ON ISOTOPY. I

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1. The following is well known. For each continuous mapping f (in the following we call it only a mapping) of an n-dimensional compactum $\mathfrak{A}^n$ [1] into an $2n+1$-dimensional Euclidean space R^{2n+1}, there exists a topological mapping of $\mathfrak{A}^n$ into R^{2n+1} which is arbitrarily near the giving mapping f [2. p. 56], [4].

In this paper we show an analogous theorem, that is, for each mapping f of $\mathfrak{A}^n$ into a combinatorial $2n+1$-dimensional manifold M^{2n+1}, there exists a topological mapping of $\mathfrak{A}^n$ into M^{2n+1} which is arbitrarily near the given mapping f (§ 2. Theorem 1) and apply it on isotopy (§ 3. Theorems 2 and 3). By a *combinatorial* $2n+1$-*dimensional manifold* we mean a compactum which is homeomorphic to a finite complex whose open star of each vertex is a combinatorial open $2n+1$-dimensional cell which is the image of an open $2n+1$-dimensional simplex under a simplicial homeomorphism with the suitable subdivisions.

Let X and Y be arbitrary compacta. X^Y denotes the set of all mappings of Y into X and it is as usual a metrizable complete space [2. pp. 55-56].

2. Lemma. *Let $A^m (m \leqq n)$ be an m-dimensional finite complex and M^{2n+1} be a combinatorial $2n+1$-dimensional manifold. For each mapping $f \in M^A$ and each number $\varepsilon > 0$, there exists a topological mapping g which belongs to M^A and $\rho(f, g) < \varepsilon$* [2].

Proof. By virtue of the simplicial approximation theorem [1. p. 319], the mapping f is indefinitely approximated by a simplicial mapping of the suitably subdivided A into the suitably subdivided M. Hence we may suppose that the mapping f is simplicial.

Let $z_1, z_2, \cdots, z_p$ be all vertices of M and $S_1, S_2, \cdots, S_p$ be their open stars. We apply a baricentric subdivision on M and let $z'_1, z'_2, \cdots, z'_q$ be all vertices of the subdivided M and $S'_1, S'_2, \cdots, S'_q$ be their closed stars. Then $\{S'_1, S'_2, \cdots, S'_q\}$ is a closed covering of M.

When it is necessary we apply the suitable subdivisions on A and on M, we may suppose that the diameter of each simplex of M does not exceed η (where $\varepsilon/3q > \eta > 0$) and an image of each simplex of A under the simplicial mapping f is contained in some S'_j (where $1 \leqq j \leqq q$).

We take some S'_j and S_i which contains S'_j and let $f^{-1}(S_i)$ be A_i and

1) When it is evident, superscripts which denote the dimensions are omitted.

2) ρ and d denote the distances.

$f^{-1}(S'_j)$ be A'_j. In general, A_i does not coincide with A'_j (otherwise our problem is reduced to Hurewicz's theorem [2. p. 56]), hence we may suppose that the closed star of A'_j in A is contained in A_i.

Since M is the combinatoral manifold, after suitable subdivisions, S_i is simplicial homeomorphic to the $2n+1$-dimensional open cell C_i which constitutes a convex set in certain Euclidean space R. We call this homeomorphism t_i. By suitable subdivisions of A_i, $t_i f: A_i \to C_i$ is a simplicial mapping.

In the sufficiently small neighborhood of $t_i f(a'_k)$ for each vertex a'_k of A'_j we correspond $\bar{a}'_k$ of C_i to a'_k one to one such that $\bar{a}'_k$ does not belong to any linear subspace of C_i which is spanned by every pair of $2r+2$ vertices which are $\bar{a}'_h(\neq \bar{a}'_k)$ and which belong to $t_i f(A_i)$, and not to $tf_i(A'_j), (r \leqq m)$. These correspondences are possible because A_i is m-dimenensional $(m \leqq n)$ and C_i is $2n+1$-dimensional [1. p. 596]. Then we correspond $t_i f(a_i)$ to a_i which belongs to A_i and not to A'_j. From these correspondences of the vertices, we can construct a mapping $\bar{g}_j$ which maps A_i into R and is linear on each simplex of A_i.

