# TWO THEOREMS ON THE RIEMANN SUMMABILITY

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**1.** The series $\sum_{\nu=1}^{\infty} a_\nu$ is said $(R_1)$-summable to zero if the series

$$F(t) = \sum_{\nu=1}^{\infty} \frac{s_\nu}{\nu} \sin \nu t, \tag{1}$$

where $s_n = \sum_{\nu=1}^{n} a_\nu$, converges in some interval $0 < t < t_0$, and if $F(t)$ tends to zero as $t$ tends to zero.

The series $\sum_{\nu=1}^{\infty} a_\nu$ is said $(R, 1)$-summable to zero if the series

$$G(t) = \sum_{\nu=1}^{\infty} a_\nu \frac{\sin \nu t}{\nu t} \tag{2}$$

converges in some interval $0 < t < t_0$, and if $G(t)$ tends to zero as $t$ tends to zero.

Recently, one of the present authors [2] proves the following theorem;

THEOREM A. *Suppose that*

$$\sum_{\nu=1}^{n} s_\nu = o(n^\alpha),$$

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-\alpha}),$$

*where* $0 < \alpha < 1$. *Then the series* $\sum_{\nu=1}^{\infty} a_\nu$ *is* $(R_1)$*-summable to zero.*

THEOREM B. *Under the assumptions of Theorem A, the series* $\sum_{\nu=1}^{\infty} a_\nu$ *is* $(R, 1)$*-summable to zero.*

The object of this paper is to generalize the above theorems.

THEOREM 1. *Let* $s_n^\beta$ *be the* $(C, \beta)$*-sum of* $\sum_{n=1}^{\infty} a_n$. *Then, if*

$$s_n^\beta = o(n^{\beta\alpha}), \tag{3}$$

and

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-\alpha}), \tag{4}$$

*where* $0<\alpha<1, 0\leqq\beta$, *the series* $\sum_{n=1}^{\infty} a_n$ *is* $(R_1)$*-summable to zero.*

THEOREM 2. *Under the assumptions of Theorem* 1, *the series* $\sum_{n=1}^{\infty} a_n$ *is* $(R, 1)$*-summable to zero.*

**2. Proof of Theorem 1.**

Firstly, we shall show that the series (1) is convergent for all $t$. If we put $r_n = \sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu}$, then $|a_n| = n(r_n - r_{n+1})$.

Since, by (4),

$$\sum_{\nu=1}^{n} |a_\nu| = \sum_{\nu=1}^{n} r_\nu - nr_{n+1} = O\left(\sum_{\nu=1}^{n} \nu^{-\alpha}\right) + O(n^{1-\alpha}) = O(n^{1-\alpha}),$$

we have

$$s_n = O(n^{1-\alpha}). \tag{5}$$

Hence

$$\sum_{\nu=n}^{\infty} \frac{|s_\nu|}{\nu^2} = O(n^{-\alpha}). \tag{6}$$

Furthermore, by (4), (6),

$$\sum_{\nu=n}^{\infty}\left|\frac{s_\nu}{\nu} - \frac{s_{\nu+1}}{\nu+1}\right| = \sum_{\nu=n}^{\infty}\left|\frac{s_\nu - s_{\nu+1}}{\nu} + \left(\frac{1}{\nu} - \frac{1}{\nu+1}\right)s_{\nu+1}\right| \leqq \sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} + \sum_{\nu=n}^{\infty} \frac{|s_{\nu+1}|}{\nu^2} = O(n^{-\alpha}). \tag{7}$$

Using the Abel's lemma, we have

$$\sum_{\nu=n}^{m} \frac{s_\nu}{\nu} \sin \nu t = \sum_{\nu=n}^{m}\left(\frac{s_\nu}{\nu} - \frac{s_{\nu+1}}{\nu+1}\right) T_\nu(t) + \frac{s_m}{m} T_m(t) - \frac{s_n}{n} T_{n-1}(t), \tag{8}$$

where

$$T_n(t) = \left\{\cos t - \cos\left(n+\frac{1}{2}\right)t\right\}/2 \sin \frac{1}{2} t.$$

