HANKEL DETERMINANTS AND BERNOULLI NUMBERS

L. CARLITZ

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1. Introduction. A *Hankel determinant* is one of the type

$$\Delta = |a_{i+j}| \qquad (i, j = 0, 1, \ldots, r), \tag{1.1}$$

where $\{a_i\}$ is an arbitrary sequence. It follows at once that

$$\Delta = |\Delta^{i+j} a_0| \qquad (i, j = 0, 1, \ldots, r), \tag{1.2}$$

where

$$\Delta a_m = a_{m+1} - a_m, \ \Delta^r a_m = \Delta^{r-1} a_{m+1} - \Delta^{r-1} a_m.$$

(See for example [4, § 51]). Now suppose the a_i are rational integers such that

$$\Delta^r a_0 \equiv 0 \pmod{M^r} \qquad (r \geqq 1), \tag{1.3}$$

where M is some fixed integer. It follows at once from (1.2) and (1.3) that

$$\Delta \equiv 0 \pmod{M^{r(r+1)}}. \tag{1.4}$$

Indeed (1.4) holds when the a_i are rational numbers that are integral (mod M) and satisfy (1.3).

We shall now construct some examples of Hankel determinants of Bernoulli and Euler numbers that satisfy congruences of the form (1.4).

2. Euler numbers. The *Euler numbers* E_m may be defined by means of

$$\frac{2}{e^x + e^{-x}} = \sum_{m=0}^{\infty} E_m \frac{x^m}{m!}.$$

It is well known that they satisfy Kummer's congruences:

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} E_{m+sb} \equiv 0 \pmod{p^{er}} \qquad (m \geqq re), \tag{2.1}$$

where p is a prime $\geqq 3$, $e \geqq 1$ and $p^{e-1}(p-1) | b$. For proof see for example [5, Chapter 14]. Hence if we put

$$a_i = E_{m+ib} \qquad (i = 0, 1, 2, \ldots),$$

it is clear that (1.3) is satisfied with $M = p^e$; we may therefore assert that

$$|E_{m+(i+j)b}| \equiv 0 \pmod{p^{er(r+1)}} \qquad (i, j = 0, 1, \ldots, r) \tag{2.2}$$

provided $m \geqq re$.

Somewhat more generally, the numbers $E_m^{(k)}$ of order k defined by [6, p. 143]

$$\left(\frac{2}{e^{x/2} + e^{-x/2}}\right)^k = \sum_{m=0}^{\infty} E_m^{(k)} \frac{x^m}{m!},$$

where k is an integer $\geqq 1$, also satisfy the congruence (2.1) so that

$$\left|E^{(k)}_{m+(i+j)b}\right| \equiv 0 \pmod{p^{er(r+1)}} \qquad (i,j=0,\dots,r), \tag{2.3}$$

provided $m \geqq er$. For $k=1$, (2.3) reduces to (2.2). A like result holds for the numbers $C^{(k)}_m$ defined by [6, p. 143]

$$\left(\frac{2}{e^x+1}\right)^k = \sum_{m=0}^{\infty} \frac{C^{(k)}_m}{2^m} \frac{x^m}{m!}.$$

Additional examples of the same kind are easily constructed.

3. Bernoulli numbers. The *Bernoulli numbers* B_m may be defined by means of

$$\frac{x}{e^x-1} = \sum_{m=0}^{\infty} B_m \frac{x^m}{m!}.$$

They satisfy [6, Chapter 14]

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}\frac{B_{m+sb}}{m+sb} \equiv 0 \pmod{p^{er}} \qquad (m > re), \tag{3.1}$$

where as above $p^{e-1}(p-1)\mid b$; in addition we must assume $p-1 \nmid m$. Hence if we put

$$a_i = B_{m+ib}/(m+ib) \qquad (i=0,1,2,\dots),$$

(1.4) implies

$$\left|\frac{B_{m+(i+j)b}}{m+(i+j)b}\right| \equiv 0 \pmod{p^{er(r+1)}} \quad (i,j=0,1,\dots,r), \tag{3.2}$$

provided $m > re$ and $p-1 \nmid m$.