We can easily see that $\bar{g}_j$ (the closed star of A'_j) is a geometrical complex in R and since C_i is convex, $\bar{g}_j$ (the closed star of A'_j) is contained in C_i.

Then we subdivide C_i such that $\bar{g}_j$ (the closed star of A'_j) is a subcomplex of subdivided C_i and subdivide S_i and A_i such that $t_i^{-1}\bar{g}_j : A_i \to S_i$ is a simplicial mapping.

We construct a mapping $g_j : A_i \to M$ such that g_j coincides with f on the rest of M by the closed star of A'_j and coinsides with $t_i^{-1}\bar{g}_j$ on the closed star of A_j. By the construction we easily see $\rho(f, g_j) < \varepsilon'$ and g_j is topological on $g_j^{-1}(S'_j)$ (where $\varepsilon' = \varepsilon/q$).

If we put $j = 1$, then we have g_1 such that $\rho(f, g_1) < \varepsilon'$ and g_1 is topological on $g_1^{-1}(S'_1)$. Next instead of f we take g_1 and put $j = 2$, then we have g_2 such that g_2 is topological on $g_2^{-1}(S'_1 \cap S'_2)$ and so on. Finally we have $g_q \equiv g$ such that g is a topological mapping of A into M and

$$\rho(f,g) \leqq \rho(f,g_1) + \rho(g_1,g_2) + \cdots + \rho(g_{q-1},g_q) < q\varepsilon' = \varepsilon \qquad \text{Q. E. D.}$$

THEOREM 1. *Let* $\mathfrak{A}^m$ $(m \leqq n)$ *be an* m*-dimensional compactum and* M^{2n+1} *be a combinatorial* $2n+1$*-dimensional manifold. For each mapping* $f \in M^{\mathfrak{A}}$ *and each number* $\varepsilon > 0$, *there exists a topological mapping* g *which belongs to* $M^{\mathfrak{A}}$ *and* $\rho(f,g) < \varepsilon$.

PROOF. By virtue of the theoorem of Hurewicz [1. p. 109], we have to show the existence of a mapping g which belongs to $M^{\mathfrak{A}}$ and $\rho(f,g) < \varepsilon$ and is an ε-mapping [3].

3) When the inverse of each point of the image space has a diameter $< \varepsilon$ then the mapping is an ε-mapping.

If we realize M in certain Euclidean space R then M is a neighborhood retract of R, that is, there exists a compactum N in R which includes M in the interior and a retraction $\theta: N \to M$, where $\theta(z) = z$, for each $z \in M$.

We put $0 < \varepsilon' < \varepsilon/2$ and take β $(< \varepsilon'/3)$ such that for every $x,y \in N$,

$$d(x,y) < \beta \text{ implies } d(\theta(x), \theta(y))^{2)} < \varepsilon'/3$$

and take α $(0 < \alpha \leqq \beta/2)$ such that for each point z of M an α-neighborhood of z, $V(z, \alpha)$, is contained in N.

By compactness of $\mathfrak{A}$ and M we can select finite points $a_1, a_2, \cdots, a_r$ of $\mathfrak{A}$ such that the η-neighborhoods U_i of a_i constitute an η-covering, $(\eta < \varepsilon)$, $\{U_1, U_2, \cdots, U_r\}$ of $\mathfrak{A}$ which has order $\leqq m+1$ and

$$f(U_i) \subset \overline{V(f(a_i), \alpha)}^{4)} \cap M, \ (1 \leqq i \leqq r).$$

Let A' be a nerve of the covering $\{U_1, U_2, \cdots, U_r\}$ and a_i' be a vertex of $\mathfrak{A}'$ corresponding to U_i $(1 \leqq i \leqq r)$ and ϕ be a canonical mapping $\phi: \mathfrak{A} \to A'$. We know ϕ is an ε-mapping [2. p. 71].

