topological differential denoted by $d\bar{f}^*(u: du)$, then the linear space V_p of linear functionals

$$L : L\bar{f} = d\bar{f}^*(f(p): Lf)$$

of $\mathfrak{F}$ into C formed by composition rule $(L + L')(f) = L(f) + L'(f)$ will be called the *general tangent vector space* to X at p, which is no other than the linear space of mappings f expressed as $Lf = df^*(x_p(p); Lx_p)$ in terms of the components Lx_p of L with respect to x_p and is evidently the generalized notion of the tangent vector space of usual manifold of finite dimension. If $p = p^*$ and the general tangent vector space V_p to X at p for any point $p \in X$ coincides with the one V_p^* to Σ_p at p^*, then Σ_p and $p = p^*$ will be called the *tangent manifold* and the *point of contact* respectively. Thus, the general homogeneous manifold W becomes a genaral manifold with connection in our wide sense such that $W = \Sigma_p$ and $\Gamma: \Sigma_p \to \Sigma_q$ are $T_g: p \to q(g \in G)$, hence

$\mathfrak{G} = G$ and $p = p^*$. The set $\mathfrak{G}$ of displacements also does not form a group but a pseudo-group and for any point $p \in X$, $H_p \equiv \{\Gamma_p \in \mathfrak{G};\ \Sigma_p \to \Sigma_p\} \subset \mathfrak{G}$ forms a group called the *general holonomy group* of X at p, which becomes the *group of isotropy* at p for the homogeneous manifold W as a general manifold of connection in above stated sense. Then, for any two points o, $p \in X$, we have $H_o \approx H_p \approx H$ (homogeneity of the general holonomy group), for every displacements carrying Σ_o onto Σ_p take the form $\Gamma\Gamma_0(\Gamma_o \in H_o)$ when one of them is Γ, hence $H_o = \Gamma^{-1} H_p \Gamma \equiv \{\Gamma^{-1}\Gamma_p\Gamma : \Gamma_p \in H_p\}$. H will be called the *general holonomy group* of X which may be considered as $H \subset G$ from (1.3). If $H = \{\mathcal{E}\}$, then connection be called *holonomic*. In this case there exists only one displacement carrying Σ_p onto Σ_q for any two points p and q. It corresponds to effectivity of the group of structure in the case of homogeneous manifolds.

Now let any transformation $\Gamma_h \in H_o(h \in H)$ be expressed as

$$\Gamma_h : \overline{y}_{o^*(h)} = \overline{y}^*_{o^*(h)}\ (y_{o^*})\ (h \in H)$$

at o^* in terms of local coordinate system y_{o^*} at o^*, where $\overline{y}_{o^*(h)} \equiv y_{o^*}\Gamma_h$ and functional $\overline{y}^*_{o^*(h)}\ (u)$ is of class C^k, the topological differential of which will be denoted by $d\overline{y}^*_{o^*(h)}\ (u; du)$, then the group dH_o of linear homogeneous transformations

$$\gamma_h : \overline{e} = d\overline{y}^*_{o^*(h)}\ \ (y_{o^*}(o^*) : e)\ \ (h \in H; e, \overline{e} \in C)$$

will be called the *general homogeneous holonomy group* of X at o, which is no other than the group of linear homogeneous transformations operating the components of general tangent vectors at o induced by differential mappings $d_{o^*}\Gamma_h$ at o^* of transformations Γ_h of H_o, where by the differential mapping $d_{o^*}\Gamma_h$ we mean the same as in [2], provided that for each point $o \in X$ each point o^* of contact is invariant under the displacements by which we shall mean that every point of contact is changed to other point of contact, if Σ's are *non-linear* by which we shall mean that general tangent vector space at every point does not coincide with each other, since any $L_{o^*} \in V_{o^*}$ takes the form $L_{o^*}(f) = df^*(y_{o^*}(o^*);\ L_{o^*}y_{o^*})$ and $d_{o^*}\Gamma_o\ L_{o^*}(y_{o^*}) = L_{o^*}(y_{o^*}\Gamma_o) = L_{o^*}(\overline{y}_{o^*(h)})$ $= \overline{y}^*_{o^*(h)}\ (y_{o^*}(o^*);\ L_{o^*}y_{o^*})$, thus $L_{o^*}(y_{o^*})$ undergoes exactly transformation of γ_h. And if Σ's are n-dimensional homogeneous spaces and linear, that is, the general tangent vector spaces at every points coincide with each other, it is easily proved that Σ's are flat affine spaces, and conversely. Therefore dH_o may be considered as the generalized notion of usual homogeneous holonomy groups, which in the case of homogeneous manifolds as our manifold with general connection becomes the generalized notion of the usual linear group of isotropy, for in that case dH_o is, in fact, no other than the group of linear homogeneous transformations induced by differential mappings at o of transformations of the group H_o of isotropy at o under which the component of the general tangent vector at o is transformed. In the following, let Σ's be tangent, and if Σ's are non-linear, we shall suppose that any point o^* of contact is invariant under every displacement.

