ON SOME INVARIANTS OF MAPPINGS

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(Received May 15, 1953)

1. Introduction. In the study of homotopy groups of sphere there are very few methods for determining whether a mapping of one sphere on another is essential or not. One such method is furnished by Brouwer degree of a mapping of S^r on itself. Another is furnished by the Hopf invariant of a mapping of S^{2r-1} on S^r[3, 4]. These methods are used only for mappings of S^n on S^r with $n = r$ or $n = 2r - 1$. Freudenthal's results [2] are applied for the case $r < n < 2r - 1$ but almost nothing are known about $\pi_n(S^r)$ for $n > 2r - 1$. L. Pontrjagin [6] succeeded in the enumeration of the homotopy classes of maps of a 3-complex K^3 on S^2 and obtained the result that Hopf's invariant determines completely homotopic classes of the maps of S^3 on S^2; Whitney [8] reformulated another Hopf's theorem [5] and introduced two deformation theorems. In this paper we shall at first attempt to generalize both Hopf's invariant and Brouwer degree. Such quantity will be used for mappings of S^n on S^r with $n = (k+1)r - k$ $(k = 0, 1, 2, 3, \ldots.)$. Secondly for this quantity we shall attempt to generalize Pontrjagin's and some other theorems.

2. Hopf [3] studied many interesting properties of maps of S^3 on S^2 and he stated [4] the generalization of these results for the maps of S^{2r-1} on S^r but he omitted these proofs.

We now consider maps f of S^m on S^n $(m > n)$ and denote by T^n any n-dimensional simplex on S^m, by τ^n a fined n-dimensional simplex on S^n. Let ξ be an interior point of τ^n. When ξ has only one interior point x in T^n as inverse image of f, we denote $\varphi_{T^n}(\xi) = \pm x$. The signs of x will be $+$ or $-$ according to whether T^n is mapped on τ^n positively or negatively. If ξ does not have inverse image of f in T^n, we define $\varphi_{T^n}(\xi) = 0$. We consider any integral complex $C^n = \Sigma a_i T_i^n$ and define by $\varphi_{C^n}(\xi) = \Sigma a_i \varphi_{T_i^n}(\xi)$ the inverse image for C^n. This integral complex is clearly 0-dimensional. From the definition the following relations are introduced:

(1) $$\varphi_{C_1^n + C_2^n}(\xi) = \varphi_{C_1^n}(\xi) + \varphi_{C_2^n}(\xi)$$

(2) $$\varphi_{-C^n}(\xi) = -\varphi_{C^n}(\xi)$$

(3) $$\varphi_0(\xi) = 0$$

Secondly we consider any r-dimensional simplex $(m \geqq r \geqq n+1)$ T^r and denote by $\varphi_{T^r}(\xi)$ the intersection of the inverse image of ξ and T^r. From the definition the following relations are introduced:

(1′) $$\varphi_{C_1^r + C_2^r}(\xi) = \varphi_{C_1^r}(\xi) + \varphi_{C_2^r}(\xi)$$

(2′) $$\varphi_{-C^r}(\xi) = -\varphi_{C^r}(\xi)$$

(3′) $\varphi_0(\xi) = 0$

(4′) $\dot{\varphi}_{C^r}(\xi) = \varphi_{\dot{C}^r}(\xi)$

From the property $\dot{S}^m = 0$ and (3′), (4′), we know $\dot{\varphi}_{S^m}(\xi) = 0$. On the other hand S^m is a manifold and $\varphi_{S^m}(\xi)$ consists of finite numbers of closed manifolds M_1^{m-n}, M_2^{m-n}, $\dots M_k^{-nm}$ which are disjoint each other. If we apply the Freudenthal's Lemma [2] for our mappings, we may assume $k = 1$ without any loss of generality. We consider $r + 1$ points $\xi_0, \xi_1, \dots \xi_r$ on S^n and denote by M_i^{m-n} the inverse image of ξ_i. Let K^m, $\overset{*}{K}{}^n$ be simplicial subdivision of S^m S^n respectively. We introduce the standard map which has at first been introduced by Whitney and which the author has defined for more general case[1]. The following Lemma is the immediate result from the definition. Its proof is similar to the one of the preceding paper [1].

LEMMA 2. 1 *Let f be a map of S^m into S^n, then there exists a standard map which is homotopic to f.*

As $M_i^{m-n} \sim 0$ in S^m, there exists a complex K_i^{m-n+1} bounded by M_i^{m-n}. We shall take useful one as K_i^{m-n+1} and following Lemma is used in studing the special one of K^{im-n+1}.

LEMMA 2. 2. *Let Z^p, Z^q be any manifold of Euclidean space R^m which are fremed each other. The regular connected complexes which are bounded by Z^p, Z^q respectively can be deformed so as to hold at most $p + q - m + 2$ dimensional-simplexes in common.*

PROOF. We can assume $p > q$ without any loss of generality. Let K^{p+1}, K^{q+1} be regularly connected complexes bounded by Z^p, Z^q respectively and holding in common $(q + 1)$-dimensional simplex $(a_0, a_1, \dots a_{q+1})$. If $q + 1 \leqq p + q - m + 2$, then this Lemma is evident. Thus we assume $q + 1 > p + q - m + 2$ and consider $(q + 2)$ − simplex $(a_0, a_1, \dots, a_{q+2})$ and its interior point b. We replace the simplex $(a_0, a_1, \dots, a_{q+1})$ of K^{q+1} by

$[(ba_1 \dots a_{q+2}) - (ba_0a_2 \dots a_{q+1}) + \dots + (-1)^{q+1}(b\, a_0a_1 \dots a_q)]$.

If this process is done for every common $(q + 1)$ − simplex of K^{p+1} and K^{q+1}, then they have common simplexes which are at most q-dimensional. If $q = p + q - m + 2$, this Lemma was completely proved. Thus we assume $q > p + q - m - 2$. Let $(a_0a_1 \dots a_q)$ be any common simplex of K^{p+1} and $(a_0a_1 \dots a_qa_{q+1})$, $(a_0a_1 \dots a_qa'_{q+1})$ be a pair of $(q + 1)$ − simplexes of K^{q+1} which have the common q-simplex $(a_0a_1 \dots a_q)$. We consider their interior points b, b' respectively. At first we replace $(a_0a_1 \dots a_qa_{q+1})$, $(a_0a_1 \dots a_qa'_{q+1})$ by

$$C_1^{q+1} = [(ba_1 \dots a_qa_{q+1}) - (ba_0a_2 \dots a_qa_{q+1}) + \dots + (-1)^q(ba_0a_1 \dots a_{q-1}a_{q+1})],$$

$$C_2^{q+1} = [(b'a_1 \dots a_qa'_{q+1}) - (b'a_0a_2 \dots a_qa'_{q+1}) + \dots + (-1)^q(ba_0a_1 \dots a_{q-1}a'_{q+1})]$$

respectively. As $q > p + q + 2 - m$, we can construct regularly connected complex C^{q+1} bounded by

$$[(b\, a_1 \dots a_q) - (b\, a_0a_2 \dots a_q) + \dots + (-1)^q(b\, a_0a_1 \dots c_{q-1})]$$

and $[(b'a_1 \cdots a^q) - (b'a_0a_2 \cdots a_q) + \cdots + (-1)^q(b'a_0a_1 \cdots a_{q-1})]$ which have common simplexes at most of $(q-1)$-dimensions with K^{p+1}. We replace $(a_0a_1 \cdots a_qa_{q+1}) + (a_0a_1 \cdots a_ia'_{q+1})$ by $C_1^{q+1} + C^{q+1} + C_2^{q+1}$. If such a process is done for every common q-simplexes of K^{p+1} and K^{q+1}, then they have common simplexes at most of $(q-1)$-dimensions. If we take care of only fact that every simplex at most of $(q-1)$-dimensions is a common one face of some $(q+1)$-simplexes (its number need not to be two for the common $(q-1)$-simplex)), we can perform similar process. By a repetition of similar processes, we can lower the dimension of common simplexes of K^{p+1} and K^{q+1} untill at most $p+q+2-m$, where the dimension of the last common simplex is calculated from the dimensions of S^m, K^{p+1} and K^{q+1}.