Let A'' be a first baricentric subdivision of A' then a vertex a_i'' of A'' is a baricentre of some simplex $a_{i1}' \cdots a_{il}'$ of A' $(l \leqq r+1)$. Hence $f(U_{ij}) \subset \overline{V(f(a_{ij}'), \alpha)}$ and U_{ij} have a nonempty intersection and $\overline{V(f(a_{ij}'), \alpha)}$ have also a nonempty intersection $(j = 1, \cdots, l)$.

We correspond $\psi'(a_i'')$ to a_i'' in the intersection of $\overline{V(f(a_{ij}'), \alpha)}$ $(j=1,\cdots,l)$. (When $a_i'' = a_i'$ we put $\psi'(a_i'') = f(a_i')$).

We extend ψ' over R linearly on each simplex of A'' and take a mapping $\psi': A'' \to R$. Let $a_1'' \cdots a_l''$ be a simplex of A'', by the construction of ψ' there exists some $f(a_i')$ such that for each k, $(1 \leqq k \leqq l)$, $\psi'(a_k') \subset \overline{V(f(a_i), \alpha)}$. Hence $\psi'(A'') \subset N$ and $\psi\phi \in M^{\mathfrak{A}}$, where $\psi \equiv \theta\psi'$.

Let a be an arbitrary point of $\mathfrak{A}$. When a belongs to $U_{a1}, U_{a2}, \cdots, U_{al}$, $\phi(a)$ is in a simplex which is spanned by $f(a_1), \cdots, f(a_l)$. By the definition of $\psi\phi(a)$, for some i, $(1 \leqq i \leqq l)$, we show $d(\psi\phi(a), f(a_i)) < 2\varepsilon'/3$.

On the other hand $f(a)$ is contained in $\overline{V(f(a_i), \alpha)}$. Hence

$$\begin{aligned} d(f(a), \psi\phi(a)) &\leqq d(f(a), f(a_i)) + d(f(a_i), \psi\phi(a)) \\ &\leqq \frac{\varepsilon'}{3} + \frac{2\varepsilon'}{3} = \varepsilon', \end{aligned}$$

that is, $\rho(f, \psi\phi) < \varepsilon'$.

From Lemma for A' and ψ, there exists a topological mapping $\overline{\psi} \in M^{A'}$ such that $\rho(\overline{\psi}, \psi) < \varepsilon'$. Hence $g \equiv \overline{\psi}\phi \in M^{\mathfrak{A}}$ is an ε-mapping and $\rho(f,g) < \varepsilon$. Q. E. D.

3. The two mappings f and g of $M^{\mathfrak{A}}$ are said to be homotopic, whenever there exists a mapping $F=\{f_t\}: \mathfrak{A} \times I \to M$ which is called homotopy such that f_0 and f_1 coincide with the giving mappings f and g respectively, where I is an

4) The upper bar of the set denotes the closure of the set.

interval $0 \leqq t \leqq 1$. When, for each t, $(t \neq 0$ and $\neq 1)$, f_t is topological, two mappings are *isotopic* and $F=\{f_t\}$ is called an *isotopy*. $M^{\mathfrak{A}}$ are classified by the homotopic relation and these classes are called homotopy classes. Equally the subspace of $M^{\mathfrak{A}}$ which is constituted by all topological mappings are also classified by the isotopic relation and these classes are called *isotopy classes*.