Since

$$\left|\sum_{\nu=n}^{m} \frac{s_\nu}{\nu} \sin \nu t\right| < 2t^{-1}\sum_{\nu=n}^{\infty}\left|\frac{s_\nu}{\nu} - \frac{s_{\nu+1}}{\nu+1}\right| + 2t^{-1}\left(\frac{|s_m|}{m} + \frac{|s_n|}{n}\right)$$

if $t \neq 0$, by (5), (7), the series (1) is convergent. If $t=0$, this fact is evident. Thus the series (1) is convergent for all $t$.

Given a positive number $\varepsilon$, put $M = [(t\varepsilon)^{-1/\alpha}]$, and

$$\sum_{\nu=1}^{\infty} \frac{s_\nu}{\nu} \sin \nu t = \left(\sum_{1}^{M} + \sum_{M+1}^{\infty}\right) = U(t) + V(t),$$

say. Then we have

$$
\begin{aligned}
|V(t)| &\leqq 2t^{-1}\sum_{M+1}^{\infty}\left|\frac{s_\nu}{\nu}-\frac{s_{\nu+1}}{\nu+1}\right|+2t^{-1}\frac{|s_M|}{M} \\
&= O(t^{-1}M^{-\alpha})+O(t^{-1}M^{-\alpha})=O(t^{-1}t\varepsilon) \qquad (9)\\
&\leqq O(\varepsilon),
\end{aligned}
$$

by (5), (7), (8).

Nextly, we show that $U(t)=o(1)$. Putting $[\beta]=\gamma$, by repeated use of Abel's transformation $\gamma$ times, we have

$$
\begin{aligned}
U(t) &= \sum_{\nu=1}^{M-\gamma} s_\nu^\gamma \Delta_\nu^\gamma(t)+s_{M-\gamma+1}^\gamma \Delta_{M-\gamma+1}^{\gamma-1}(t)+\cdots \\
&\qquad \cdots + s_{M-1}^2\Delta_{M-1}^1(t)+s_M^1\Delta_M^0(t) \qquad (10)\\
&= W(t)+\sum_{\nu=1}^{\gamma}U_\nu(t),
\end{aligned}
$$

say, where

$$\Delta_n^0(t)=\sin nt/n,\quad \Delta_n^k(t)=\Delta_n^{k-1}(t)-\Delta_{n+1}^{k-1}(t),$$

and

$$U_\nu(t)=S_{M-\nu+1}^\nu\Delta_{M-\nu+1}^{\nu-1}(t).$$

Since

$$\Delta_n^{2k}(t)=(-1)^{k+1}2^{2k}\int_0^t\left(\sin\frac{t}{2}\right)^{2k}\cos(n+k)t\,dt, \tag{11, a}$$

$$\Delta_n^{2k+1}(t)=(-1)^{k+1}2^{2k+1}\int_0^t\left(\sin\frac{t}{2}\right)^{2k+1}\sin\left(n+\frac{2k+1}{2}\right)t\,dt, \tag{11, b}$$

for $k=0,\ 1,\ 2,\ \ldots,$ we have

$$\Delta_n^k(t)=O(t^k/n) \tag{12}$$

by the second mean value theorem. From (3), (5), using the Riesz convexity theorem [1], we have

$$s_n^\nu=O\{(n^{1-\alpha})^{1-\nu/\beta}(n^{\beta\alpha})^{\nu/\beta}\}=O(n^{\{(1-\alpha)(\beta-\nu)+\alpha\beta\nu\}/\beta}), \tag{13}$$
$$(\nu=1,2,3,\ldots\ldots).$$