LEMMA 2.3. *Let Z^p be any manifold of Euclidean space R^m. If we construct a complex projecting Z^p from a fix point, the resulted $(p+1)$-dimensional complex may be deformed so as to have singular simplexes [1] at most of $(2p+2-m)$-dimension.*

PROOF. By the similar deformations of Lemma 2.2, we can prove immediately.

THEOREM 2.1. *K_i^{m-n+1} may be chosen as a manifold which has some singular simplexes at most of $(m-2n+2)$-dimension.*

PROOF. Let us consider that a fixed point of S^m is a point at infinity, then S^m may be regarded as the sum of the point at infinity and a m-dimensional Euclidean space R^m. Of course, we don't take the point at infinity on M_i^{m-n}. We project M_i^{m-n} from a suitable point O and denote by $[O, M_i^{m-n}]$ the resulted sets. $[O, M_i^{m-n}]$ are special complexes bounded by M_i^{m-n} and are one of K_i^{m-n+1}. By Lemma 2.3, the dimension of singular simplex of $[O, M_i^{m-n}]$ is at most

$$2(m-n+1) - m = m - 2n + 2.$$

LEMMA 2.4. *$\bigcap_{i=1}^{r} [O, M_i^{m-n}]$ is a sum of finite manifolds which are at most of $(m - rn + r)$-dimension having some singular simplexes at most of $[m-(r+1)n + r + 1]$-dimension.*

PROOF. By $\xi_i \neq \xi_j (i, j = 0, 1, 2, \ldots, r, i \neq j)$, we know that

$$M_i^{m-n} \bigcap M_j^{m-n} = 0 \ (i, j = 0, 1, 2, \ldots, r, i \neq j)$$

If we consider $[O, M_2^{m-n}]$ and $[O, M_2^{m-n}]$ except for singular simplexes, $[O, M_1^{m-n}] \bigcap [O, M_2^{m-n}]$ are of at most of dimensions

$$2(m-n+1) - m = m - 2n + 2.$$

By Theorem 2.1, the singular simplexes of $[O, M_i^{m-n}]$ are at most of $(m-2n+2)$-dimensions. Therefore the dimensions of singular simplexes of $[O, M_1^{m-n}] \cap [O, M_2^{m-n}]$ are at most of dimensions

$$(m-2n+2)+(m-n+1)-m=m-3n+3.$$

On account of singularity of $[O, M_i^{m-n}]$, $[O, M_1^{m-n}] \cap [O, M_2^{m-n}]$ are a sum of some manifolds having singular simplexes.

By Lemma 2.2 and Theorem 2.1, the intersection of $[O, M_1^{m-n}] \cap [O, M_2^{m-n}]$ and $[O, M_3^{m-n}]$ is studied. As the general case $\bigcap_{i=1}^{r} [O, M_i^{m-n}]$ are a sum of some manifolds whose dimension are at most

$$r(m-n+1)-(r-1)m = m-rn+r,$$

where the dimensions of those singular simplexes are at most

$$(m-2n+2)+(r-1)(m-n+1)-(r-1)m = m-(r+1)n+r+1.$$

LEMMA 2.5 *$M_k^{m-n} \cap \{\bigcap_{i=1}^{k-1} [O, M_i^{m-n}]\} \cap \{\bigcap_{j=k+1}^{r} [O, M_j^{m-n}]\}$ are a sum of finite manifolds which are at most of $(m-rn+r-1)$ — dimensions having some singular simplexes at most of $(m-(r+1)n+r)$-dimension.*

PROOF. If we replace r by $r-1$ in Lemma 2.4, $\left\{\bigcap_{i=1}^{k-1} [O, M_i^{m-n}]\right\} \cap \left\{\bigcap_{j=k+1}^{r} [O, M_j^{m-n}]\right\}$ are at most of $[m-(r-1)n+(r-1)]$-dimensions having some singular simplexes at most of $[m-rn+r]$-dimensions. Therefore the dimension of $M_k^{m-n} \cap \left\{\bigcap_{i=1}^{k-1} [O, M_i^{m-n}]\right\} \cap \left\{\bigcap_{j=k+1}^{r} [O, M_j^{m-n}]\right\}$ is at most $[m-(r-1)n+(r-1)]+(m-n)-m = m-rn+r-1$.

The dimensions of the singular simplexes of those complexes are at most

$$[m-rn+r]+(m-n)-m = n-(r+1)n+r.$$

LEMMA 2.6. *If $m=(r+1)n-r$, the intersection number $\phi(M_0^{m-n}, \bigcap_{i=1}^{r} [O, M_i^{m-n}])$ can be defined uniquely.*

PROOF. For $\bigcap_{i=1}^{r} [O, M_i^{m-n}]$ of Lemma 2.4, we give an orientation as follows: We know an orientation for the faces of simplexes of $[O, M_1^{m-n}]$ and by this orientation we introduce an orientation on $\bigcap_{i=1}^{r} [O, M_i^{m-n}]$. This orientation does not depend upon the singular simplexes of $\bigcap_{i=1}^{r} [O, M_i^{m-n}]$. Similarly we introduce an orientation for $M_k^{m-n} \cap \left\{\bigcap_{i=0,1,2,\ldots,\hat{k},\ldots,r} [O, M_i^{m-n}]\right\}$ by the faces of simplexes of M_k^{m-n}. This orientation does not depend upon the singular simplexes of $M_k^{m-n} \cap \left\{ \bigcap_{i=0,1,2,\ldots,\hat{k},\ldots,r} [O, M_1^{m-n}]\right.$. Then, by lemma 2.4,

$$\dim \left\{ \bigcap_{i=1}^{r} [O, M_i^{m-n}] \right\} = m - rn + r = n.$$

The intersection of M_0^{m-n} and $\bigcap_{i=1}^{r} [O, M_0^{m-n}]$ are at most of dimensions $n + (m-n) - m = 0$.

From the orientations of R^m, M_0^{m-n} and $\bigcap_{i=1}^{r} [O, M_i^{m-n}]$ we know the intersection number $\phi(M_0^{m-n}, \bigcap_{i=1}^{r} [O, M_i^{m-n}])$.

LEMMA 2.7. $\phi\left(M_0^{m-n}, \bigcap_{i=1}^{r} [O, M_i^{m-n}]\right)$ *does not depend on a choice of a fixed point O.*

PROOF. We consider m-dimensional Euclidean space R^m as in the proof of Theorem 2.1 and fixed point O' which is in $R^m - \bigcup_i M_i^{m-n} - O$.

By the projection from O', we get a similar complex $[O', M_i^{m-n}]$. If the bounded set containing O and O' are covered by sufficiently fine open set, we can choose a finite covering, which refine the given covering. If we consider point-pairs in the same element of covering at first and remove from one element to the adjacent secondly, we can remove from O to O' by finite processes. Therefore we can assume without any loss of generality that O' exist in sufficiently small neighborhood of O. We shall prove

$$\phi\left(M^{m-n}, \bigcap_{i=1}^{r} [O, M_i^{m-n}]\right) = \phi\left(M_0^{m-n}, \bigcap_{i=1}^{r} [O', M_i^{m-n}]\right).$$

As O and O' lie in sufficiently near, for $k = 0, 1$, $M_k^{m-n} \cap \left(\bigcap_{i=2}^{r} [O, M_i^{m-n}]\right)$ and $M_k^{m-n} \cap \left(\bigcap_{i=2}^{r} [O', M_i^{m-n}]\right)$ are situated sufficiently near in R^m and are fremed each other. We consider a complex X which is bounded by $M_0^{m-n} \cap \left(\bigcap_{i=2}^{r} [O, M_i^{m-n}]\right) - M_0^{m-n} \cap \left(\bigcap_{i=2}^{r} [O', M_i^{m-n}]\right)$ and is fremed from $M_1^{m-n} \cap \left(\bigcap_{i=2}^{r} [O, M_i^{m-n}]\right)$ and $M_1^{m-n} \cap \left(\bigcap_{i=2}^{r} [O', M_i^{m-n}]\right)$. Similarly we consider a complex Y which is bounded by $M_1^{m-n} \cap \left(\bigcap_{i=2}^{r} [O, M_i^{m-n}]\right) - M_1^{m-n} \cap \left(\bigcap_{i=2}^{r}\right.$