THEOREM 2. *Let $\mathfrak{A}^m$ $(m \leqq n)$ be an m-dimensional compactum and M^{2n+3} be a combinatorial $2n+3$-dimensional manifold. When the two topological mappings of $M^{\mathfrak{A}}$ are homotopic then they are isotopic.*

PROOF. Let $f_0, f_1 \in M^{\mathfrak{A}}$ be homotopic topological mappings and $F=\{f_t\}$ be their homotopy. In the space $M^{\mathfrak{A}\times I}$, we consider a subspace $M_0^{\mathfrak{A}\times I}$ which is constituted by all homotopies which coincide with f_0 and f_1 for $t=0$ and $t=1$ respectively. We know easily that $M^{\mathfrak{A}\times I}$ is also a metrizable complete space. Let M_j^* (where j is an integer $\geqq 3$) be a subset of $M_0^{\mathfrak{A}\times I}$ which is constituted by all homotopies $G=\{g_t\}$ of $M_0^{\mathfrak{A}\times I}$ which are, for $1/j \leqq t \leqq (j-1)/j$, $1/j$-mappings. If $\bigcap_{j=}^{\infty} M_j^*$ is not empty then there exists a required isotopy. Since M_j^* is open set of $M_0^{\mathfrak{A}\times I}$, by the well known theorem [1. p. 108] we have to show that for each homotopy $F=\{f_t\} \in M_0^{\mathfrak{A}\times I}$ there exists a homotopy $G=\{g_t\} \in M_j^*$ such that $\rho(F,G) < \varepsilon$, where $\varepsilon > 0$ is an arbitrary number.

For each point z of M we consider an α-neighborhood $V(z,\alpha)$, and as the above, we can take α such that $x, y \in V(z,\alpha)$ means $d(\theta(x),\theta(y)) < \varepsilon/2 = \varepsilon'$.

Since $\mathfrak{A} \times I$ is $n+1$-dimensional, by Theorem 1 there exists a topological mapping $\overline{G}=\{\overline{g}_t\} \in M_0^{\mathfrak{A}\times I}$ such that for sufficiently small $\eta > 0$, $\rho(f_0, \overline{g}_\eta) < \alpha$ $(\rho(f_1, \overline{g}_{1-\eta}) < \alpha)$.

For each $a \in \mathfrak{A}$, $\overline{g}_\eta(a)$ and $f_0(a)$ $(f_1(a)$ and $\overline{g}_{1-\eta}(a))$ are in the same $V(z,\alpha)$. We join these two points by the segment and divide it by $\overline{\overline{g}}_t(a)$ $(\overline{\overline{g}}'_t(a))$ with the ratio $t : 1-t$ and put

$$G=\{g_t\}=\begin{cases} \theta\, \overline{\overline{g}}_{\frac{t}{\eta}} & (0 \leqq t \leqq \eta) \\ \overline{g}_t & (\eta \leqq t \leqq 1-\eta) \\ \theta\, \overline{\overline{g}}'_{\frac{t-(1-\eta)}{\eta}} & (1-\eta \leqq t \leqq 1). \end{cases}$$

By the construction $G \in M^{\mathfrak{A}\times I}$ and, for $\eta \leqq t \leqq 1-\eta$, g_t is topological and $\rho(F,G) < 2\alpha + \varepsilon'$. Hence if we take $\eta < 1/j$ and $\alpha < \varepsilon/4$, G is a required homotopy. Q.E.D.

THEOREM 3. *Let $\mathfrak{A}^m$ $(m \leqq n)$ be an m-dimensional compactum and M^{2n+3} be a combinatorial $(2n+3)$-dimensional maifold. In this case the homotopy classes correspond one to one to the isotopy classes.*

PROOF. since $M^{\mathfrak{A}}$ is a locally contractible space [4. p 113], for each $f \in M^{\mathfrak{A}}$ there exists an $\varepsilon > 0$ such that $g \in M^{\mathfrak{A}}$ and $\rho(f,g) < \varepsilon$ mean that f and g is homotopic.

Hence by Theorem 1 we know there exists an isotopy class in each homtopy class.

On the other hand by Theorem 2 the two isotopy classes which are in the same homtopy class are identical. Q.E.D.

REMARK. We can easily show that the above theorems even hold without restriction of finiteness of combinatorial manifold.

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