Hence by (11, a), (12),

$$
\begin{aligned}
U_\nu(t) &= O(M^{\{(1-\alpha)(\beta-\nu)+\alpha\beta\nu\}/\beta}t^{\nu-1}/M)\\
&= O(t^{-(\beta-\alpha\beta-\nu+\alpha\nu+\alpha\beta\nu)/(\alpha\beta)+\nu-1+1/\alpha})\\
&= O(t^{\nu(1-\alpha)/(\alpha\beta)})\\
&= o(1)
\end{aligned}
$$

for $\nu=1, 2, 3, \ldots\gamma$. Thus

$$\sum_{\nu=1}^{\gamma}U_\nu(t)=o(1). \tag{14}$$

Nextly, we shall prove that $W(t)=o(1)$. By the well-known formula

$$s_\nu^\gamma=\sum_{n=0}^{\nu}(-1)^{\nu-n}\binom{\beta-\gamma}{\nu-n}s_n^\beta\ ,\quad (s_0=0), \tag{15}$$

where $\binom{m}{n} = \dfrac{m(m-1)\ldots(m-n+1)}{n!}$ and $\binom{0}{0} = 1$, we have

$$W(t) = \sum_{\nu=1}^{M-\gamma} s_\nu^\gamma \Delta_\nu^\gamma(t)$$
$$= \sum_{\nu=1}^{M-\gamma} \left\{\sum_{n=0}^{\nu} (-1)^{\nu-n}\binom{\beta-\gamma}{\nu-n} s_n^\beta\right\}\Delta_\nu^\gamma(t)$$
$$= \sum_{n=0}^{M-\gamma} s_n^\beta \sum_{\nu=n}^{M-\gamma} (-1)^{\nu-n}\binom{\beta-\gamma}{\nu-n}\Delta_\nu^\gamma(t).$$

Here, we consider the two case, that is, i) $\gamma$ is even; ii) $\gamma$ is odd.

i). By (11, a), we have

$$W(t) = \sum_{n=0}^{M-\gamma} s_n^\beta \sum_{\nu=n}^{M-\gamma} (-1)^{\nu-n}\binom{\beta-\gamma}{\nu-n}\int_t^0 (-1)^{\frac{\gamma}{2}+1} 2^\gamma\left(\sin\frac{t}{2}\right)^\gamma \cos\left(\nu+\frac{\gamma}{2}\right)t\,dt$$

(16) $$= \sum_{n=0}^{M-\gamma} s_n^\beta\left\{(-1)^{\frac{\gamma}{2}+1}2^\gamma\int_0^t \sum_{\nu=n}^{M-\gamma}(-1)^{\nu-n}\binom{\beta-\gamma}{\nu-n}\cos\left(\nu+\frac{\gamma}{2}\right)t\left(\sin\frac{t}{2}\right)^\gamma dt\right\}$$
$$= \sum_{n=0}^{M-\gamma} s_n^\beta\left\{(-1)^{\frac{\gamma}{2}+1}2^\gamma\int_0^t \sum_{\nu=0}^{M-\gamma-n}(-1)^{\gamma}\binom{\beta-\gamma}{\nu}\cos\left(\nu+n+\frac{\gamma}{2}\right)t\left(\sin\frac{t}{2}\right)^\gamma dt.\right.$$

Since

$$\sum_{\nu=0}^{\infty}(-1)^{\gamma}\binom{\beta-\gamma}{\nu}\cos\left(\nu+n+\frac{\gamma}{2}\right)t$$

(17) $$= \Re\left\{\sum_{\nu=0}^{\infty}(-1)^{\nu}\binom{\beta-\gamma}{\nu}e^{int}e^{i}\left(\nu+\frac{\gamma}{2}\right)t\right\}$$
$$= 2^{\beta-\gamma}\left(\sin\frac{t}{2}\right)^{\beta-\gamma}\cos\left\{\left(\frac{\beta}{2}+n\right)t+\frac{\beta-\gamma}{2}\pi\right\},$$