2. (H)-invariant V-differential forms. Consider now a *ξ-V-exterior differential form* (briefly ξ-V-differential form) Ω^{ξ} namely a mapping under which to each point $p \in X$ ξ-linear alternating function Ω^{ξ}_p on the general tangent vector space V_p having the value in Banach space V, namely, a ξ-V-exterior form $\Omega^{\xi}_p \in A^{\xi}_p (V_p, V)$ on V_p corresponds. Then the value Ω^{ξ}_p of Ω^{ξ} at p which is a ξ-V-exterior form on $V_p{}^* = V_p$ from our assumption may be considered as a ξ-V-differential form with constant coefficients on Σ if Σ's are linear. Now Ω^{ξ} will be called *invariant under the holonomy group H* or *(H)-invariant* if for any two points o and p of X and any displacement $\Gamma \in \mathfrak{C}$: $\Sigma_p \to \Sigma_o$, it follows that $\Omega_p = \Omega_o \Gamma$, and the additive group formed by all (H) invariant ξ-V-differential forms on X denoted by $D^{\xi}_{(H)}(X, V)$. Then for the exterior differential operator δ defined by M. Kerner [4], if we put as $Z^{\xi}_{(H)}(X, V) \equiv Z^{\xi}(X, V) \cap D^{\xi}_{(H)}(X, V) \equiv \{\delta\Theta^{\xi-1} \in D^{\xi}_{(H)}(X, V);\ \delta\Omega^{\xi} = 0\}$ and $B^{\xi}_{(H)}(X, V) \equiv \delta D^{\xi-1}_{(H)}(X, V) \cap D^{\xi}_{(H)}(X, V) \equiv \{\delta\Theta^{\xi-1} \in D^{\xi}_{(H)}(X, V), \Theta^{\xi-1} \in D^{\xi-1}_{(H)}(X, V)\}$, factor moduls $H^{\xi}_{(H)}(X, V) \equiv Z^{\xi}_{(H)}(X, V)/B^{\xi}_{(H)}(X, V)$ $(\xi = 0, 1, 2, \cdots)$ can be defined, which will be called the *ξ-dimensional cohomology group* of X *obtained using (H)-invariant V-differential forms.* In the case of the homogeneous manifolds, the above condition of (H)-invariance of Ω^{ξ} becomes $\Omega^{\xi} T_g = \Omega^{\xi} (g \in G)$, thus the (H)-invariant V-differarantial form in our sense becomes the invariant V-differential form, hence we have $\delta D^{\xi}_{(H)} \subset D^{\xi+1}_{(H)}$ and $B^{\xi}_{(H)} = \delta D^{\xi-1}_{(H)}$.

Now, hereafter, let us fix a point $o \in X$ and for any $\Omega^{\xi} \in D^{\xi}_{(H)}(X, V)$, consider the values $\Omega^{\xi}_0 \equiv \{\Omega^{\xi}\} \in A^{\xi}_0(V_o, V)$ at o, then from the definition we have $\{\Omega\}\Gamma_o = \{\Omega\}(\Gamma_o \in H)$, that is, $\{\Omega\}$ is (H)-invariant, therefore since we have $\{\Omega\}\Gamma_o(L_{o1}, \cdots, L_{o\xi}) = \{\Omega\}(d_o\Gamma_o L_{o1}, \cdots, d_o\Gamma_o L_{o\xi})$ $(L_{oi} \in V_o, i = 1, \cdots, \xi)$ and $d_o\Gamma_o L_{oi}(L_{oi} \in V_o; i = 1, \cdots, \xi)$ are the vectors resulted by γ_h under which the component is transformed, if $d_o\Gamma_o L_{oi}$ are denoted by $\gamma_h L_{oi}$ and define $\gamma_h\{\Omega\}$ by $\gamma_h\{\Omega\}(L_{o1}, \cdots, L_{o\xi}) = \{\Omega\}(\gamma_h L_{o1}, \cdots, \gamma_h L_{o\xi})$, we have $\gamma_h\{\Omega\} = \{\Omega\}$, therefore $\{\Omega\}$ also may be said to be (dH_o)-invariant. Conversely, let us consider any (dH_o)-invariant ξ-V-exterior form $\omega \in A^{\xi}_0(V_o, V)$ on V_o and denote the additive group formed by them by $D^{\xi}_{(dH_0)}(V_o, V)$ and for ω define the ξ-V-differential form $<\omega>$ on X as $<\omega>_p = \omega\Gamma(\Gamma \in \mathfrak{C}; \Sigma_p \to \Sigma_o)$, then this definition is independent of the choice of Γ, since every displacement $\Sigma_p \to \Sigma_o$ takes the form $\Gamma_o\Gamma(\Gamma_o \in H_o)$ when one of them is Γ, and we have $\omega(\Gamma_o\Gamma) = (\omega\Gamma_o)\Gamma = \omega\Gamma$. Moreover, $<\omega>$ thus obtained is (H)-invariant, since we have $<\omega>_p\Gamma'$