A result like (3.2) for the determinant $|B_{m+(i+j)b}|$ can also be obtained. Indeed Nielsen has proved [5, Chapter 14] the congruence

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}B_{m+sb} \equiv 0 \pmod{p^{e(r-1)}} \qquad (r \geqq 1), \tag{3.3}$$

provided $p-1 \nmid m$ and $m > er$. Hence modifying (1.4) slightly we get

$$|B_{m+(i+j)b}| \equiv 0 \pmod{p^{er(r-1)}} \qquad (i,j=0,1,\dots,r), \tag{3.4}$$

provided $p-1 \nmid m$ and $m > er$.

The condition $p-1 \nmid m$ may be waived in certain cases. Vandiver [7] has proved the congruence

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}B_{(m+s)p-1} \equiv 0 \qquad \pmod{p^{r-1}}, \tag{3.5}$$

where $m \geqq 1$, $r \geqq 1$, $m+r < p-1$. This result evidently implies

$$|B_{(m+i+j)p-1}| \equiv 0 \pmod{p^{r(r-1)}} \qquad (i,j=0,1,\dots,r), \tag{3.6}$$

provided $m \geqq 1$, $m+r<p-1$. The writer [1] has extended (3.5) in several directions. We quote two such extensions. In the first place

$$\sum_{s=0} (-1)^{r-s} \binom{r}{s} \sigma_{(m+s)(p-1)} \equiv 0 \pmod{p^r}, \tag{3.7}$$

where $m \geqq 1$, $r \geqq 1$, $m + r \geqq p - 1$, and

$$\sigma_{m(p-1)} = (B_{m(p-1)} + \frac{1}{p} - 1)/m;$$

we remark that $\sigma_{m(p-1)}$ is integral (mod p). It evidently follows from (3.7) that

$$|\sigma_{(m+i+j)(p-1)}| \equiv 0 \pmod{p^{r(r+1)}} \qquad (i, j = 0, 1, \ldots . r), \tag{3.8}$$

provided $m \geqq 1$, $m + r \geqq p - 1$.

Secondly we have

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{m+sb} \equiv 0 \pmod{p^{re-h}}, \tag{3.9}$$

where $(p-1)p^{e-1} | b, p - 1 | m$, $m > re$, and $h = e$ for $r < p$ (except perhaps when $r = p - 1$, $e = 1$ and $h = 2$), while for $r \geqq p$, h is the least integer $\geqq (re + 1)/p$. In particular therefore (3.9) implies

$$|B_{m+(i+j)b}| \equiv 0 \pmod{p^{er(r-1)}} \qquad (i, j = 0, 1, \ldots, r), \tag{3.10}$$

provided $m > re$, $r < p - 1$. For $e = 1$, (3.10) evidently includes (3.6).

Turning next to the Bernoulli numbers $B_m^{(k)}$ of order k defined by [6, p. 143]

$$\left(\frac{x}{e^x - 1}\right)^k = \sum_{m=0}^{\infty} B_m^{(k)} \frac{x^m}{m!},$$

we quote the results [2, Theorems 5, 6]

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} T_{m+sb}^{(k)} \equiv 0 \pmod{p^{re}}, \tag{3.11}$$

where

$$T_m^{(k)} = B_m^{(k)}/(m)_k, \qquad (m)_k = m(m-1) \ldots . (m - k + 1);$$

$$\sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} B_{m+sb}^{(k)} \equiv 0 \pmod{p^{(r-k)e}}. \tag{3.12}$$

In both (3.11) and (3.12) it is assumed that

$$k < p - 1; \ m \not\equiv 0, 1, \ldots, \ k - 1 \pmod{p - 1}; \ m \geqq re + k. \tag{3.13}$$

(Note that the condition $m \geqq rb + k$ in Theorems 4, 5, 6 of [2] may be replaced by $m \geqq re + b$.) In (3.12) it is also assumed that $r \geqq k$.