we write $W(t)$ in the form

$$W(t) = \sum_{n=0}^{M-r} s_n^\beta\left\{(-1)^{\frac{\gamma}{2}}2^\beta\int_0^t\left(\sin\frac{t}{2}\right)^\beta\cos\left[\left(\frac{\beta}{2}+n\right)t+\frac{\beta-\gamma}{2}\pi\right]dt\right.$$
$$-(-1)^{\frac{\gamma}{2}}2^\beta\int_0^t \sum_{\nu=M-\gamma-n+1}^{\infty}(-1)^{\nu}\binom{\beta-\gamma}{\nu}\cos\left(\nu+n+\frac{\gamma}{2}\right)t\left(\sin\frac{t}{2}\right)^\gamma dt$$
$$\equiv W_1(t) + W_2(t),$$

say. By second mean value theorem

$$\int_0^t\left(\sin\frac{t}{2}\right)^\beta\cos\left\{\left(\frac{\beta}{2}+n\right)t+\frac{\beta-\gamma}{2}\pi\right\}dt = O(t^\beta/n),$$

If $\gamma=0$, we pnt $U(t)=W(t)$.

and then

$$W_1(t) = o\left(\sum_{n=1}^{M-\gamma} n^{\beta\alpha} \frac{t^\beta}{n}\right)$$

$$= o(M^{\beta\alpha} t^\beta)$$

(18) $$= o(\varepsilon^{-\beta} t^{-\beta} t^\beta)$$

$$= o(1).$$

Now we have

$$W_2(t) = o\left(\sum_{n=0}^{M-\gamma} n^{\beta\alpha} \sum_{\nu=M-\nu-n+1}^{\infty} \nu^{-(\beta-\gamma+1)} t^\gamma \frac{1}{\nu+n}\right)$$

$$= o\left(\frac{(M-\gamma)^{\beta\alpha}}{M-\gamma+1} \sum_{n=0}^{M-\gamma} (M-\gamma n+1)^{-\beta+\gamma} t^\gamma\right)$$

(19) $$= o\left(t^\gamma M^{\beta\alpha-1} \sum_{n=1}^{M-\gamma+1} n^{-\beta+\gamma}\right)$$

$$= o(t^\gamma M^{\beta\alpha-\beta+\gamma})$$

$$= o(t^\gamma t^{-(\beta\alpha-\beta+\alpha)/\alpha})$$

$$= o(1).$$

Thus, from (14), (18), (19)

$$U(t) = o(1).$$

Therefore, given arbitrarily fixed $\varepsilon > 0$, from (9)

$$|F(t)| \leqq |U(t) + V(t)| \leqq \varepsilon \quad (t \to 0).$$

Since $\varepsilon$ is arbitrarily small,

$$F(t) \to 0 \quad \text{as} \quad t \to 0.$$

Thus, if $\gamma$ is even, the proof is complete.

ii) Nextly, we consider the 2nd case, i.e., $\gamma$ is odd. The proof is similar to the former case. In this case, if we replace (11, a) by (11, b), we get

$$W(t) = \sum_{n=0}^{M-\gamma} s_n^\beta \left\{(-1)^{\frac{\gamma+1}{2}} 2^\gamma \int_0^t \sum_{\nu=0}^{M-\gamma-n} (-1)^\nu \binom{\beta-\gamma}{\nu} \sin\left(\nu+n+\frac{\gamma}{2}\right) t \, dt\right\}$$

and similar as (17)

$$\sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta-\gamma}{\nu} \sin\left(\nu+n+\frac{\gamma}{2}\right) t$$

$$= I\left\{\sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta-\gamma}{\nu} e^{int} e^{i(\nu+\gamma/2)t}\right\}$$

$$= 2^{\beta-\gamma}\left(\sin\frac{t}{2}\right)^{\beta-\gamma} \sin\left\{\left(\frac{\beta}{2}+n\right)t + \frac{\beta-\gamma}{2}\pi\right\}.$$

Therefore, we can proceed the proof similarly as in former case. Thus, the proof of theorem is complete.

**3. Proof of Theorem 2.**
The method of proof is similar to the former section. We first show the series (2) is convergent for all positive $t < t_0$.

Since, by (4),

$$|G(t)| \leq \frac{1}{t}\sum_{n=1}^{\infty}\frac{|a_n|}{n} < +\infty,$$

the series (2) is convergent for all such $t$.