$[O', M_i^{m-n}]\Big)$ and is fremed from $M_0^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big)$ and $M_0^{m-n} \cap \Big(\bigcap_{i=2}^{r}$ $[O, M_i^{m-n}]\Big)$.

$$R\partial\Big\{[O, M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big)\Big\} = M_1^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big),$$

$$R\partial\Big\{[O', M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\Big)\Big\} = M_1^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\Big).$$

Then

$$R\partial\Big\{[O, M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big) - Y - [O', M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\Big) = 0,$$

$$\phi\Big\{M_0^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big),\ [O, M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big) - Y - [O', M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\Big)\Big\} = 0.$$

On the other hand,

$$\phi\Big(M_0^{m-n}, \bigcap_{i=1}^{r}[O, M_i^{m-n}]\Big)$$

$$= \varepsilon\phi\Big(M_0^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big),\ [O, M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big)\Big)$$

$$\phi\Big\{M_0^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}], Y\Big\} = 0. \quad (\varepsilon = \pm 1)$$

Therefore,

$$\phi\Big\{M_0^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big),\ [O, M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big)\Big\}$$

$$= \phi\Big\{M_0^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big),\ [O', M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\Big)\Big\}.$$

Similarly,

$$\phi\Big\{[O, M_0^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big), M_1 \cap \Big(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\Big)\Big\}$$

$$= \phi\Big\{[O', M_0^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\Big),\ M_1 \cap \Big(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\Big)\Big\}.$$

Then

$$\phi\Big\{M_0^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big), [O, M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big)\Big\}$$

$$= \phi\Big\{M_0^{m-n} \cap \Big(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\Big),\ [O', M_1^{m-n}] \cap \Big(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\Big)\Big\}$$

$$= (-1)^{n-1}\phi\left\{[O, M_0^{m-n}] \cap \left(\bigcap_{i=2}^{r}[O, M_i^{m-n}]\right), M_1^{m-n} \cap \left(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\right)\right\}$$

$$= (-1)^{n-1}\phi\left\{[O', M_0^{m-n}] \cap \left(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\right), M_1^{m-n} \cap \left(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\right)\right\}$$

$$= \phi\left\{M_0^{m-n} \cap \left(\bigcap_{r=2}^{i}[O', M_i^{m-n}]\right), [O', M_1^{m-n}] \cap \left(\bigcap_{i=2}^{r}[O', M_i^{m-n}]\right)\right.$$

$$= \frac{1}{\varepsilon}\,\phi\left\{M_0^{m-n}, \bigcap_{i=1}^{r}[O', M_i^{m-n}]\right\},$$

$$\phi\left(M_0^{m-n}, \bigcap_{i=1}^{r}[O, M_i^{m-n}]\right) = \phi\left(M_0^{n-n}, \bigcap_{i=1}^{r}[O', M_i^{m-n}]\right),$$

DEFINITION 2.1. When $m = n$, we define $\phi\,[M_0^0, \theta]$ (θ means the empty set) as follows;

Let t_+^0, or t^0_- be the numbers of simplexes of K^m which are mapped on τ_0^n positively or negatively respectively. Then;

$$\phi[M_0^0, \theta] = t_+^0 - t_-^0.$$

By Lemma 2.7 we know that $\phi\left\{M_0^{m-n}, \bigcap_{i=1}^{r}[O, M_i^{m-n}]\right\}$ does not depend on a choice of a fixed point O. By these reason and definition 2.1 we define as follows:

DEFINITION 2.2. In cases $m = (r+1)n - r$ or mn:

$$W_r(f, \xi_0, \xi_1, \ldots, \xi_r) = \left[\phi\left\{M_0^{m-n}, \bigcap_{i=1}^{r}[O, M_i^{m-n}]\right\}\right]^r, \quad W_0(f, \xi_0) = \phi[M_0^0, \theta].$$

LEMMA 2.8. *$W_r(f, \xi_0, \xi_1, \ldots, \xi_r)$ does not depend on a choice of $\xi_0, \xi_1, \ldots, \xi_r$. in the following cases:*

(i) $m = n$, *or*

(ii) $m = (r+1)n - r$ *and n is even.*

PROOF. In case (i) $W_r(f, \xi_0)$ means the Brouwer degree, and Lemma 2.8 is well known.

We shall prove this Lemma in case (ii).

$$\phi\left(M_0^{m-n}, \bigcap_{i=1}^{r}[O, M_i]\right) = \varepsilon\phi\left\{M_0^{m-n} \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right), [O, M_1]\left(\bigcap_{i=2}^{r}[O, M_i]\right)\right\}$$

$$= (-1)^n\varepsilon\phi\left\{[O, M_0] \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right), M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)\right\}$$

$$= (-1)^n(-1)^{n(n-1)}\varepsilon\phi\left\{M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right), [O, M_0] \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)\right\}$$

$$= (-1)^n(-1)^{n(n-1)}\,\phi\left\{M_1^{m-n}, [O, M_0] \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)\right\}.$$

As n is even,

$$W_r(f,\xi_0,\xi_1,\dots,\xi_r)=W_r(f,\xi_1,\xi_0,\dots,\xi_r). \qquad (1)$$

$\phi\left(M_0^{m-n},\ \bigcap_{i=1}^{r}[O,M_i^{m-n}]\right)$ means a degree of $f\Big|\bigcap_{i=1}^{r}[O,M_i^{m-n}]$, then $W_r(f,\xi_0,\dots,\xi_r)$ does not depend on a choice of ξ_0. Then

$$W_r(f,\xi_0,\dots,\xi_1,\xi_r)=W_r(f,\xi_0',\xi_1,\dots,\xi_r). \qquad (2)$$

By using of (1) and (2), we can introduce the following calculations:

$$W_r(f,\xi_0,\xi_1,\dots,\xi_i,\dots,\xi_r)=\varepsilon W_r(f,\xi_0,\xi_i,\xi_2,\dots,\xi_1,\dots,\xi_r)$$
$$=\varepsilon W_r(f,\xi_i,\xi_0,\xi_2,\dots,\xi_1,\dots,\xi_r)$$
$$=\varepsilon W_r(f,\xi_i',\xi_0,\xi_2,\dots,\xi_1\dots,\dots,\xi_r) \qquad (\varepsilon=\pm 1)$$
$$=W_r(f,\xi_1,\xi_0,\xi_2,\dots,\xi_i',\dots,\xi_r)$$
$$=W_r(f,\xi_0,\xi_1,\xi_2,\dots,\xi_i',\dots\xi_r),$$

that is,

$$W_r(f,\xi_0,\xi_1,\dots,\xi_i,\dots,\xi_r)=W_r(f,\xi_0,\xi_1,\dots,\xi_i',\dots,\xi_r). \qquad (3)$$

Furthermore,

$$W_r(f,\xi_0,\xi_1,\dots,\xi_i',\dots,\xi_r)=W_r(f,\xi_i,\xi_1,\dots,\xi_i,\dots,\xi_r)$$
$$=W_r(f,\xi_i,\xi_1,\dots\xi_0,\dots\xi_r)$$

From this relation, we can introduce the following relation by using of (3):

$$W_r(f,\xi_0,\xi_1,\dots,\xi_i,\dots,\xi_r)=W_r(f,\xi_i,\ \xi_1,\dots,\xi_0,\dots,\xi_r). \qquad (4)$$

When we use (1), (2) and (4), successively,

$$W_r(f,\xi_0,\xi_1,\dots,\xi_r)=W_r(f,\xi_0',\xi_1',\dots,\xi_r').$$

By Lemma 2.8 we know that $W_r(f,\xi_0,\xi_1,\dots,\xi_r)$ does not depend on a choise of $\xi_0,\xi_1,\dots,\xi_r$. Then we define as follows.

