ON GENERALIZATIONS OF HOPF'S CLASSIFICATION THEOREMS

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Hopf's brilliant results have been generalized to various general cases by H. Whitney [13][1], P. Alexandroff [1], S. Eilenberg [3]. C. H. Dowker [2] and S. T. Hu [4].

The purpose of this paper is to generalize the Hu's work [4] and to obtain Hopf's classification theorems which contain the results of Dowker [2] and Hu [4].

1. FUNDAMENTAL BRIDGE THEOREMS.

In this section we shall provide Hu's fundamental bridge theorems in more general forms which are the main tools of his paper [4]. Therefore the majority of his results in [4] will be generalized.

Let X be a normal space and X_0 be its closed subset. By a covering [finite covering] α of X we shall always means a locally finite [finite] open covering of X. The nerve of α with the weak topology (see [8, Definition 3.1]) is denoted by A. The nerve A_0 of the covering $\alpha \cap X_0$[2] *may be regarded* as a subcomplex of A. Let denote a canonical mapping (see [8, Definition 3.2]) of (X, X_0) into (A, A_0). Since X is normal and coverings considered are locally finitet, here always exist canonical mappings [7_2, Lemma 2].

Let f be a given mapping of X_0 into an arbitrary space Y and α a covering of X. A mapping $\psi_\alpha : A_0 \to Y$ is called a bridge mapping for f, if the partial mapping $\psi_\alpha\phi_\alpha | X_0$ is homotopic to f for each canonical mapping $\phi_\alpha : (X, X_0) \to (A, A_0)$. If such a bridge mapping exists, α is said to be a bridge for the mapping f. In particular, if Y is compact and α is a finite covering and $\psi_\alpha\phi_\alpha | X_0$ is uniformly homotopic [references 7_1, §2, p.86] to f, then ψ_α and α are called a finite bridge mapping and a finite bridge for f respectively.

BRIDGE REFINEMENT THEOREM. *For a given mapping $f: X_0 \to Y$, any locally finite refinement β of a bridge α is a bridge.*

PROOF. Let $\psi_\alpha : A_0 \to Y$ be a bridge mapping for f; $\phi_\alpha : (X, X_0) \to (A, A_0)$ be an arbitrary canonical mapping. Since β is a refinement of α there exists a simplicial projection $p_{\beta\alpha} : (B, B_0) \to (A, A_0)$ [8, § 9]. By the first half of Corollary 9.3 [8], $\phi_\alpha \simeq p_{\beta\alpha}\phi_\beta$. Hence

$$f \simeq \psi_\alpha\phi_\alpha | X_0 \simeq \psi_\alpha p_{\beta\alpha}\phi_\beta | X_0.$$

Thus β is a bridge and $\psi_\alpha p_{\beta\alpha} : B_0 \to Y$ is a brige mapping for f.

1) Numbers in square brackets refer to the references cited at the end of this paper.

2) $\alpha \cap X_0$ means a covering of X_0 consisting of all intersections of X_0 and elements of α.

FINITE BRIDGE REFINEMENT THEOREM. *Let Y be a compact space. For a given mapping : $X_0 \to Y$, any finite refinement β of a finite bridge α is a finite bridge.*

PROOF. Let ψ_α, ϕ_α and $p_{\beta\alpha}$ be the same as in the preceding theorem. In the present case A and B are both finite complexes. By the second part of Corollary 9.3 [8], $\phi_\alpha \overset{u}{\simeq} p_{\beta\alpha}\phi_\beta$, hence we have

$$f \overset{u}{\simeq} \psi_\alpha\phi_\alpha | X_0, \ \overset{u}{\simeq} \psi_\alpha p_{\beta\alpha}\phi_\beta | X_0.$$

Thus β is a finite bridge with a bridge mapping $\psi_\alpha p_{\beta\alpha}$.