$= (\omega\Gamma)\,\Gamma' = \omega(\Gamma\Gamma') = <\omega>_{p'}$ for any two points p and p' of X and any Γ : $\Sigma_p \rightarrow \Sigma_o$; Γ': $\Sigma_{p'} \rightarrow \Sigma_p$. And for any $\Omega \in D^{\xi}_{(H)}(X, V)$ and any $\omega \in D^{\xi}_{(dH_o)}(V_o, V)$, we have $<\{\Omega\}>_p = \{\Omega\}\,\Gamma = \Omega_o\Gamma = \Omega_p(\Gamma: \Sigma_p \rightarrow \Sigma_o)$, that is, $<\{\Omega\}> = \Omega$, and $\{<\omega>\} = <\omega>_o = \omega\Gamma_o = \omega(\Gamma_o \in H_o)$, that is $\{<\omega>\} = \omega$, hence the correspondcnce $\Omega \longleftrightarrow \{\Omega\}$ or $\omega \longleftrightarrow <\omega>$ gives a one-to-one correspondence between $D^{\xi}_{(H)}(X, V)$ and $D^{\xi}_{(dH_o)}(V_o, V)$. Therefore, if for $\omega \in D^{\xi}_{(dH_c)}(V_o, V)$ we define the exterior differential as $\delta\omega = \{\delta<\omega>\}$, then we have $\delta\Omega = <\delta\{\Omega\}>$, hence the direct sum $D_{(dH_o)}(V_o, V)$ of $D^{\xi}_{(dH_o)}(V_o, V)$ for $\xi = 0, 1, 2, \cdots$ forms a graded group with the derived group $H_{(dH_o)}(V_o, V)$, the components $H^{\xi}_{(dH_o)}(V_o, V)$ of which are isomorphic with $H^{\xi}_{(H)}(X, V)$ and δ as the differential operator of degree one. Therefore, if we shall call $H^{\xi}_{(dH_o)}(V_o, V)$ *ξ-dimensional cohomology groups of V_o obtained using exterior ξ-V-forms invariant under the general homogeneous holonomy group dH_o* at o, thus we have the following

THEOREM *The cohomology groups of the general manifold X of class C^k $(k \geq 2)$ with general connection, Banach coordinate, a topological group G as the fundamental group and the tangent manifolds Σ obtained using V-exterior differential forms invariant under the holonomy group H, where V is a certain Banach space, are isomorphic with the cohomology groups of the general tangent vector space V_o at a point o of contact of the tangent manifold Σ_o obtained using ξ-V-exterior forms invariant under the general homogeneous holonomy group dH_o, provided that if every tangent manifolds are non-linear, each points of contact supposed to be invariant under every displacements*:

$$H^{\xi}_{(H)}(X, V) \approx H^{\xi}_{(dH_o)}(V_o, V)\ (\xi = 0, 1, 2, \cdots)$$

COROLLARY (*Generalized theorem of E. Cartan*). *The cohomology groups of the general homogeneous manifold W of class C^k $(k \geq 2)$ with Banach coordinate and a topological group G of transformations as the group of structure obtained using invariant ξ-V-differential forms are isomorphic with the cohomology groups of the general tangent vector space V_o at a point $o \in X$ obtained using ξ-V-exterior forms invariant under the general linear group of isotropy dH_o at o*:

$$H^{\xi}_{(G)}(W, V) \approx H^{\xi}_{(dH_o)}(V_o, V)\ \ (\xi = 0, 1, 2, \cdots).$$

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