DEFINITION 2.3. $W_r(f)=W_r(f,\xi_0,\xi_1,\dots,\xi_r)$.

THEOREM 2.2. *Let $m=n$ or $m=(r+1)n-r$ and n be even. If f, g are maps of S^m into S^n and f is homotopic to g, then $W_r(f)$ equals to $W_r(g)$.*

PROOF. When $K^m,\overset{*}{K}{}^n$ are simplicial subdivision of S^m, S^n respectively, we may assume that f and g are simplicial maps without any loss of generality. As f is homotopic to g, there exist a shar of maps $f_r(1\leqq r\leqq 2)$, of K^m into $\overset{*}{K}{}^n$, where $f_1=f$ and $f_2=g$. Then we can consider f_r to be a map of $S^m\times I$ into S^n and use it as $F(x,r)$. We define $S^m\times(1)=S_1^m$, $S^m\times(2)=S_2^m$.

$f_r(x)$ also may be assumed as simplicial map. Let ξ_1 a interior point of fixed n-simplex τ_0^n of K^n and $\varphi_{S^m\times I}(\xi_1)$, $\varphi_{S_1^m}(\xi_1)$, $\varphi_{S_2^m}(\xi_1)$ inverse images of ξ_1 for f_r, f, g respectively.

As we have $(S^m\times \dot{I})=S_2^m-S_1^m$, by (1)′, (2)′ in § 2,

$$\varphi_{S^m\times\dot{I}}(\xi_1)=\varphi_{S_2^m}(\xi_1)-\varphi_{S_1^m}(\xi_1). \qquad (1)$$

We denote $\varphi_{S_2^m}(\xi_1)$, $\varphi_{S_1^m}(\xi_1)$ by M_1, M_1' respectively. When we fix r, we denote the inverse image of ξ_1 for f_r by $\varphi_{S^m\times(r)}(\xi_1)$, then

$$\varphi_{S^m\times I}(\xi_1)=\bigcup_{1\leqq r\leqq 2}\varphi_{S^m\times(r)}(\xi_1)$$

Let O, O' be fixed points of $S^m\times(1), S^m\times(2)$ respectively. We connect O with O' by an arc in $S^m\times I$ which intersect with $S^m\times(r)$ at only one point. These intersections will be denoted by O_r.

In $S^m\times(r)$ we project $\varphi_{S^m\times(r)}(\xi_1)$ from O_r and denote the resulted complex by $[O_r, \varphi_{S^m\times(r)}(\xi_1)]$. Then,

$$\left\{\bigcup_{1\leqq r\leqq 2}[O_r, \varphi_{S^m\times(r)}(\xi_1)]\right\}=[O,M_1]-[O',M_1']-\varphi_{S^m\times I}(\xi_1)$$

We denote this relation by $\dot{C}_2=[O,M_1]-[O',M_1']-A_2$ and a second interior point of τ_0^n by ξ_2, then

$$\dot{\varphi}_{c_2}(\xi_2)=\varphi_{[O,M_1]}(\xi_2)-\varphi_{[O',M_1']}(\xi_2)-\varphi_{A_2}(\xi_2).$$

As f_r has an uniquely determined image, $\varphi_{A_2}(\xi_2)=0$ (for $\xi_1\neq\xi_2$).

We denote $\varphi_{[O,M_1]}(\xi_2), \varphi_{[O',M_1']}(\xi_2)$ by M_2, M_1' respectively. Then

$$\dot{\varphi}_{c_2}(\xi_2)=M_2-M_2'$$

$[O,M_1]\cap[O,M_2]$ and $[O',M_1']\cap[O',M_2']$ are complexes bounded by $\varphi_{[O,M_1]}(\xi_2)$, $\varphi_{O,M_1'}(\xi_2)$ on $[O,M_1]$, $[O,M_1']$ respectively. Then $R\partial\{\varphi_{c_2}(\xi_2)-[O,M_1]\cap[O,M_2]+[O',M_1']\cap[O', M_2']\}=0$, therefore $\varphi_{c_2}(\xi_2)-[O,M_1]\cap[O,M_2]+[O',M_1']\cap[O',M_2']=Z_2$ is a cycle.

$\dot{\varphi}_{c_2}(\xi_2)=\varphi_{[O,M_1]}(\xi_2)-\varphi_{[O,M_1']}(\xi_2)$ is analogous to (1), then we consider a third interior point ξ_3 of τ_0^n and similar relation as above and so on. Therefore we obtain the following relation:

$$\dot{\varphi}_{c_r}(\xi_r)=\varphi_{\bigcap_{i=1}^{r-1}[O,M_i]}(\xi_r)-\varphi_{\bigcap_{i=1}^{r-1}[O,M_i]}(\xi_r)$$

where C_r is defined to be similar to C_2.

$\bigcap_{i=1}^{r}[O,M_i]$ and $\bigcap_{i=1}^{r}[O',M_i']$ are complexes bounded by $\varphi_{\bigcap_{i=1}^{r-1}[O,M_i]}(\xi_r)$, $\varphi_{\bigcap_{i=1}^{r-1}[O',M'_i]}(\xi_r)$ on $\bigcap_{i=1}^{r-1}[O,M_i]$, $\bigcap_{i=1}^{r-1}[O',M_i']$ respectively.

Then $R\partial\left\{\varphi_{c_r}(\xi_r)-\bigcap_{i=1}^{r}[O,M_i]+\bigcap_{i=1}^{r}[O',M_i']\right\}=0$, and $\varphi_{c_r}(\xi_r)-\bigcap_{i=1}^{r}[O, M_i]+\bigcap_{i=1}^{r}[O',M_i']=Z_r^n$ is a cycle.

When we denote the projection of Z_r^n on S_1^m by $\overset{*}{Z}{}_r^n$, $Z_r^n\sim\overset{*}{Z}{}_r^n$ in $S^m\times I$. Then $F(Z_r^n)\sim f(\overset{*}{Z}{}_r^n)$ in S^n, therefore

$$F(Z_r^n)=f(\overset{*}{Z}{}_r^n)\ \text{in}\ S^n.$$

On the other hand,

$$\overset{*}{Z}{}_r^n\sim 0\ \text{in}\ S_1^m$$

$$F(\overset{*}{Z}{}_r^n) = f(\overset{*}{Z}{}_r^n) \sim 0 \text{ in } S^n,$$
$$f(\overset{*}{Z}{}_r^n) = 0 \text{ in } S^n.$$

Therefore

$$F(Z_r^n) = 0,$$
$$F(\varphi_{c_r}(\xi_r)) - f\Big(\bigcap_{i=1}^{r}[O, M_i]\Big) + g\Big(\bigcap_{i=1}^{r}[O', M_i']\Big) = 0,$$
$$F(\varphi_{c_r}(\xi_r)) = 0,$$
$$f\Big(\bigcap_{i=1}^{r}[O, M_i]\Big) = g\Big(\bigcap_{i=1}^{r}[O', M_i']\Big)$$
$$W_r(f) = W_r(g).$$

3. In this chapter we shall investigate that $W_r(f)$ is used for determining whether a mapping of one sphere on another is essential or not. In case $m = n$, $W_r(f)$ means Brouwer degree and it is well known that $W_r(f)$ is only used for determining whether the mapping is essential or not. In case $m = (r+1)n - r$ and n is even, if $r = 1$, $W_r(f)$ is the Hopf invariant and we showed in my preceding paper that the Hopf invariant is used for such purpose. In this chapter we consider the case $W_r(f)$ is determined, that is (i) $m = n$, or (ii) $m = (r+1)n - r$ and n is even. In the case (ii), by Lemma 2.4, $\bigcap_{i=1}^{r}[O, M_i^{m-n}]$ is a sum of finite manifolds which are at most of n dimension having some singular simplexes at most of one-dimension. By a similar method as in Frendenthal's Lemma [2], we can consider that $\bigcap_{i=1}^{r}[O, M_i^{m-n}]$ is a manifold M^n which is at most of n-dimensions having some singular simplexes at most of one-dimention. Let τ_0^n be a fixed n-simplex and ξ_0 an interior point of τ_0^n, $\bar{\xi}_0$ be the antipodal point of S^n. We assume that $\sigma = (a_0 a_1 \cdots a_n)$, $\sigma' = (a_0' a_1 \cdots a_n)$ are oriented n-simplexes of M^n with the common $(n-1)$-face $\tau = (a_1 \cdots a_n)$. Then we obtain the following Lemma in my preceding paper.