An immediate consequence of (3.11) and (1.4) is

$$|T_{m+(i+j)b}^{(k)}| \equiv 0 \pmod{p^{er(r+1)}} \qquad (i, j = 0, \ldots, 1, r), \tag{3.14}$$

provided (3.13) holds. Making use of (3.12) we get

$$|B_{m+(i+j)b}^{(k)}| \equiv 0 \pmod{p^{e(r-k)(r-k+1)}} \quad (i, j = 0, 1, \ldots, r), \tag{3.15}$$

provided (3.13) holds and $r \geqq k$. For $r < k$ we can only assert that the left member of (3.15) is integral (mod p).

4. **Coefficients of the Jacobi elliptic functions.** Not only the Euler and Bernoulli numbers satisfy Kummer's congruences but certain other sequences as well. In particular if

$$\operatorname{sn} x = \operatorname{sn}(x, u) = \sum_{m=1}^{\infty} A_m(u) x^m / m!$$

denotes the Jacobi elliptic function, then it is familiar that the $A_m(u)$ are polynomials in u with integral coefficients. The writer has proved [3] that the $A_m(u)$ satisfy the congruence

$$(4.1) \qquad \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} A_p^{r-s}(u)\, A_{m+s(p-1)}(u) \equiv 0 \qquad (\text{mod } p^r) \quad (m \geqq r),$$

and indeed

$$(4.2) \qquad \sum_{s=0} (-1)^{r-s} \binom{r}{s} A^{(r-s)b/(p-1)p}(u) A_{m+sb}(u) \equiv 0 \ (\text{mod } p^{er}) \ (m \geqq er),$$

where $p > 2$, $p^{e-1}(p-1) | b$ and u is an indeterminant. Both (4.1) and (4.2) are to be understood as meaning that after expansion each coefficient in the left member $\equiv 0$. Hence modifying (1.4) slightly it is clear that (4.2) implies

$$(4.3) \qquad | A_{m+(i+j)b}(u) | \equiv 0 \quad (\text{mod } p^{er(r+1)}) \qquad (i, j = 0, 1, \ldots, r),$$

provided $m \geqq er$. In particular if we put u equal to a rational number c which is integral (mod p) and let $a_m = A_m(c)$, (4.3) becomes

$$| a_{m+(i+j)b} | \equiv 0 \quad (\text{mod } p^{er(r+1)}) \quad (m \geqq er).$$

In the next place if we define $\beta_m(u)$ by means of

$$\frac{x}{\operatorname{sn} x} = \sum_{m=0}^{\infty} \beta_m(u) \frac{x^m}{m!},$$

then $\beta_m(u)$ is a polynomial in u with rational coefficients. Then we have

$$(4.4) \qquad \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} A^{(r-s)b/(p-1)p}(u) \tau_{m+sb}(u) \equiv 0 \ (\text{mod } p^{er}),$$

where

$$(4.5) \qquad (p-1)p^{e-1} | b, \quad p - 1 \nmid m, \quad m > er,$$

and

$$\tau_m(u) = \beta_m(u)/m.$$

We have also

$$(4.6) \qquad \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} A^{(r-s)b/(p-1)}(u) \beta_{m+sb}(u) \equiv 0 \quad (\text{mod } p^{e(r-1)})$$

provided (4.5) holds. It evidently follows from (4.4) that

$$(4.7) \qquad |\tau_{m+(i+j)b}(u)| \equiv 0 \pmod{p^{er(r+1)}},$$

while (4.6) implies

$$(4.8) \qquad |\beta_{m+(i+j)b}(u)| \equiv 0 \pmod{p^{er(r-1)}},$$

where in both (4.7) and (4.8) it is assumed that (4.5) holds.

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DUKE UNIVERSITY.