BRIDGE EXISTENCE THEOREM. *If Y is dominated*[3] *by a CW-complex*[4], *every mapping f: $X_0 \to Y$ has a bridge.*

FINITE BRIDGE EXISTENCE THEOREM. *If Y is compact and is dominated by a CW-complex, then every mapping f: $X_0 \to Y$ has a finite bridge.*

PROOF. Any CW-complex is of the same homotopy type as its singular complex [12, Theorem 23] and the singular complex of a space has a simplicial decomposition [5, (7.1), p.172]. Hence Y is dominated by a simplex. Furthermore, if Y is compact then Y is dominated by a finite simplicial complex [12, Appendix A, p.107].

Let $\lambda: Y \to P$ and $\mu: P \to Y$ be mappings such that $\mu\lambda \simeq 1$, where P is a simplicial complex, and in the case where Y is compact, P is finite simplicial complex. Let $\{st\, p_j\}$ be the covering consisting of all open stars of vertices $p_j \in P$, and set $\alpha_0' = \{(\lambda f)^{-1}(st\, p_j)\}$. Then α_0' is a covering of A_0 and if Y is compact, then α_0' is a finite covering. Let $\alpha = \{a_0, a_j\}$ be a family of open sets in X such that $a_0 = X - X_0$, $a_j \cap X = (\lambda f)^{-1}(stp_j)$. Let $\phi^\alpha: (X, X_0) \to (A, A_0)$ be a canonical mapping and τ the simplicial mapping defined by the vertices correpsondence $a_j \to p_j$. Since $\phi_\alpha | X_0$ is a canonical mapping of X_0 into the nerve A_0 of the covering $\alpha \cap X_0$, by Lemma 3.4 [8], $\lambda f \simeq \tau\ \phi_\alpha | X_0$ and if A is finite then $\lambda\ f \overset{u}{\simeq} \tau\phi_\alpha | X_0$. Hence $f \simeq \mu\lambda f \simeq \mu\tau\phi_\alpha | X_0$, and if Y is compact and P is finite then $f \overset{u}{\simeq} \mu\lambda\ f \overset{u}{\simeq} \mu\tau\phi_\alpha | X_0$ because $\mu\lambda$ is uniformly continuous. Thus α and $\psi_\alpha = \mu\tau$ are a bridge and a bridge mapping for f and in the case where Y is compact, these are a finite bridge and a finite bridge mapping for f. Thus the above two theorems are established.

BRIDGE HOMOTOPY THEOREM. *Let X be a normal space and X_0 its closed paracompact*[5] *space. Let α and β be two bridges for a given mapping f of X_0 into a space Y which is dominated by a CW-complex, and let $\psi_\alpha: A_0 \to Y$, $\psi_\beta: B_0 \to Y$ be bridge mappings. Then there exists a common refinement*

3) See [11, p.214].

4) See [11, §5, p.223].

5) A topological space X_0 is said to be paracompact if any open covering (not necessary locally finite) of X has a locally finite refinement.

γ of α and β such that $\psi_\alpha p_{\gamma\alpha}|\, C_0$ and $\psi_\beta p_{\gamma\beta}|C_0$ are homotopic, where $p_{\gamma\alpha}$: $(C, C_0)\to(A, A_0)$, $p_{\gamma\beta}$; $(C, C_0)\to(B, B_0)$ are arbitrary projections.

Proof. By the hypothesis, there exist a simplicial complex P and mappings λ: $Y\to P$, μ: $P\to Y$ such that $\mu\lambda\simeq 1$. Let us put $g=\lambda f$, $\psi'_\alpha=\lambda\psi_\alpha$, $\psi'_\beta=\lambda\psi_\beta$. Since ψ_α, ψ_β are bridge mappings for f, we have

$$\psi'_\alpha\phi_\alpha|X_0\simeq g\simeq\psi'_\beta\phi_\beta X_0.$$

By the proof of the first half of Theorem 9.4 [8], it is easily seen that there exists a common refinement γ of α and β such that

$$\psi'_\alpha p_{\gamma\alpha}|\, C_0\simeq\psi'_\beta p_{\gamma\beta}|C_0.$$

Hence

$$\psi_\alpha p_{\gamma\alpha}|C_0\simeq\mu\lambda\,\psi_\alpha p_{\gamma\alpha}|C_0=\mu\psi'_\alpha p_{\gamma\alpha}|C_0$$
$$\simeq\mu\psi'_\beta p_{\gamma\beta}|C_0=\mu\lambda\psi_\beta p_{\gamma\beta}|C_0\simeq\psi_\beta p_{\gamma\beta}|C_0.$$

This completes the proof.