LEMMA 3.1. *Let f be a standard map of M^n into S^n and $f(\sigma) = +S^n$, $f(\sigma') = \bar{\xi}_0$, then there is a standard map g which is homotopic to f and $g(\sigma) = \bar{\xi}_0$, $g(\sigma') = +S^n$ leaving the degree of $M^n - (\sigma + \sigma')$ fixed.*

LEMMA 3.2. *Let f be a standard map of M^n into S^n and $f(\sigma) = +S^n$ $f(\sigma') = -S^n$, then there is a standard map g which is homotopic to f and $g(\sigma) = \bar{\xi}_0$, $g(\sigma') = \bar{\xi}_0$ leaving the degree of $M^n - (\sigma + \sigma')$ fixed.*

A map f of S^m on S^n may be considered a simplicial map of K^m on $\overset{*}{K}{}^n$. Let P_0 be an interior point of a fixed simplex τ_0^n of $\overset{*}{K}{}^n$. We may assume that inverse image $\varphi_{S^m}(P_0)$ of P_0 for f is a manifold M^{m-n}. The m-

dimensional simplexes T^m_α of K^m which are mapped on τ^n_0 necessarily intersect M^{m-n} and $M^{m-n} \cap T^m_\alpha$ are $(m-n)$-simplexes. $\xi_i (i = 0, 1, 2, \ldots, r)$ in § 2 may be considered in τ^n_0 without any loss of generality. M-dimensional simplexes T^m_α of K^m which are mapped on τ^n_0 necessarily intersect $M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)$ and $M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right) \cap T_\alpha'$ are $(r-1)$-simplexes. We denote it by $(a^{\alpha_k}_0, a^{\alpha_k}_1, \ldots, a^{\alpha_k}_{n-1})$. Let $(e^{\alpha_k}_0, e^{\alpha_k}_1, \ldots, e^{\alpha_k}_{n-1})$ be an $(n-1)$-face of T^{m-n}_α such that any two of its vertices are not mapped by f on the same vertex of τ^n_0, then $\{a^{\alpha_k}_0, a^{\alpha_k}_1, \ldots, a^{\alpha_k}_{n-1} \ldots, e^{\alpha_k}_0, \ldots, e^{\alpha_k}_{n-1}\}$ is a $(2n-1)$-simplex which we denote by $\overset{*}{T}{}^{2n-1}_{\alpha_k}$. $\overset{*}{T}{}^{2n-1}_{\alpha_k}$ is a face of T^m_α. Such a $(2n-1)$-dimensional simplex $\overset{*}{T}{}^{2n-1}_{\alpha_k}$ is considered for all $(n-1)$-dimensional-simplex $\overset{*}{T}{}^{2n-1}_{\alpha_k}$ of $M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right) \cap T^m_\alpha$. We consider all $(2n-1)$-simplexes $\overset{*}{T}{}^{2n-1}_{\alpha_k}$ which involve some n-simplex of $M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)$. Then $\sum_\alpha \sum_{\alpha_k} (e^{\alpha_k}_0, e^{\alpha_k}_1, \ldots, e^{\alpha_k}_{n-1})$ is $(n-1)$-manifold with singularity. We denote is by M^{n-1} and consider a following complex:

$$(a^{\alpha_k}_0 a^{\alpha_k}_1 \ldots a^{\alpha_k}_{n-1}, e^{\alpha_k}_0(+ (-1)^n (a^{\alpha_k}_1, \ldots, a^{\alpha_k}_{n-1}, e^{\alpha_k}_0, e^{\alpha_k}_1) + \ldots$$
$$+ (-1)^{n(n-1)} (a^{\alpha_k}_{n-1}, e^{\alpha_k}_0, e^{\alpha_k}_1, \ldots, e^{\alpha_k}_{n-1}).$$
$$\sum_\alpha \sum_{\alpha_k} [(a^{\alpha_k}_0 a^{\alpha_k}_1 \ldots a^{\alpha_k}_{n-1}, e^{\alpha_k}_0) + (-1)^n (a^{\alpha_k}_1 \ldots, a^{\alpha_k}_{n-1}, e^{\alpha_k}_0, e^{\alpha_k}_1) + \ldots$$
$$\ldots + (-1)^{n(n-1)} (a^{\alpha_k}_{n-1}, e^{\alpha_k}_0, e^{\alpha_k}_1, \ldots, e^{\alpha_k}_{n-1})]$$

is n-dimensional manifold with some singular simplexes bounded by $M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)$ and M^{n-1}. We denote it by $\overline{K}{}^n_0$.

We replace $(e^{\alpha_k}_0, e^{\alpha_k}_1, \ldots, e^{\alpha_k}_{n-1})$ by $(e^{\alpha_k}_p, e^{\alpha_k}_{p+1}, \ldots, e^{\alpha_k}_{n-1}, e^{\alpha_k}_0, \ldots, e^{\alpha_k}_{p-1})$, then

$$\sum_\alpha \sum_{\alpha_k} [(a^{\alpha_k}_0, a^{\alpha_k}_1, \ldots, a^{\alpha_k}_{n-1}, e^{\alpha_k}_p) + (-1)^n (a^{\alpha_k}_1, a^{\alpha_k}_2, \ldots, a^{\alpha_k}_{n-1}, e^{\alpha_k}_p, e^{\alpha_k}_{p+1}) +$$
$$\ldots + (-1)^{n(n-1)} (a^{\alpha_k}_{n-1}, e^{\alpha_k}_p, e^{\alpha_k}_{p+1}, \ldots, e^{\alpha_k}_{n-1}, e^{\alpha_k}_0, e^{\alpha_k}_1, \ldots, e^{\alpha_k}_{p-1})]$$

is n-dimensional manifold with some singular simplexes bounded by $M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)$ and M^{n-1}. We denote it by $\overline{K}_p$. Since M^{n-1} is homologous zero in S^m, there exists an n-complex $\overline{\overline{K}}{}^n$ bounded by M^{n-1}. $K^n_p = \overline{K}{}^n_p + \overline{\overline{K}}{}^n$ is an n-complex bounded by $M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)$. As p_0 is an interior point of τ^n_0, p_0 and vertices of each $(n-1)$-face of τ^n_0 form n-simplexes τ^n_{00}, $\tau^n_1, \ldots, \tau^n_{0n}$. If we replace τ^n_0 by $\tau^n_{00}, \ldots, \tau^n_{0n}$ in $\overset{*}{K}{}^n$, then the complex thus obtained is a finer simplicial subdivision than $\overset{*}{K}{}^n$. We may denote the resulted complex by the same notation $\overset{*}{K}{}^{n}$ for brevity, without any confusion. Similarly we can consider a fine simplicial triangulation of K^m by inverse image of $\overset{*}{K}{}^{n}$ by f and a suitable additional subdivision. We shall also denote it by K^m. $[O, M_1] \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)$ which is bounded by $M_1 \cap \left(\bigcap_{i=2}^{r}[O, M_i]\right)$ may be considered as one of K_p^n. When M_i is mapped on ξ by f we denote the complex which is mapped on τ_0^n by $R(\xi_i)$. Evidently $R(\xi_i) = R(\xi_j)$ $(i \neq j, i. j = 0, 1, 2, \dots, r)$. We denote this common complex by R and its n-skeleton by R^n.

