SOME TRIGONOMETRICAL SERIES VII.

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It is well known that if (n_k) is a sequence of integers with the Hadamard gap:

$$n_{k+1}/n_k > \theta > 1 \ (k = 1, 2, \cdots),$$

then the sequence $(\cos n_k x)$, which is a subsequence of $(\cos nx)$, is "almost independent". Then arises the problem: does any sequence of functions contain an "almost independent" subsequence? To treat this problem, the definition of the "almost independence" is the key point. We give two definitions of such concept.

1. Let $(f_n(t))$ be a sequence of integrable functions defind in the interval $(0, 1)$. If the sequence $(f_n(t))$ is independent, then for any m and n, and for any interval (a, b) in $(0, 1)$,

$$\int_b^a f_m(t) f_n(t) dt = \int_b^a f_m(t) dt \int_b^a f_n(t) dt \quad (m \neq n)$$

Let $(f_n(t))$ be a sequence of positive integrable functions defined in $(0, 1)$. We define that $(f_n(t))$ is quasi-independent in $(0, 1)$, if for any $\lambda \, (0 < \lambda < 1)$ and for any interval (a, b) in $(0, 1)$, there exists an integer N such that

$$\int_b^a f_m(t) f_n(t) dt > \lambda \int_b^a f_m(t) dt \int_b^a f_n(t) dt$$

for any $m, n \geqq N$, $m \neq n$.

Positive independent sequence is evidently quasi-independent. In the case of uniformly bounded independent sequence, its sum with an adequate constant is also quasi-independent. For example, $(1 + \nu_n(t))$, $\nu_n(t)$ being Rademacher function, is so.

For example, the sequence $(1 + \cos 2\pi n_k t)$ is quasi-independent when (n_k) has the Hadamard gap.

Hence our problem becomes: under what condition a sequence $(f_n(t))$ contains a quasi-independent subsequence? The solution is given in Theorem 2 in the following.

2. We prove the following

THEOREM 1. *If $(f_n(t))$ is a sequence of positive and uniformly bounded measurable functions such that*

$$\limsup_{n \to \infty} \int_b^a f_n(t) \, dt > 0 \tag{1}$$

for any interval (a, b) in $(0, 1)$, then there is a subsequence $(f_{n_k}(t))$ such that, *for any* λ $(0 < \lambda < 1)$ *and for any interval* (a, b) *in* $(0, 1)$, *there exists an integer* N *such that*

$$\int_a^b \min(f_{n_j}(t), f_{n_k}(t))dt > \lambda \int^b f_{n_j}(t)dt \int^b f_{n_k}(t)dt \tag{2}$$

for any $j, k \geqq N$.

PROOF.[1] Let $M \geqq f_n(t) \geqq 0$ for all n and t, and let μ be the left member of (1) where we can replace lim sup by lim, since otherwise it is sufficient to consider a suitable subsequence. We denote by $f \cap g$ the minimum of $f(t)$ and $g(t)$, and we put

$$g_1 \equiv f_\nu, g_2 \equiv f_{\nu+1} - f_{\nu+1} \cap f_\nu,\ g_3 \equiv f_{\nu+2} - f_{\nu+2} \cap (f_\nu \cap f_{\nu+1}),$$
$$g_4 \equiv f_{\nu+3} - f_{\nu+3} \cap f_\nu \cup f_{\nu+1} \cup f_{\nu+2})$$

where ν is taken such that

$$\int_a^b f_\nu\, dt = \int_a^b g_1 dt \geqq \mu/2.$$

If we suppose that

$$\int^b (f_i \cap f_k)dt < p \quad (\nu \leqq i, k \leqq \nu + n - 1),$$

then we have

$$\int_a^b g_1 dt \geqq \mu_2,\ \int_a^b g_2 dt > \mu/2 - p,\ \int_a^b g_3 dt > \mu/2 - 2p, \ldots .$$

Since $\sum_{i=1}^n g_i(t) \leqq M$, we get

$$M \geqq M(b-a) \geqq \int^b \left(\sum_{i=1}^n g_i\right) dt = \sum_{i=1}^n \int_a^b g_i\, dt > n\mu/2 - \frac{1}{2}n(n-1)p.$$

If we take $\mu/2p < n \leqq \mu/2p + 1$, then

$$M \geqq \frac{\mu^2}{8p} - \frac{\mu}{4}, \text{ i.e. } p \geqq \frac{\mu^2}{8M + 2\mu} \geqq \frac{\mu^2}{10M}.$$

Hence there is a pair of $\nu \leqq i, k \leqq n + \nu - 1$ such that

$$\int_a^b (f_i \cap f_k)dt \geqq \mu^2/10M, \tag{3}$$

and then there are infintely many pair of such i, j.