FINITE BRIDGE HOMOTOPY THEOREM. *Let X be a normal space and X_0 its closed subset. Let α and β be two finite bridges for a given mapping f of X_0 into a compact normal space Y which is dominated by a CW-complex, and let $\psi_\alpha: A_0\to Y$, $\psi_\beta: B_0\to Y$, be finite bridge mappings. Then there exists a common finite refinement γ of α and β such that $\psi_\alpha p_{\gamma\alpha}|C_0$ and $\psi_\beta p_{\gamma\beta}|C_0$ are uniformly homotopic, where $p_{\gamma\alpha}$: $(C, C_0)\to(A, A_0)$, $p_{\gamma\beta}$: $(C, C_0)\to(B, B_0)$ are arbitrary projections.*

PROOF. By the hypothesis, there exist a finite simplicial complex P and mappings λ: $Y\to P$, μ: $P\to Y$ such that $\mu\lambda\simeq 1$. Since P and Y are compact we know that $\mu\lambda\overset{u}{\simeq}1$. Since ψ_α, ψ_β are finite bridge mappings for f, and λ is uniformly continuous, we have

$$\psi'_\alpha\phi_\alpha|X_0\overset{u}{\simeq}g\simeq\psi'_\beta\phi_\beta|X_0.$$

where $g=\lambda f$, $\psi'_\alpha=\lambda\psi_\alpha$, $\psi'_\beta=\lambda\psi_\beta$.

By the proof of the second half of Theorem 9.4 [8], there exists a common finite refinement γ of α and β such that

$$\psi'_\alpha p^\gamma_\alpha|C_0\overset{u}{\simeq}\psi'_\beta p_{\gamma\beta}|C_0.$$

Hence we have

$$\psi_\alpha p_{\gamma\alpha}|C_0\overset{u}{\simeq}\psi_\beta p_{\gamma\beta}|C_0.$$

This completes the proof.

2. HOPF'S CLASSIFICATION THEOREMS.

In this section we shall assume that Y is a connected space dominated by a CW-complex and satisfying $\pi_r(Y)=0$ for each $1\leqq r\leqq n$, where $\pi_r(Y)$ denotes r^{th} homotpoy group of Y. If $n>1$, the latter condition implies $\pi_1(Y)=0$ and hence the i-simplicity of Y for all i. If $n=1$ we assume the i-simplicity of Y for each $i\leqq m$, where m is an integer to be specified in the sequel.

Let α be an arbitrary bridge for $f: X \to Y$ with $\psi_\alpha: A \to Y$ as a bridge mapping. Since $\pi_r(Y) = 0$ for each $r < n$, we may assume that $\psi_\alpha(A^{n-1}) = y_0$, where A^q denotes the q-skeleton of A and y_0 is a fixed point in Y. For each oriented n-simplex $\sigma_i^n \in A$, the partial mapping $\psi_\alpha | \sigma_i^n$ determines an element $(\psi_\alpha, \sigma_i^n)$ of homotopy group $\pi_n(Y)$. Since ψ_α is defined throughout A, the n-cochain

$$k^n(\psi_\alpha) = \Sigma (\psi_\alpha, \sigma_i^n)\sigma_i^n$$

is clearly a cocyle of A.

By the same way as in the proof of (9.1) [4, p. 353], we know that all the possible cocycles $k^n(\psi_\alpha)$ represent a unique element $\kappa^n(f)$ of $H^n(X, \pi_n(Y))$[6], where we assumed that X is paracompact.