THEOREM 3.1. *If* $W_r(f) = 0$, *then* $f \mid R^n$ *is homotopic to zero.*

PROOF. The map f can be considered as a standard map without any loss of generality by Lemma 2.1, when we use Lemma 3.1 and Lemma 3.2. As $W_r(f) = 0$, $\phi\left(M_0^{r-n}, \bigcap_{i=1}^{r}[O, M_i]\right) = 0$ and there is a set of n-simplexes $\sigma_1, \sigma_2, \dots, \sigma_s; \sigma_1', \sigma_2', \dots, \sigma_s'$, on $\bigcap_{i=1}^{r}[O, M_i]$, where σ_i and σ' are mapped on S^n positively and negatively, respectively. $\bigcap_{i=1}^{r}[O, M_i]$ may be considered as one of K_p^n. For σ_i and σ_i' $(i = 1, 2, \dots, s)$, there are regularly connected chain $\sigma_i + \sigma_{i_1} + \sigma_{i_2} + \dots + \sigma_{i_k} + \sigma_i'$ on K_p^n. It may be supposed that $d_f(\sigma_{i1}) = +1, d_f(\sigma_{i1}) = \dots = d_f(\sigma_{i_k}) = 0$, $d_f(\sigma_i') = -1$. Using Lemma 3.1, we deform f in $\sigma_i + \sigma_{i_1}$, next in $\sigma_{i_1} + \sigma_{i_2}$, etc.; then using Lemma 3.2, we deform the map in $\sigma_{i_k} + \sigma_i'$. The new map f' has as its degree $d_{f'}(\sigma_i) = d_f(\sigma_{i1}) = \dots = d_f(\sigma_i') = 0$. We continue in this manner untill no simplexes are mapped positively and none are mapped negatively over S^n. Then $f \mid K_p^n$ is homotopic to zero, fixing the image of M_1 on ξ_1.

By Definition, $K_p^n = K_p^n + \overline{\overline{K}}{}^n$ and $\overline{K}_p^n$ have no common n-simplex for any p. When f is simplicial mapping, we consider the state where K_p^n are mapped on τ_{00}^n. $\overline{K}_p^n$ consist of the following complex:

$$(a_0^{\alpha_k}, a_1^{\alpha_k}, \dots, a_{n-1}^{\alpha_k}, e_p^{\alpha_k}) + (-1)^n(a_1^{\alpha_k}, a_2^{\alpha_k}, \dots, a_{n-1}^{\alpha_k}, e_p^{\alpha_k}, e_{p+1}^{\alpha_k}) + \dots$$
$$+ (-1)^{n(n-1)}(a_{n-1}^{\alpha_k}, e_p^{\alpha_k}, e_{p+1}^{\alpha_k}, \dots, e_{nn-1}^{\alpha_k}, e_0^{\alpha_k}, e_1^{\alpha_k}, \dots, e_{p-1}^{\alpha_k}).$$

If $(a_{n-1}^{\alpha_k}, e_p^{\alpha_k}, e_{p+1}^{\alpha_k}, \dots, e_{n-1}^{\alpha_k}, e_0^{\alpha_k}, e_1^{\alpha_k}, \dots, e_{p-1}^{\alpha_k})$ is mapped on τ_{00}^n, other simplexes are mapped on faces of τ_{00}^n and their dimensions depend on numbers of e_i. $(a_{n-1}^{\alpha_k}, e_p^{\alpha_k}, e_{p+1}^{\alpha_k}, \dots, e_{n-1}^{\alpha_k}, e_0^{\alpha_k}, e_1^{\alpha_k}, \dots, e_{p-1}^{\alpha_k})$ is mapped on τ_{00}^n in the same manner for each p except for orientation.

We may neglect this orientation when we take care of this similar property for all α_k. If $(a_{n-1}^{\alpha_k}, e_p^{\alpha_k}, e_{p+1}^{\alpha_k}, \dots, e_{n-1}^{\alpha_k}, e_0^{\alpha_k}, e_1^{\alpha_k}, \dots, e_{p-1}^{\alpha_k})$ is not mapped on τ_{00}^n, we may neglect T_α^m. In other words, we may consider that $(a_0^{\alpha_k}, a_1^{\alpha_k}, \ldots, a_{n-1}^{\alpha_k})$ contract tos a point and T_α^m is empty. As $\overline{\overline{K^n}}$ is not mapped on τ_{00}^n, we may consider only $\overline{K}_p$ for the degree based on τ_{00}^n. On the other hand, deformations of Lemma 3.1 and Lemma 3.2 can be introduced leaving the degree of $\overline{\overline{K^n}}$ fixed. If we deform f to the standard map f' by Lemma 2.1, f' maps $\sum_p K_p^n$ on $\overline{\xi}_0$ and f maps $R^n - \sum_p K_p^n$ on $\overline{\xi}_0$ by the above remark. Hence $f|R^n \simeq 0$.

When M_i is mapped on ξ_i by f_1 and f_2 we denote the complex which is mapped on τ_0^n by R_1 and R_2 respectively. We denote their n-dimentional skeleton by R_i^n $(i = 1, 2)$.

THEOREM 3.2. *Let f_1 and f_2 be continuous mappings and $W_r(f_1)$ be equal to $W_r(f_2)$, then $f_1|R_1^n + R_2^n \simeq f_2|R_1^n + R_2^n$.*

PROOF. We consider Cartesian $(m+1)$-space $\mathfrak{C}^{m+1}$ and its subsets:

$$S^m = \left\{x \in \mathfrak{C}^{m+1};\ \sum_{i=1}^{m+1} x_i^2 = 1\right\},$$
$$E_+^n = \{x \in S^n;\ x_{m+1} \geqq 0\},$$
$$E_-^m = \{x \in S^m;\ x_{m+1} \leqq 0\},$$
$$S_0^{m-1} = \{x \in S^m;\ x_{m+1} = 0\}.$$

We and define φ_1 as follows:

φ_1 maps E_+^m onto S^m,

φ_1 is a homeomorphism on $E_+^m - S_0^{m-1}$,

$\varphi(S_0^{m-1}) = P$, where P is a fixed point on S^n, $d(\varphi_1) = 1$.

We also define φ_2 as follows:

φ_2 maps E_-^m onto S^m,

φ_2 is a homeomorphism on $E_-^m - S_0^{n-1}$,

$\varphi_2(S_0^{m-1}) = P$,

$d(\varphi_2) = -1$.

We may assume $f_1(P) = f_2(P) = Q$ without any loss of generality. We construct a map F of S^m into S^n as follows:

$$F = \begin{cases} f_1\ \varphi_1 & \text{on } E_+^m, \\ f_2\ \varphi_2 & \text{on } E_-^m. \end{cases}$$

From $W_r(f_1) = W_r(f_2)$ we know $W_r(F) = 0$. If we denote by $[F]$, $[f_1]$, $[f_2]$ the homotopy classes of F, f_1 and $f_2$respectively, $[F] = [f_1] - [f_2]$.

By Theorem 3.1, $F|R_1^n + R_2^n \simeq 0$, then $f_1|R_1^n + R_2^n \simeq f_2|R_1^n + R_2^n$.

Pontrjagin theorem[6] may be obtained from Theorem 3.2 as its special case when we put $r = 1$, and $n = 2$. For the proof, see my paper.

THEOREM 3.3. (Pontrjagin's theorem). *If f_1 and f_2 are maps of S^3 on S^2 and $W_1(f_1)$ is equal to $W_1(f_2)$, then f_1 is homotopic to f_2.*

4. Freudenthal introduced the idea of suspension in the well known paper [2]. We shall investigate in this section some relations of the suspension and $W_r(f)$.

Let f be a map of S^m into S^n. We denote the equators of S^{m+1} and S^{n+1} by S^m and S^n respectively and extend f to a map Ef of S^{m+1} into S^{n+1} as follows:

A point of S^{m+1} is represented by (P, β), where P is a point of S^m and $-1 \leqq \beta \leqq 1$. Similarly, a point of S^{n+1} is represented by (P', β), where P' is a point of S^n and $-1 \leqq \beta \leqq 1$. We define

$$Ef(P, \beta) = (f(P), \beta).$$

If $m+1 = (r+1)(n+1) - r$, and $n+1$ is even, it is trivial that $W_r(Ef) = 0$. Secondly we investigate its inverse. We denote a subset of $\pi_{m+1}(S^{n+1})$ whose elements have 0 as $W_r(f)$ invariant by $[\pi_{m+1}(S^{n+1})]_0$.