Now, considering the sequence of functions defined in the π-dimensional

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[1] Method of proof depends on the idea of J. Visser, On Poincaré's recurrence theorem, Bull. Amer. Math. Soc., 42(1936).

unit cube:

$$f_n(t_1)\cdot f_n(t_2)\cdot\dots\cdot f_n(t_n)$$

in place of $f_n(t)$, we can show that

$$\int_a^b (f_i \cap f_k)\,dt \geqq \left(\frac{1}{10M}\right)^{1/\pi}\mu^2,$$

instead of (3), for infinitely many pair of i, k. Thus, taking π sufficiently large, we see that

$$\int_a^b (f_i \cap f_k)dt \geqq \lambda'\mu^2 \tag{4}$$

for infinitely many pair of i, k and for $\lambda', 1 > \lambda' > \lambda$.

We can further find a subsequence $(f_{k_p}(t))$ such that

$$\int_a^b (f_{k_1} \cap f_{k_p})\,dt \geqq \lambda'\mu^2 \quad \text{for all } p. \tag{5}$$

For, if otherwise, there is an integer p_n, for any n, such that

$$\int_a^b (f_n \cap f_m)dt < \lambda'\mu^2 \quad \text{for all } m \geqq n + p_n.$$

Let us put

$$n_1 = 1, n_2 = n_1 + p_{n_1}, n_3 = n_2 + p_{n_2}, \dots,$$

then

$$\int_a^b (f_{n_i} \cap f_{n_k}) < \lambda'\mu^2 \quad \text{for all } i, k.$$

Thus contradicts (4), applied to $(f_{n_k}(t))$. Thus we have proved (5).

Hence we can see that the sequence $(f_{n_k}(t))$ such that

$$\lim_{k\to\infty} \int_a^b f_{n_k}(t)\,dt = \mu \tag{6}$$

contains a subsequence $(f_{k_p}(t))$ such that

$$\int_a^b (f_{k_i} \cap f_{k_j})\,dt \geqq \lambda'\mu^2 \text{ for all } i, j. \tag{7}$$

By (6), (7) may be written as

$$\int_a^b (f_{k_i} \cap f_{k_j})\,dt \geqq \lambda \int_a^b f_{k_i}\,dt \int_a^b f_{k_j}\,dt$$

for sufficiently large i and j.

By the diagonal method we get the theorem.

3. THEOREM 2. *If* $(f_n(t))$ *is a sequence of functions positive and uniformly bounded measurable such that there is a sequence* (n_k) *of integers, satisfying*

$$\liminf_{k\to\infty} f_{n_k}(t) \geqq \mu > 0,$$

then the sequence contains a quasi-independent subsequence.

PROOF. Since we can suppose $\mu = 1$, we have

$$f_{n_i}(t) f_{n_j}(t) \geqq f_{n_i}(t) \cap f_{n_j}(t).$$

Hence Theorem 2 follows from Theorem 1.

4. We shall give the second (stronger) definition of quasi-independence.

Let $(f_n(t))$ be a sequence of measurable functions defined in $(0,1)$. We define that $(f_n(t))$ is quasi-independent, if for any $\lambda\,(0 < \lambda < 1)$ and for any intervals (a,b) and (c,d) there exists an integer N such that

$$\begin{aligned} &\text{meas}\ (t;\ a < f_m(t) < b, c < f_n(t) < d) \\ &\qquad > \lambda\ \text{meas}\ (t;\ a < f_m(t) < b) \cdot \text{meas}\ (t;\ c < f_n(t) < d) \end{aligned}$$

for any $m,\ n \geqq N$.

This definition is closely related to that of A. Renyi[2]. His definition reads as follows: if for any intervals (a,b) and (c,d),

$$\left| \frac{\text{meas}\ (t; a < f_m < b, c < f_n < d)}{\text{meas}\ (t; a < f_m < b)\ \text{meas}\ (t; c < f_n < d)} - 1 \right| < \delta_m \delta_n,$$

where $\sum \delta_n < \infty$.

For example, $(\cos n_k x)$ is quasi-independent (in the second sense) when $n_{k+1}/n_k \to \infty$.

Then we have

THEOREM 3. *Let* $(f_n(t))$ *be a sequence of measurable functions such that*

$$\limsup_{n\to\infty} \text{meas}\ (t; a < f_n(t) < b) > 0$$

for any interval (a,b). *Then* $(f_n(t))$ *contains a quasi-independent sequence (in thè second sence).*

PROOF. Let $\varphi_{n,a,b}(t)$ be a characteristic function of the set $(t; a < f_n(t) < b)$ and let $F_{2n-1}(t) = \varphi_{n,a,b}(t)$, $F_{2n}(t) = \varphi_{n,c,d}(t)$ $(n = 1, 2, \cdots\cdots)$.
Applying Theorem 1 to the sequence $(F_n(t))$, we get the theorem.

We can easily see that, under the hypothesis of Theorem 3, $(f_n(t))$ contains a quasi-independent sequence in the A. Renyi sense.

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[2] A. Renyi, Journal de Mathématique pure et applique, 28(1949), 137–149.