Nextly, choose $M \equiv \left[\left(\frac{1}{t\varepsilon}\right)^{\frac{1}{\alpha}}\right]$ and write

$$G(t) = \sum_{n=1}^{\infty} a_n \frac{\sin nt}{nt} = \left(\sum_{1}^{M} + \sum_{M+1}^{\infty}\right) \equiv U(t) + V(t),$$

say. Then, by (4),

$$|V(t)| \leqq t^{-1}\sum_{M+1}^{\infty}\frac{|a_\nu|}{\nu} = O(t^{-1}t\varepsilon) \leqq \varepsilon.$$

Putting $[\beta] = \gamma$, by repeated use of Abel's transformation $(\gamma+1)$ times, we have

$$\begin{aligned} U(t) &= t^{-1}\left(\sum_{n=1}^{M} a_n \frac{\sin nt}{n}\right) \\ &= t^{-1}\left\{\sum_{\nu=1}^{M-\gamma-1} s_\nu^\gamma \Delta_\nu^{\gamma+1}(t) + s_{M-\gamma}^{\gamma}\Delta_{M-\gamma}^{\gamma}(t) + \cdots\right. \\ &\qquad \left.\cdots + s_{M-1}^{1}\Delta_{M-1}^{1}(t) + s_M^0 \Delta_M^0(t)\right\} \\ &\equiv t^{-1}(W(t) + U(t)), \end{aligned}$$

say, where $\Delta_n^\nu(t)$ is same in § 2.

In the same method in § 2 we obtain, by (12), (13),

$$\begin{aligned} U^\nu(t) &= OM^{\{(1-\alpha)(\beta-\nu)+\alpha\beta\nu\}}\frac{t^\nu}{M} \\ &= o(t) \qquad (\nu = 1, 2, 3, \ldots, \gamma) \end{aligned}$$

and

$$U_0(t) = O\left(M^{1-\alpha}\frac{1}{M}\right) = O(M^{-\alpha}) = O(t\varepsilon) \leqq \varepsilon t.$$

Now, we shall prove that $W(t) = o(t)$. Using (15),

$$W(t) = \sum_{n=0}^{M-\gamma-1} s_n^\beta \sum_{\nu=n}^{M-\gamma-1} (-1)^{\nu-n}\binom{\beta-\gamma}{\nu-n}\Delta_\nu^{\gamma+1}(t).$$

Dividing the method into two case as in § 2, we shall prove the case in which $\gamma$ is odd. Using (17),

$$W(t) = \sum_{n=0}^{M-\gamma-1} s_n^\beta\left\{(-1)^{(\gamma+1)/2}2^{\gamma+1}\int_0^t \sum_{\gamma=0}^{M-\gamma-n-1}(-1)^\nu\binom{\beta-\gamma}{\nu}\cos\left(\nu+n+\frac{\gamma+1}{2}\right)t\right\}$$

$$= \sum_{n=0}^{M-\gamma-1} s_n^\beta \Big\{ (-1)^{(\gamma+1)/2} 2^\beta \int_0^t \Big(\sin\frac{t}{2}\Big)^{\beta+1} \cos\Big[\Big(\frac{\beta}{2}+n\Big)t + \frac{\beta-\gamma-1}{2}\pi\Big] dt$$

$$- (-1)^{\frac{\gamma}{2}} 2^\beta \int_0^t \sum_{\nu=M-\gamma-n}^{\infty} (-1)^\nu \binom{\beta-\gamma}{\nu} \cos\Big(\nu+n+\frac{\gamma}{2}\Big)t \Big(\sin\frac{t}{2}\Big)^{\gamma+1} dt \Big\},$$

therefore, similarly as (18), (19), we obtain

$$W(t) = o(t).$$

If we use the same method as in § 2, we obtain

$$G(t) \to 0 \quad \text{as} \quad t \to 0$$

The case in which $\gamma$ is even is similar.

Thus, the theorem is proved.

## REFERENCES

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