THEOREM 4.1. $E(\pi_m(S^n)) = [\pi_{m+1}(S^{n+1})]_0$.

PROOF. We consider a map f of S^{m+1} into S^{n+1} where $W_r(f) = 0$. If we can prove that the inverse image of a point P' of S^{n+1} consists of only one point P, we can prove this theorem as follows: Let V_1^{m+1} be a closed neighborhood of P on S^{m+1} and V_2^{m+1} be the closure of the complement of V_1^{m+1}. Then $f(V_1^{m+1})$ or $f(V_2^{m+1})$ does not completely cover S^{n+1} and we can deform f to a form of Eg. In order to prove that the inverse image of P' consists of only one point it is sufficient to prove two following properties: 1° $\bigcap_{i=1}^{r} [O, M_i]$ is mapped on P' by f_1 which is homotopic to f. 2° $\bigcap_{i=1}^{r} [O, M_i]$ is contractible to one point on itself. We shall prove them. By $W_r(f) = 0$, we have $\phi(M_0, \bigcap_{i=1}^{r} [O, M_i]) = 0$. As we know in the proof of Theorem 3.1, the image of $\bigcap_{i=1}^{r} [O, M_i]$ can be contractible to a point fixing the image of $M_i \cap \left(\bigcap_{i=2}^{r} [O, M_i]\right)$. Then the image of $\bigcap_{i=1}^{r} [O, M_i]$ by f_1' can be considered to be P'. At first we consider any point R in distance $\rho \leqq \varepsilon$ of $\bigcap_{i=1}^{r} [O, M_i]$, where ε is sufficiently small and denote by Q the fixed point of the segment PO for which $RQ/QO = (\varepsilon - \rho)/\rho$. Let $\varphi(R, \tau)$ move linearly along the segment RQ as τ move from 0 to 1. For the point R in distance $\rho \geqq \varepsilon$, we set $\varphi(R, \tau) = R$. We denote the inverse of $\varphi(R, \tau)$ by $\psi(R, \tau)$, and define $f_{1+\tau} = f_1(\psi(R, \tau))$, then f_2 maps O and only O on P'. The proof of (2) is complete.

We define some special sets which are used for the proof of Theorem 4.2.

Let $\mathfrak{C}^{m+2}$ be the Cartesian $(m+2)-$space. We define its subsets as follows:

$$S^{m+1} = \left\{x \in \mathfrak{C}^{m+2} : \sum_{i=1}^{m+2} x_i^2 = 1\right\},$$
$$S^m = \{x \in S^{m+1} : x_{m+1} = 0\},$$
$$S^m_{\beta_0} = \{x \in S^{m+1} : x_{m+2} = \beta_0\},\ (-1 \leqq \beta_0 \leqq 1)$$
$$V^{m+2} = \left\{x \in \mathfrak{C}^{m+2} : \sum_{i=1}^{m+2} x_i^2 \leqq 1\right\},$$
$$E^{m+1}_{\beta_0} = \{x \in V^{m+2} : x_{m+2} = \beta_0\},$$
$$V^{m+1}_{\geqq \beta_0} = \{x \in S^{m+1} : x_{m+2} \geqq \beta_0\},$$
$$V^{m+1}_{\leqq \beta_0} = \{x \in S^{m+1} : x_{m+2} \leqq \beta_0\}.$$
$$V^{m+1}_{\beta_1, \beta_2} = \{x \in S^{m+1} : \beta_1 \leqq x_{m+2} \leqq \beta_2\},$$

Similarly we define for $\mathfrak{C}^{n+2}$.

THEOREM 4.2. *If $m = (r+1)n - r$ and n is even, E is an isomorphism of $[\pi_m(S^n)]_3$.*

PROOF. Let g be a map of S^m into S^n and be $f = Eg = 0$. Then f can be extended to a map of $V^{m+2}(S^{m+1} = R\partial V^{m+2})$ into S^{n+1}. Let τ_0^{n+1} be a fixed simplex of S^{n+1} and $\xi_i (i = 1, 2, 3, \dots, r)$ be the fixed interior points of τ_0^{n+1} and let ξ_i exist on S^n_β $(-1 < \beta < 1)$. We denote the inverse image of ξ_i in S^{m+1} by M_i^{m-n} and the inverse image of ξ_i in V^{m+2} by Y_i^{m-n+1}.
Then, we have

$$f\left\{\bigcap_{i=1}^{r} [O, M_i]\right\} = W_r(f) S^n_{\beta_0},$$

and $$R\partial\, Y_i^{m-n+1} = M_i^{m-n}.$$

On the other hand M_i^{m-n} belongs to $S^m_{\beta_0}$ essentially and there exists a complex K_i^{m-n+1} bounded by M_i^{m-n} in $S^m_{\beta_0}$. When we denote the fixed point of $S^m_{\beta_0}$ by O, we can consider $[O, M_i^{m-n}]$as K_i^{m-n+1}. We define

$$Z^n = Y_1^{m-n+1} \cap \bigcap_{i=2}^{r} [O, M_i] - \bigcap_{i=1}^{r} [O, M_i].$$

There exist a complex K^{n+1} which is bounded by Z^n in V^{m+2}.

$$R\partial f(K^{n+1}) = f(R\partial K^{n+1}) = f\left\{Y_i^{m-n+1} \cap \left(\bigcap_{i=2}^{r} [O, M_i]\right)\right\} - f\left\{\bigcap_{i=2}^{r} [O, M_i]\right\}$$
$$= -W_r(f) S^n,$$

$f(K^{n+1})$ covers c'-times over $V^{n+1}_{\geqq \beta_0}$ and c''-times over $V^{n+1}_{\leqq \beta_0}$.

$$-W_r(f) = c' - c''.$$

c' and c'' have the following properties:

(i) c' and c'' are the same value when we use Y_i^{n+1} $(i = 2, 3, \dots r)$.

In the relation of Z^n we replace Y_1^{m-n+1} by Y_i^{m-n+1}, then it is similar to the proof of Lemma 2.8 that c' and c'' are invariant.

(ii) c' and c'' are not depend on a choice of K^{n+1}.

In $f(K^{n+1})$ the difference of K^{n+1} give us the difference only of the image of bounding cycle.

(iii) c' and c'' do not depend on a choice of O. The proof is similar as in Lemma 2.7.

(iv) c' and c'' do not depend on a choice of ξ_i. We shall prove it.

Let ξ_i and $\overset{*}{\xi}_i$ be the fixed interior points of τ_0^{n+1}, where ξ_i, $\overset{*}{\xi}_i$ exist on $S^n_{\beta_0}$ and $S^n_{\beta_2}$ respectively. We introduce the following complexes:

$$M^{m-n},\ Y_1^{m-n+1},\ [O, M_1],\ Z^n,\ K^{n+1};$$
$$\overset{*}{M}^{m-n},\ \overset{*}{Y}_1^{m-n+1},\ [\overset{*}{O}, \overset{*}{M}_1],\ \overset{*}{Z}^n,\ \overset{*}{K}^{n+1}.$$