If Y is a compact normal space we restrict ourself to all finite bridges α and finite bridge mappings ψ_α, then it is also seen that all the possible cocycles $k^n(\psi^\alpha)$ represent a unique element $\kappa_F^n(f)$ of $H_F^n(X, \pi_n(Y))$[7]. In this case we do not assume the paracompactness of X. The elements $\kappa^n(f)$ and $\kappa_F^n(f)$ are both called the characteristic elements of f.

Now we can state the following generalized Hopf's classification theorems.

THEOREM I. *If X is a paracompact normal space with $\dim X \leqq m$[8] and $H^r(X, \pi_r(Y)) = 0 = H^{r+1}(X, \pi_r(Y))$ for each $n < r \leqq m$, then the elements of $H^n(X, \pi_n(Y))$ are in a $(1-1)$-correspondence with the homotopy classes of the mappings $f: X \to Y$. The correspondence is determined by the operation $\kappa^n(f)$.*

THEOREM II. *If X is a normal space with $\dim X \leqq n$, and $H^r(X, \pi_r(Y)) = 0 = H^{r+1}(X, \pi_r(Y))$ for each $n < r \leqq m$, then the elements of $H^n(X, \pi_n(Y))$ are in a $(1-1)$-correspondence with the uniform homotopy classes of the mapping $f: X \to Y$. The correspondence is determined by the operation $\kappa_F^n(f)$, where Y is compact.*

If Y is paracompact and normal, then the characteristic element $\kappa^n(\tau) \in H^n(Y, \pi_n(Y))$ of the identity mapping $\tau: Y \to Y$ can be considered. And if Y is compact, $\kappa_F^n(\tau)$ may be regarded as an element of $H_F^n(Y, \pi_n(Y))$ $(\approx H^n(Y, \pi_n(Y)))$.

THEOREM I'. *If X is a paracompact normal space with $\dim X \leqq n$ and $\pi_r(Y) = 0$ except n, then the homotopy classes of mappings $f: X \to Y$ are in a $(1-1)$-correspondence with the group $H^n(X, \pi_n(Y))$. The correspondence is determined by the operation $f \to f^*(\kappa^n(\tau))$, where $f^*: H^n(Y, \pi_n(Y))$ is the homomorphism induced by the mapping f.*

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6) $H^n(X, G)$ denotes the n^{th} Čech cohomology group of X with coefficients in G based on all open cpverings (not necessary locally finite) of X. See [2, §4, p. 213].

7) $H_F^n(X, G)$ denotes the n th Čech cohomology group of X with coefficients in G based on all finite open coverings of X.

8) For the definition on dim X see [2, §3, p. 206].

THEOREM II'. *If X is a normal space with dim $X \leqq n$, Y is a compact normal space and $\pi_r(Y)=0$ except n, then the uniform homootpy classes of mappings $f\colon X \to Y$ are in a $(1-1)$-correspondence with the group $H^n(X, \pi_n(Y))$. The correspondence is determined by the operation $f \to f^{*F}(\kappa^n(\tau))$, where f^{*F}: $H^n(Y, \pi_n(Y)) \to H^n(X, \pi_n(Y))$ is the homomorphism induced by the mapping f.*

These Theorems I, II, I' and II' can be easily obtained by the same arguments as in [4] if we use our fundamental bridge and finite brige theorems instead of Hu's fundamental bridge theorems and using of Theorem 3.5 [2]. And so we shall omit the complete proof.

REMARK. Theorems I' and II' contain Theorem 7.5 and 9.3 [2] as special cases. It is immediately follows by a theorem proved in Appendix that our theorem I', II, I' and II' are generalizations of the Hu's results [4, p. 356]. We shall also notice that the majority of Hu's work [4] can be generalized in our general cases.

APPENDIX

A metric [separable metric] space Y is said to be a ANR [separable ANR] if for any metric [separable metric] space Z which contains Y as its closed subset, Y is a neighborhood retract[9] of Z. We shall prove the following theorem[10]

THEOREM. *If Y is a ANR or separable ANR then Y is dominated by a simplicial complex with the weak topology.*

PROOF. According to a theorem due to Wojdyslawski [14, p.186], Y can be imbedded as a closed set of a convex subset Z of a Banach space W, and W is separable when Y is separable. Since Y is ANR [or separable ANR], there exist an open set V of Z containing Y and a retraction $\theta\colon V \to Y$. For each point $y \in Y$, let $S(y)$ denote an open spherical neighborhood of y in Z such that $S(y) \subset V$. Since Z is convex, $S(y)$ is also convex. Y is metric and hence paracompact[11]. Therefore there exists a locally finite open covering $\{U_\alpha\}$ of Y which is a refinement of $\{S(y) \cap Y\}$. Let K be the nerve of the covering $\{U_\alpha\}$ and let u denote the vertex corresponding to a element U_α. For a finite points $y_0, \cdots, y_n \in Y$ let $[y_0, \cdots, y_n]$ denote the minimal convex set containing $y_0, \cdots, y_n$ in W. Hence $[y_0, \cdots, y_n]$ consists of points $a_0y_0 + \cdots + a_n y_n$, where $a_0, \cdots, a_n$ are non-negative real numbers such that $a_0 + \cdots + a_n = 1$.

For each vertex $u_\alpha \in K$, let us choose a point $y_\alpha \in Y$ such that $U_\alpha \subset S(y_\alpha)$.

9) See [6, § 5, p. 58].

10) This theorem for compact separable ANR is well-known. See [6, Theorems 12.2, 16.2, pp. 93, 99].

11) See [9, Theorem 1] and [10, Theorem 8, 14, p. 53].

For each n-simplex $\sigma = (u_{\alpha_0}, \cdots, u_{\alpha_n}) \subset K$ we define a mapping $\nu_\sigma : \bar{\sigma} \to [y_{\alpha_0}, \cdots, y_{\alpha_n}]$ by taking $\nu_\sigma(x)\ a_0 y_{\alpha_0} + \cdots + a_n y_{\alpha_n} (x \in \bar{\sigma})$, where $a_0, \cdots a_n$ are barycentric coordinates of x with respect to the vertices $u_{\alpha_0}, \cdots, u_{\alpha_n}$. Then ν_σ define a continuous mapping

$$\nu : K \to W.$$

If $u_{\alpha_0}, \cdots, u_{\alpha_n}$ are vertices of a simplex in K, then $U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \neq 0$, hence $S(y_{\alpha_0}) \cap \cdots \cap S(y_{\alpha_n}) \neq 0$. Hence $S(y_{\alpha}) \cup \cdots \cup S(y_{\alpha_n}) \supset [y_{\alpha_0}, \cdots, y_{\alpha_n}]$. Therefore $\nu(K) \subset V$.

We put $\mu = \theta\nu$. Let $\lambda : Y \to K$ be a canonical mapping. It is remains to show that $\mu\lambda \simeq 1$.

For each point $y \in Y$ let $U_{\alpha_0}, \cdots, U_{\alpha_n}$ be all elements of $\{U_\alpha\}$ which contain the point y. Then $\lambda(y) \in \overline{(u_{\alpha_0}, \cdots, u_{\alpha_n})}$. Hence $\nu\lambda(y) \in [y_{\alpha_0}, \cdots, y_{\alpha_n}]$. On the other hand $y \in U_{\alpha_0} \cap \cdots \cap U_{\alpha_n} \subset S(y_{\alpha_0}) \cap \cdots \cap S(y_{\alpha_n})$. Hence a point $\rho_t(y)$ which divides the segment joining $\nu\lambda(y)$ and y into a ratio $1 - t : t (0 \leqq t \leqq 1)$ always belongs to som $S(y_{\alpha_j})$ and hence belongs to V. Hence

$$\xi_t = \theta\rho_t : Y \to Y$$

is a well-defined homotopy between $\xi_1(y) = \theta\rho_1(y) = \theta\nu\lambda(y)$ and $\xi_0(y) = \theta\rho_0(y) = \theta(y) = y$. Hence $\mu\lambda \simeq 1$. This completes the proof.

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