We consider a segment x_i in $V^{n+1}_{\beta_1,\beta_2}$ whose end points are $\overset{*}{\xi}_i$ and ξ_i and denote the inverse image of x_1 in S^{m+1} by Z_{12}^{m-n+1}, the inverse image of x_1 in V^{m+2} by Y_{12}^{m-n+2}. Besides we denote $V^{m+1}_{\beta_1,\beta_2}$ by $S^m\times I$ where I is an interval $\beta_1\leqq t\leqq\beta_2$. We connect O with $\overset{*}{O}$ by an arc in $S^m\times I$ which intersect with $S^m\times(t)$ on only one point. These points of intersection will be denoted by O_t. We also denote $x_i\cap S^n\times(t)$ by ξ_{it} in $S^m\times(t)$ by M_{it} and the inverse image of ξ_{it} in E_t^{m+1} by C_{it}^{m-n+1}. Then we have

$$R\partial\Big[Y_{12}^{m-n+2}\cap\Big\{\bigcap_{\beta_1\leqq\ \leqq\beta_2}\Big(\bigcap_{i=2}^{r}[O_t, M_{it}]\Big)\Big\}\Big]$$
$$=\overset{*}{Y}_1^{m-n+1}\cap\Big(\bigcap_{i=2}^{r}[\overset{*}{O},\overset{*}{M}_i]\Big)-Y_1^{m-n+1}\cap\Big(\bigcap_{i=2}^{r}[O, M_i]-$$
$$Z_{12}^{m-n+1}\cap\Big\{\bigcup_{\beta_1\leqq t\leqq\beta_2}\Big(\bigcap_{i=2}^{r}[O_t, M_{it}]\Big)\Big\},$$

and

$$R\partial\Big[Z_{12}^{m-n+1}\cap\Big\{\bigcup_{\beta_1\leqq t\leqq\beta_2}\Big(\bigcap_{i=2}^{r}[O_t, M_{it}]\Big)\Big\}\Big]$$
$$=\overset{*}{M}^{m-n}\cap\Big(\bigcap_{i=2}^{r}[\overset{*}{O},\overset{*}{M}_i]\Big\}-M^{m-n}\cap\Big(\bigcap_{i=2}^{r}[O, M_i]\Big\}-$$
$$\Big\{\bigcup_{\beta_1\leqq\ \leqq\beta_2}(M_i\cap\Big(\bigcap_{i=2}^{r}[O_t, M_{it}]\Big)\Big\}.$$

We define

$$L^{n+1}=\overset{*}{K}^{n+1}-K^{n+1}-Y_{12}^{m-n+2}\cap\Big\{\bigcup_{\beta_1\leqq t\leqq\beta_2}\Big(\bigcap_{i=2}^{r}[O_t, M_{it}]\Big)\Big\}.$$

Then

$$R\partial\ L^{n+1}=\overset{*}{Z}^n-Z^n+Y_1\cap\Big(\bigcap_{i=2}^{r}[O, M_i]\Big)-\overset{*}{Y}_1\cup\Big(\bigcap_{i=2}^{r}[\overset{*}{O},\overset{*}{M}_i]\Big)+Z_{12}^{m-n+1}$$

$$\Big\{\bigcup_{\beta_1\leqq l\leqq\beta_2}\Big(\bigcap_{i=2}^{r}[O_i, M_{il}]\Big)\Big\}$$

$$=\bigcap_{i=1}^{r}[O, M_i]-\bigcap_{i=1}^{r}[\overset{*}{O}, \overset{*}{M}_i]+Z_{12}^{m-r+1}\cap\Big\{\bigcup_{\beta_1\leqq l\leqq\beta_2}\Big(\bigcap_{i=2}^{r}[O_i, M_{il}]\Big)\Big\}.$$

This complex is bounded by a cycle M^{n+1} included in V^{m+1}. Therefore $R\partial L^{n+1}=R\partial M^{n+1}$. By the fact of $M^{n+1}\subset V^{m+1}_{\beta_1,\beta_2}$, we know $f(M^{n+1})\subset V^{n+1}_{\beta_1.\beta_2}$ and $f(M^{n+1})$ cover the north pole and south pole with the degree 0. As n is even, $c'+c''=0$, therefore $-W_r(f)=c'-c''=2c'>0$.

To prove Theorem 4.2, using $W_r(f)=c'=0$ and $Eg=0$, it is sufficient to prove $g=0$. Now, by $W_r(f)=0$ we can deform f in S^m_β so as to map $\bigcap_{i=1}^{r}[O, M_i]$ to one point and also $f(R\partial K^{n+1})$ to one point. By $c'=0$, the image of K^{n+1} is contractible to one point fixing $R\partial K^{n+1}$ and also K^{n+1} can be contractible to one point. Then, if $f(V^{m+2})\subset S^{n+1}$, one point (for example north pole) has one point (for example north pole) and only one point as inverse image, therefore g is not essential.

5. Product theorem.

THEOREM 5.1. *Let g be a map of one m-sphere S^m_1 into another n-sphere S^m of degree c. If f is a map of S^m into S^n, then $W_r(fg)=c^rW_r(f)$.*

PROOF. As c and $W_r(f)$ are constant for the homotopy class of maps respectively, we can assume that f and g are simplicial mappings. Let ξ_i be a fixed point of S^n and $\varphi(\xi_i)$ and $\psi(\xi_i)$ the inverse image of ξ_i for f and fg respectively. We also assume that m-simplex T^m of S^m intersects with $\varphi(\xi_i)$ and the number of m-simplexes of S^m_1 mapped on T^m positively or negatively by g is p or q respectively. Then $p-q=c$ and the number of $(m-n)$-simplex of $\psi(\xi_i)$ are mapped on $(m-n)$-simplex of $\psi(\xi_i)$ which are included in T^m positively or negatively is p or q. Conversely the image of each $(m-n)$-simplex of $\psi(\xi_i)$ are $(m-n)$-simplex of $\varphi(\xi_i)$. Therefore we have

$$g(\psi(\xi_i))=c\cdot\varphi(\xi_i).$$

We define L_i^{m-n+1}, K_i^{m-n+1} as follows:

$$R\partial L_i^{m-n+1}=\psi(\xi_i),$$
$$R\partial K_i^{m-n+1}=\varphi(\xi_i).$$

Then,

$$R\partial(g(L_i^{m-n+1}))=g(R\partial(L_i^{m-n+1})=g(\psi(\xi))=c\cdot\varphi(\xi)=c\cdot R\partial K^{m+n-1},$$

and, $g(L_i^{m-n+1})-c\cdot K_i^{m-n+1}$ is a cycle of S^m. On the other hand, $g(L_i^{m-n+1})-c\cdot K_i^{m-n+1}\sim 0$ in S^m and $fg\,(L_i^{m-n+1})-cf\,(K_i^{m-n+1})\sim 0$ in S^n.

Then, $fg(L_i^{m-n+1})=c\cdot f(K_i^{m-n+1})$.

As in the proof of Theorem 2.1, we may use $[O, \varphi(\xi_i)]$, $[O, \psi(\xi_i)]$ for K_i^{m-n+1}, L_i^{m-n+1} respectively. Then $fg\Big(\bigcap_{i=1}^{r}[O, \psi(\xi_i)]\Big)=c^r\cdot f\Big(\bigcap_{i=1}^{r}$

$[O, \varphi(\xi_i)]\Big)$, therefore $W_r(fg) = c^r \cdot W_r(f)$.

THEOREM 5.2. *Let h be a map of one n-sphere S^n into another n-sphere S_1^n of degree c. If f is a map of S^m into S^n, then*

$$W_r(hf) = c^{r+1} W_r(f).$$

PROOF. As c and $W_r(f)$ are constant for the homotopy class of maps respectively, we can assume that h and f are simplicial mappings. Let ξ be a fixed point of S^n and φ and ψ the inverse image for f and hf respectively. The inverse image of ξ for h are denoted by, $\eta_1, \dots, \eta_p$; $\zeta_1, \zeta_2, \dots, \zeta_q$; where n-simplexes including η_i and ζ_i are mapped on n-simplex including ξ positively and negatively respectively. It is clear that

$$c = p - q \text{ and } \qquad \psi(\xi) = \Sigma\varphi(\eta_i) - \Sigma\varphi(\zeta_j).$$

We define K_i^{m-n+1}, L_j^{m-n+1} as follows:

$$R\partial K_i^{m-n+1} = \varphi(\eta_i)$$
$$R\partial L_j^{m-n+1} = \varphi(\zeta_j).$$

Therefore we have

$$R\partial\Big(\Sigma K_i^{m-n+1} - \sum_j L_j^{m-n+1}\Big) = \psi(\xi)$$

$\sum K_i^{\ -n+1} - \sum L_j^{m-n+1}$ is used for $W_r(hf)$, analogously

$\sum K_i^{m-n+1} - \sum L_j^{m-n+1}$ is used for $(p-q)\ W_r(f)$. The degree c above considered is the degree of for h. Therefore we have.

$$W_r(hf) = c^r \cdot c \cdot W_r(f) = c^{r+1}\ W_r(f).$$

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